Braiding and Mixing

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Experiment of Boyland et al.

Four Basic Operations

\[ \sigma_1 \quad \sigma_2 \]

\[ \sigma_1^{-1} \quad \sigma_2^{-1} \]

\[ \sigma_1 \] and \[ \sigma_2 \] are referred to as the generators of the 3-braid group.
Two Stirring Protocols

$\sigma_1 \sigma_2$ protocol

$\sigma_1^{-1} \sigma_2$ protocol

Braiding

$\sigma_1 \sigma_2$ protocol  \hspace{1cm} $\sigma_1^{-1} \sigma_2$ protocol

Matrix Representation of $\sigma_2$

Let $I$ and $II$ denote the lengths of the two segments. After a $\sigma_2$ operation, we have

$$
\begin{pmatrix}
I' \\
II'
\end{pmatrix} = \begin{pmatrix} I + II \\
II
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix} I \\
II
\end{pmatrix} = \sigma_2 \begin{pmatrix} I \\
II
\end{pmatrix}.
$$

Hence, the matrix representation for $\sigma_2$ is

$$\sigma_2 = \begin{pmatrix} 1 & 1 \\
0 & 1
\end{pmatrix}.$$
Matrix Representation of $\sigma_1^{-1}$

Similarly, after a $\sigma_1^{-1}$ operation we have

\[
\begin{pmatrix}
I' \\
I + II
\end{pmatrix} = 
\begin{pmatrix}
I \\
I + II
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} 
\begin{pmatrix}
I \\
II
\end{pmatrix} = \sigma_1^{-1} 
\begin{pmatrix}
I \\
II
\end{pmatrix}.
\]

Hence, the matrix representation for $\sigma_1^{-1}$ is

\[
\sigma_1^{-1} = 
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}.
\]
Matrix Representation of the Braid Group

We now invoke the faithfulness of the representation to complete the set,

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};
\]

\[
\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \quad \sigma_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

Our two protocols have representation

\[
\sigma_1 \sigma_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \quad \sigma_1^{-1} \sigma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.
\]
The Difference between the Protocols

- The matrix associated with each generator has unit eigenvalues.

- The first stirring protocol has eigenvalues on the unit circle.
- The second has eigenvalues $\sqrt{5} = 2.236180...$ for the larger eigenvalue.
- So for the second protocol the length of the lines I and II grows exponentially!
- The larger eigenvalue is a lower bound on the growth factor of the length of material lines.
- That is, material lines have to stretch by at least a factor of $\sqrt{5} = 2.236180...$ each time we execute the protocol.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).
The Difference between the Protocols

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  \[ \frac{2}{2^{6180}} \]
  
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The Difference between the Protocols

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Freely-moving Rods in a Cavity Flow

[A. Vikhansky, Physics of Fluids 15, 1830 (2003)]
Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).

Material lines must bend around the orbit: it acts just like a “rod”!
The idea: pick any three fluid particles and follow them.

How do they braid around each other?
In the second case there is no net braid: the two elements cancel each other.
Random Sequence of Braids

We end up with a sequence of braids, with matrix representation

$$\sum^{(N)} = \sigma^{(N)} \cdots \sigma^{(2)} \sigma^{(1)}$$

where $\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and $N$ is the number of braiding events detected after a time $t$. 
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where \(\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}\) and \(N\) is the number of braiding events detected after a time \(t\).

The largest eigenvalue of \(\Sigma^{(N)}\) is a measure of the complexity of the braiding motion, called the braiding factor.

Random matrix theory says that the braiding factor can grow exponentially! We call the rate of exponential growth the braiding Lyapunov exponent or just braiding exponent.
First consider the motion of three points in concentric circles with irrationally-related frequencies.

The braiding factor grows linearly, which means that the braiding exponent is zero. Notice that the eigenvalue often returns to unity.
To demonstrate good braiding, we need a chaotic flow on a bounded domain (a spatially-periodic flow won’t do).

Aref’s blinking-vortex flow is ideal.

The only parameter is the circulation $\Gamma$ of the vortices.
Blinking Vortex: Non-braiding Motion

For $\Gamma = 0.5$, the blinking vortex has only small chaotic regions.

One of the orbits is chaotic, the other two are closed.
For $\Gamma = 13$, the blinking vortex is globally chaotic.

The braiding factor now grows exponentially. In the same time interval as for $\Gamma = 0.5$, the final value is now of order $10^{20}$ rather than 80!
Averaging over many Triplets

\[ \log(\text{eigenvalue of } \Sigma(N)) \]

slope = 0.187
\[ \Gamma = 13 \]

Averaged over 100 random triplets.
Comparison with Lyapunov Exponents

\[ \Gamma \text{ varies from 8 to 20.} \]
Beyond Three Particles

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But does it Saturate?

\[
\Gamma = 16.5
\]

Braiding exponent

\[
\begin{align*}
& n \\
& \begin{array}{c}
0 \\
20 \\
40 \\
60
\end{array}
\end{align*}
\]
Conclusions

- Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
- The complexity of a braid can be represented by the largest eigenvalue of a product of matrices—the braiding factor.
- Any collection of $n$ particles can potentially braid.
- The complexity of the braid is a good measure of chaos.
- No need for infinitesimal separation of trajectories or derivatives of the velocity field.
- For instance, can use all the floats in a data set (J. La Casce).
- Test in 2D turbulent simulations (F. Paparella).
- Many issues to investigate: faithfulness of representation, lower-bound for topological entropy...