DEFORMATION THEORY OF SCALAR-FLAT KÄHLER ALE SURFACES

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Abstract. We prove a Kuranishi-type theorem for deformations of complex structures on ALE Kähler surfaces. This is used to prove that for any scalar-flat Kähler ALE surface, all small deformations of complex structure also admit scalar-flat Kähler ALE metrics. A local moduli space of scalar-flat Kähler ALE metrics is then constructed, which is shown to be universal up to small diffeomorphisms (that is, diffeomorphisms which are close to the identity in a suitable sense). A formula for the dimension of the local moduli space is proved in the case of a scalar-flat Kähler ALE surface which deforms to a minimal resolution of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of U(2) without complex reflections.

Contents

1. Introduction

This article is concerned with the following class of metrics:

Definition 1.1. *An ALE Kähler surface (X, g, J) is a Kähler manifold of complex dimension 2 with the following property. There exists a compact subset $K \subset M$ and a diffeomorphism $\psi : X \setminus K \to (\mathbb{R}^4 \setminus \overline{B})/\Gamma$, such that for each multi-index $\mathcal{I}$ of order $|\mathcal{I}|$*

$$\partial^{\mathcal{I}}(\psi_*(g) - g_{Euc}) = O(r^{-\mu-|\mathcal{I}|}), \tag{1.1}$$

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as $r \to \infty$, where Γ is a finite subgroup of $\mathrm{U}(2)$ containing no complex reflections, B denotes a ball centered at the origin, and g_{Euc} denotes the Euclidean metric. The real number μ is called the order of g.

Remark 1.2. In this paper, henceforth Γ will always be a finite subgroup of $\mathrm{U}(2)$ containing no complex reflections.

We are interested in the class of scalar-flat Kähler ALE metrics. These are interesting since they are *extremal* in the sense of Calabi [Cal85], and they arise as "bubbles" in gluing constructions for extremal Kähler metrics [ALM15a, ALM15b, AP06, APS11, BR15, RS05, RS09]. In the case of scalar-flat Kähler ALE metrics, it is known that there exists an ALE coordinate system for which the order of such a metric is at least 2 [LM08].

We note that for an ALE Kähler metric of order μ, there exist ALE coordinates for which

$$\partial^{\mathcal{I}}(J - J_{Euc}) = O(r^{-\mu-|\mathcal{I}|}), \tag{1.2}$$

for any multi-index $\mathcal{I}$ as $r \to \infty$, where J_{Euc} is the standard complex structure on Euclidean space [HL15].

There are many known examples of scalar-flat Kähler ALE metrics:

- SU(2) case: when $\Gamma \subset \mathrm{SU}(2)$, Kronheimer has constructed families of hyperkähler ALE metrics [Kro89a] on manifolds diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$. In [Kro89b], Kronheimer also proved a Torelli-type theorem classifying hyperkähler ALE surfaces. In the A_k case, these metrics were previously discovered by Eguchi-Hanson for $k = 1$ [EH79], and by Gibbons-Hawking for all $k \geq 1$ [GH78].
- Cyclic case: For the $\frac{1}{p}(1, q)$-action, Calderbank-Singer constructed a family of scalar-flat Kähler ALE metrics on the minimal resolution of any cyclic quotient singularity [CS04]. These metrics are toric and come in families of dimension $k - 1$, where k is the length of the corresponding Hirzebruch-Jung algorithm. For $q = 1$ and $q = p - 1$, these metrics are the LeBrun negative mass metrics and the toric multi-Eguchi-Hanson metrics, respectively [LeB88, GH78].
- Non-cyclic non-SU(2) case: The existence of scalar-flat Kähler metrics on the minimal resolution of $\mathbb{C}^2/\Gamma$, was shown by Lock-Viaclovsky [LV14].

The question we address in this paper is whether the scalar-flat Kähler property is preserved under small deformations of complex structure. In the cases where $\Gamma \subset \mathrm{SU}(2)$, the hyperkähler quotient construction produces hyperkähler metrics for the minimal resolution complex structure as well as for all small deformations of the minimal resolution complex structure. In the case of the LeBrun negative mass metrics, using arguments from twistor theory, Honda has shown that all small deformations of the complex structure on $\mathcal{O}(-n)$ also admit scalar-flat Kähler metrics [Hon13, Hon14]. For the case of general Γ, in [LV14], employing Honda's result, it was shown that *some* of small deformations of complex structure admit scalar-flat Kähler ALE metrics.

Our main result in this paper shows that for *any* scalar-flat Kähler ALE surface, *all* small deformations of complex structure admit scalar-flat Kähler ALE metrics. Our proof is analytic in nature, and in particular, gives a new analytic proof of Honda's result on $\mathcal{O}(-n)$ mentioned above.

To state our result precisely, we next recall some basic facts regarding deformations of complex structures. For a complex manifold (X, J), let $\Lambda^{p,q}$ denote the bundle of (p, q)-forms, and let Θ denote the holomorphic tangent bundle. The deformation complex

$$\Gamma(\Theta) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta) \tag{1.3}$$

corresponds to a real complex

$$\Gamma(TX) \xrightarrow{\mathfrak{L}_X J} \Gamma(End_a(TX)) \xrightarrow{N'_J} \Gamma\big((\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}}) \otimes TX\big), \tag{1.4}$$

where $\mathfrak{L}_X J$ is the Lie derivative of J,

$$End_a(TX) = \{I \in End(TX) : IJ = -JI\} \tag{1.5}$$

(the subscript a is used here since these are exactly the endomorphisms which extend to $\mathbb{C}$-antilinear endomorphisms of the complexified tangent bundle), and N'_J is the linearization of Nijenhuis tensor

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\} \tag{1.6}$$

at J. The isomorphism from the complex (1.3) to (1.4) is simply taking the real part of a section. If g is a Hermitian metric compatible with J, then let $\Box$ denote the $\bar{\partial}$-Laplacian

$$\Box \equiv \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*, \tag{1.7}$$

where $\bar{\partial}^*$ denotes the formal L^2-adjoint. Each bundle in the complex (1.3), admits a $\Box$-Laplacian, and these correspond to real Laplacians on each bundle in the complex (1.4). We will use the same $\Box$-notation for these real Laplacians.

To state our most general result, we need the following definition. This is necessary because there is a gauge freedom of Euclidean motions in the definition of ALE coordinates.

Definition 1.3. Let (X, g, J) be a Kähler ALE surface. For any bundle E in the complex (1.3) or (1.4), and $\tau \in \mathbb{R}$, define

$$\mathcal{H}_\tau(X, E) = \{\theta \in \Gamma(X, E) : \Box\theta = 0, \theta = O(r^\tau) \text{ as } r \to \infty\}. \tag{1.8}$$

Define

$$\mathbb{W} = \{Y \in \mathcal{H}_1(X, TX) \mid \mathfrak{L}_Y g = O(r^{-1}),\ \mathfrak{L}_Y J = O(r^{-3}),\ \text{as } r \to \infty\}. \tag{1.9}$$

Finally, define the real subspace

$$\mathcal{H}_{ess}(X, End_a(TX)) \subset \mathcal{H}_{-3}(X, End_a(X)) \tag{1.10}$$

(*ess* is short for essential) to be the L^2-orthogonal complement in $\mathcal{H}_{-3}(X, End_a(X))$ of the subspace

$$\mathbb{V} = \{\theta \in \mathcal{H}_{-3}(X, End_a(TX)) \mid \theta = \mathfrak{L}_Y J,\ Y \in \mathbb{W}\}. \tag{1.11}$$

Our main result is the following.

Theorem 1.4. *Let (X, g, J) be a scalar-flat Kähler ALE surface. Let $-2 < \delta < -1$, $0 < \alpha < 1$, and k an integer with $k \geq 4$ be fixed constants. Define*

$$j = \dim_{\mathbb{R}} \big(\mathcal{H}_{ess}(X, End_a(TX))\big), \tag{1.12}$$

and let b_2 denote the second Betti number of X. Let

$$B = B^1 \times B^2, \tag{1.13}$$

where B^1 is an ϵ_1-ball in $\mathcal{H}_{ess}(X, End_a(TX))$, B^2 is an ϵ_2-ball in $\mathcal{H}_{-3}(X, \Lambda^{1,1})$ (both using the L^2-norm). Define

$$d = j + b_2. \tag{1.14}$$

Then there is a family $\mathfrak{F}$ of scalar-flat Kähler metrics near g, parametrized by B, that is, there is a differentiable mapping

$$F : B^1 \times B^2 \to Met(X), \tag{1.15}$$

into the space of smooth Riemannian metrics on X, with $\mathfrak{F} = F(B^1 \times B^2)$ satisfying the following "versal" property: there exists a constant $\epsilon_3 > 0$ such that for any scalar-flat Kähler metric $\tilde{g} \in B_{\epsilon_3}(g)$, there exists a diffeomorphism $\Phi : X \to X$, $\Phi \in C^{k+1,\alpha}_{loc}$, such that $\Phi^* \tilde{g} \in \mathfrak{F}$, where

$$B_{\epsilon_3}(g) = \{g' \in C^{k,\alpha}_{loc}(S^2(T^*X)) \mid \|g - g'\|_{C^{k,\alpha}_{\delta}(S^2(T^*X))} < \epsilon_3\}. \tag{1.16}$$

Remark 1.5. The norm in (1.16) is a certain weighted Hölder norm, see Section 2 for the precise definition. For a more precise description of the diffeomorphism Φ, see Theorem 9.2 below.

In order to state our next result, we must recall some facts about vector fields and diffeomorphisms. If (X, g) is an ALE metric, and Y is a vector field on X, the Riemannian exponential mapping $\exp_p : T_pX \to X$ induces a mapping

$$\Phi_Y : X \to X \tag{1.17}$$

by

$$\Phi_Y(p) = \exp_p(Y). \tag{1.18}$$

If $Y \in C^{k,\alpha}_s(TX)$ has sufficiently small norm, ($s < 0$ and k will be determined in specific cases) then Φ_Y is a diffeomorphism. We will use the correspondence $Y \mapsto \Phi_Y$ to parametrize a neighborhood of the identity, analogous to [Biq06].

Definition 1.6. We say that $\Phi : X \to X$ is a *small diffeomorphism* if Φ is of the form $\Phi = \Phi_Y$ for some vector field Y satisfying

$$\|Y\|_{C^{k+1,\alpha}_{\delta+1}} < \epsilon_4 \tag{1.19}$$

for some $\epsilon_4 > 0$ sufficiently small which depends on ϵ_3.

The family $\mathfrak{F}$ is not necessarily "universal", because some elements in $\mathfrak{F}$ might be isometric. However, the following theorem shows that after taking a quotient by an action of the holomorphic isometries of the central fiber (X, g, J), the family $\mathfrak{F}$ is in fact universal (up to small diffeomorphisms).

Theorem 1.7. *Let (X, g, J) be as in Theorem 1.4, and let $\mathfrak{G}$ denote the group of holomorphic isometries of (X, g, J). Then there is an action of $\mathfrak{G}$ on $\mathfrak{F}$ with the following properties.*

- *Two metrics in $\mathfrak{F}$ are isometric if they are in the same orbit of $\mathfrak{G}$.*
- *If two metrics in $\mathfrak{F}$ are isometric by a small diffeomorphism then they must be the same.*

Since each orbit represents a unique isometry class of metric (up to small diffeomorphism), we will refer to the quotient $\mathfrak{M} = \mathfrak{F}/\mathfrak{G}$ as the "local moduli space of scalar-flat Kähler ALE metrics near g." The local moduli space $\mathfrak{M}$ is not a manifold in general, but its dimension is in fact well-defined, and we define

$$m = \dim(\mathfrak{M}). \tag{1.20}$$

Remark 1.8. We should point out that our local moduli space of metrics contains small rescalings, i.e, $g \mapsto \frac{1}{c^2} g(c\cdot, c\cdot)$ for c close to 1. If one considers scaled metrics as equivalent (which we do not), then the dimension would decrease by 1.

1.1. **Deformations of the minimal resolution.** As mentioned above, there are families of examples of scalar-flat Kähler ALE metrics on minimal resolutions of isolated quotient singularities. For convenience, we next recall the definition of a minimal resolution.

Definition 1.9. Let $\Gamma \subset \mathrm{U}(2)$ be as above. Then, a smooth complex surface X is called a *minimal resolution* of $\mathbb{C}^2/\Gamma$ if there is a mapping $\pi : X \to \mathbb{C}^2/\Gamma$ such that the restriction $\pi : X \setminus \pi^{-1}(0) \to \mathbb{C}^2/\Gamma \setminus \{0\}$ is a biholomorphism, and $\pi^{-1}(0)$ is a divisor in X containing no -1 curves. The divisor $\pi^{-1}(0)$ is called the *exceptional divisor*.

In the cyclic case, the exceptional divisor is a string of rational curves with normal crossing singularities. In the case that Γ is non-cyclic, it was shown by Brieskorn, see [Bri68], that the exceptional divisor is a tree of rational curves with normal crossing singularities. There are three Hirzebruch-Jung strings attached to a single curve, called the *central rational curve*. The self-intersection number of this curve will be denoted $-b_\Gamma$, and the total number of rational curves will be denoted by k_Γ.

A specialized version of Theorem 1.4 is the following.

Theorem 1.10. *Let (X, g, J) be any scalar-flat Kähler ALE metric on the minimal resolution of $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{U}(2)$ is as above. Define*

$$j_\Gamma = 2 \sum_{i=1}^{k_\Gamma} (e_i - 1), \tag{1.21}$$

where $-e_i$ is the self-intersection number of the ith rational curve, and k_Γ is the number of rational curves in the exceptional divisor, and let

$$d_\Gamma = j_\Gamma + k_\Gamma. \tag{1.22}$$

Then there is a family, $\mathfrak{F}$, parametrized by a ball in $\mathbb{R}^{d_\Gamma}$, of scalar-flat Kähler metrics near g with the "versal" property stated in Theorem 1.4. The dimension m_Γ of the local moduli space $\mathfrak{M}$ is given in Table 1.1.

TABLE 1.1. Dimension of local moduli space of scalar-flat Kähler metrics

$\Gamma \subset \mathrm{U}(2)$	d_Γ	m_Γ
$\frac{1}{3}(1,1)$	5	2
$\frac{1}{p}(1,1), p \geq 4$	$2p-1$	$2p-5$
$\frac{1}{p}(1,q), q \neq 1, p-1$	$j_\Gamma + k_\Gamma$	$j_\Gamma + k_\Gamma - 2$
non-cyclic, not in SU(2)	$j_\Gamma + k_\Gamma$	$j_\Gamma + k_\Gamma - 1$

Remark 1.11. The dimension of the moduli space of hyperkähler metrics is known to be $3k-3$ in the A_k, D_k and E_k cases for $k \geq 2$, and equal to 1 in the A_1 case [Kro86]. Our method of parametrizing by complex structures and Kähler classes overcounts in this case (i.e., F is not injective), since a hyperkähler metric is Kähler with respect to a 2-sphere of complex structures, see Section 11 for some further remarks. For other related results in the Ricci-flat case, see [CH14, CMR15].

Our final result applies to a generic deformation of the minimal resolution.

Theorem 1.12. *Let (X, g, J) be any scalar-flat Kähler ALE surface which deforms to the minimal resolution of $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{U}(2)$ is as above, through a smooth path of ALE Kähler metrics. If $\mathfrak{G}(g) = \{e\}$ then the local moduli space $\mathfrak{F}$ is smooth near g and is a manifold of dimension $m = m_\Gamma$.*

Remark 1.13. The assumption that X deforms to the minimal resolution of $\mathbb{C}^2/\Gamma$ is reasonable. In [Joy00, Section 8.9], Joyce remarks that any Kähler ALE surface with group $\Gamma \subset \mathrm{U}(2)$ is birational to a deformation of $\mathbb{C}^2/\Gamma$. There are several possible components of the deformation of such a cone, here we consider surfaces in the "Artin component" of deformations of $\mathbb{C}^2/\Gamma$. We note however that there are some known examples of scalar-flat Kähler metrics on non-Artin components, which are free quotients of hyperkähler metrics of A_k-type, see for example [Şuv12].

We end the introduction with an outline of the paper. In Section 2, we begin with the definitions of the weighted Hölder spaces which will be used throughout the paper. Then, in Section 3 we give some analysis of the complex analytic compactifications, due to Hein-LeBrun-Maskit [HL15, LM08], of Kähler ALE spaces. In Section 4, we study the deformation of complex structures using an adaptation of Kuranishi's theory to ALE spaces. The main point is that since the manifold is non-compact, the sheaf cohomology group $H^1(X, \Theta)$ should be replaced by an appropriate space of decaying harmonic forms. In Section 5, several key results about gauging and diffeomorphisms are proved, which are used to prove "versality" of the family constructed in Section 4. In Section 6, a refined gauging procedure is carried out, to construct the Kuranishi family of "essential" deformations. In Section 7, we generalize Kodaira-Spencer's stability theorem for Kähler structures to the ALE setting, using some arguments of Biquard-Rollin [BR15]. In Section 8, we adapt the LeBrun-Singer-Simanca theory of deformations of extremal Kähler metrics to the ALE setting [LS93, LS94]. In Section 9, we prove the versal property of the family $\mathfrak{F}$, using a local slicing theorem, and prove Theorem 1.7. In Section 10, we restrict attention to the minimal resolution, and prove Theorems 1.10 and 1.12, and discuss the dimensions given in Table 1.1. In Section 11, we discuss several examples to illustrate the theory, and give more explicit formulas for the dimension of the moduli space m_Γ. Finally, the Appendix contains some technical results about minimal resolutions needed in Section 10.

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2. Notation

We begin with the definition of weighted Hölder spaces, which will be used throughout the paper.

Definition 2.1. Let E be a tensor bundle on X, with Hermitian metric $\|\cdot\|_h$. Let φ be a smooth section of E. We fix a point $p_0 \in X$, and define $r(p)$ to be the distance between p_0 and p. Then define

$$\|\varphi\|_{C^0_\delta} := \sup_{p\in X}\Big\{\|\varphi(p)\|_h \cdot (1+r(p))^{-\delta}\Big\} \tag{2.1}$$

$$\|\varphi\|_{C^k_\delta} := \sum_{|\mathcal{I}|\leq k}\sup_{p\in X}\Big\{\|\nabla^{\mathcal{I}}\varphi(p)\|_h \cdot (1+r(p))^{-\delta+|\mathcal{I}|}\Big\}, \tag{2.2}$$

where $\mathcal{I} = (i_1, \ldots, i_n)$, $|\mathcal{I}| = \sum_{j=1}^n i_j$. Next, define

$$[\varphi]_{C^\alpha_{\delta-\alpha}} := \sup_{0<d(x,y)<\rho_{inj}}\Big\{\min\{r(x), r(y)\}^{-\delta+\alpha}\frac{\|\varphi(x)-\varphi(y)\|_h}{d(x,y)^\alpha}\Big\}, \tag{2.3}$$

where $0 < \alpha < 1$, ρ_{inj} is the injectivity radius, and $d(x, y)$ is the distance between x and y. The meaning of the tensor norm is to use parallel transport along the unique minimal geodesic from y to x, and then take the norm of the difference at x. The weighted Hölder norm is defined by

$$\|\varphi\|_{C^{k,\alpha}_\delta} := \|\varphi\|_{C^k_\delta} + \sum_{|\mathcal{I}|=k}[\nabla^{\mathcal{I}}\varphi]_{C^\alpha_{\delta-k-\alpha}}, \tag{2.4}$$

and the space $C^{k,\alpha}_{\delta}(X,E)$ is the closure of $\{\varphi \in C^{\infty}(X,E) : \|\varphi\|_{C^{k,\alpha}_{\delta}} < \infty\}$.

Remark 2.2. The dual space of Hölder space is not a Hölder space, but for the purpose of computing the dimension of cokernel of a Fredholm operator of order o,

$$H : C^{k,\alpha}_{\delta} \to C^{k-o,\alpha}_{\delta-o}, \tag{2.5}$$

we consider the adjoint operator as mapping between Hölder spaces

$$H^* : C^{k,\alpha}_{-4-\delta+o} \to C^{k-o,\alpha}_{-4-\delta}, \tag{2.6}$$

since the adjoint weight to weight δ is $-4-\delta$, and using elliptic regularity.

3. ALE Kähler surfaces

In this section, we will prove several results about ALE Kähler surfaces which will be needed later. We note that the results in this section do not use the scalar-flat assumption.

In [LM08, HL15], LeBrun-Maskit and Hein-LeBrun analyzed the asymptotic behavior of the metric and complex structure near infinity of ALE Kähler surfaces. The next proposition gives a summary of their results. (See also [Li14] for other related results on complex analytic compactifications).

Proposition 3.1 (Hein-LeBrun-Maskit)**.** *Let X_{∞} be an end of an ALE Kähler surface (X,g,J) with metric g asymptotic to Euclidean metric at rate*

$$|\nabla^{\mathcal{I}}_{Euc}(g_{j,k}-\delta_{j,k}))| = O(\rho^{-|\mathcal{I}|-1-\epsilon}) \tag{3.1}$$

for $|\mathcal{I}| = 0,1$ if $\epsilon > 1/2$ or for $|\mathcal{I}| = 0,\dots,4$ if $\epsilon \in (0,1/2]$; here ∇_{Euc} denotes the Euclidean derivative. Then there is a surface S containing an embedded holomorphic curve $C \simeq \mathbb{CP}^1$ with self-intersection 1, such that the universal cover $\tilde{X}_{\infty}$ of X_{∞} is biholomorphic to $S \setminus C$.

Let (X,g,J) be a Kähler ALE surface, then X has one end and can be analytically compactified to a smooth compact surface $\hat{X}$ by adding a tree of rational curves E_{∞}. The surface $\hat{X}$ is a rational surface or a ruled surface. Also, X can be compactified to an orbifold surface $\hat{X}_{orb}$ by adding a single rational curve at ∞.

Remark 3.2. The orbifold compactification $\hat{X}_{orb}$ has cyclic singularities on the rational curve at ∞. The smooth compactification $\hat{X}$ is obtained from $\hat{X}_{orb}$ by resolving these singularities.

Remark 3.3. When Γ is trivial, the metric is asymptotically flat, in which case X is biholomorphic to $Bl_{p_1,\dots,p_k}\mathbb{C}^2$, see [HL15].

Consider the space of holomorphic vector fields on X with at most growth of order τ

$$\mathscr{Z}_{\tau}(X,\Theta) = \{Z \in \Gamma(X,\Theta) \mid \bar{\partial} Z = 0,\ Z = O(r^{\tau}) \text{ as } r \to \infty\}. \tag{3.2}$$

The first vanishing result we need is the following.

Proposition 3.4. *Let (X, g, J) be an ALE Kähler surface. If $\tau < 0$ then*

$$\dim \mathcal{Z}_\tau(X, \Theta) = 0. \tag{3.3}$$

Let $0 < \tau < 1$. If X is biholomorphic to $\mathbb{C}^2$, then

$$\dim \mathcal{Z}_\tau(X, \Theta) = 2, \tag{3.4}$$

otherwise,

$$\dim \mathcal{Z}_\tau(X, \Theta) = 0. \tag{3.5}$$

Proof. Let $X_\infty = X \setminus \overline{B(R)}$. Let R be sufficiently large, then by Proposition 3.1, the universal cover $\tilde{X}_\infty$ can be analytically compactified by adding a rational curve C at infinity, where C has self-intersection $+1$. By [HL15], there exists a smooth map $[\xi_1, \xi_2, f]$ that maps $S = \tilde{X}_\infty \cup C$ to $\mathbb{CP}^2$, which maps C holomorphically to $\{f = 0\}$ as a curve $\mathbb{CP}^1 \subset \mathbb{CP}^2$, where

$$\xi_j = \{\text{holomorphic part}\} + O(|f|^3), \tag{3.6}$$

as $f \to 0$. Let

$$z_j = x_j + \sqrt{-1} y_j = \frac{\xi_j}{f}, (j = 1, 2) \tag{3.7}$$

where $\{x_1, y_1, x_2, y_2\}$ gives an ALE coordinate on $\tilde{X}_\infty$, and $|J - J_{Euc}| = O(|z|^{-3})$ as $z \to \infty$. Let

$$(v, w) = \left(\frac{1}{z_1}, \frac{z_2}{z_1}\right) \tag{3.8}$$

where $\{v = 0\}$ represents the complement of one point in C.

First, let σ be a decaying holomorphic vector field on X, which can be lifted to $\tilde{\sigma}$ on $\tilde{X}_\infty$. Note that $\tilde{\sigma}$ can be extended to $\mathbb{C}^2$ smoothly by using a cut-off function. Since $\bar{\partial} - \bar{\partial}_{Euc} = O(|z|^{-3})$ and $\bar{\partial}\tilde{\sigma} = 0$ on $\tilde{X}_\infty$,

$$\bar{\partial}_{Euc}\tilde{\sigma} = \sum_{i,j} h_{i,j} d\bar{z}_i \otimes \frac{\partial}{\partial z_j} = O(|z|^{-3}|\sigma|), \tag{3.9}$$

as $z \to \infty$. By using the $\bar{\partial}_{Euc}$-Poincaré lemma, there exist

$$p_j = O(|z|^{-2}\sigma) \in C^\infty(\mathbb{C}^2),\ \bar{\partial} p_j = \sum_i h_{i,j} d\bar{z}_i. \tag{3.10}$$

The formula for p_j can be written out explicitly as follows. Let

$$\begin{aligned} q_j &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} h_{2,j}(z_1, \zeta_2) \frac{d\zeta_2 \wedge d\bar{\zeta}_2}{\zeta_2 - z_2} \\ q_j' &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} (h_{1,j} - \bar{\partial}_1 q_j)(\zeta_1, z_2) \frac{d\zeta_1 \wedge d\bar{\zeta}_1}{\zeta_1 - z_1}, \end{aligned} \tag{3.11}$$

where the integral formula is valid since $h_{i,j} = o(|z|^{-3})$. Then $p_j = q_j + q_j'$. Then we have $\tilde{\sigma} - \sum_j p_j \frac{\partial}{\partial z_j}$ is a decaying $\bar{\partial}_{Euc}$-holomorphic vector field on $\mathbb{C}^2$, which must vanish identically. The growth rate of σ can be dropped by 2 iteratively by this argument, so we can assume that σ decays at any rate in the coordinates $\{z_1, z_2\}$.

Let $\sigma^* \in C^\infty(X, \Omega^1)$ denote the dual of σ with respect to the Kähler metric g, $\tilde{\sigma}^*$ be the lifting of σ^* on $\tilde{X}_\infty$. The section $\tilde{\sigma}^*$ is also holomorphic and we have $\tilde{\sigma}^* = O(|z|^{-10})$ as $z \to \infty$ since we can assume $\tilde{\sigma}$ decays to any order. The coordinate change gives: $dz_1 = \frac{-1}{v^2}dv, dz_2 = \frac{1}{v}dw - \frac{w}{v^2}dv$, $|z|^2 = \frac{1}{|v|^2}(1+|w|^2)$. Then $\tilde{\sigma}^* = O(|v|^8)$ near C and can be extended to C with $\tilde{\sigma}^* = 0$ on C. Consider the exact sequence

$$0 \to N_C^* \to \Omega^1|_C \to \Omega^1(C) \to 0, \tag{3.12}$$

where $\Omega^1|_C$ is the restriction of the bundle Ω^1 on C, $\Omega^1(C)$ is the cotangent bundle of C and N_C is the normal bundle of C in S. Then we have $\Omega^1|_C = \mathcal{O}(-2) \oplus \mathcal{O}(-1)$. This implies $\tilde{\sigma}^*$ vanishes on any rational curve with self-intersection $+1$ on S. Since C has self-intersection $+1$, $H^1(C, N_C^* \otimes \Theta(C)) = 0$, so C is rigid in S. Then there is an open neighborhood $U \subset S$ of C, such that $\tilde{\sigma}^*$ vanishes identically over U. Then σ^* vanishes on an open subset of X, which implies that σ^* and σ vanishes identically on X.

Finally, assume that $\sigma \in \mathcal{Z}_\tau(X, \Theta)$ where $0 < \tau < 1$. In particular $\Box\sigma = 0$. Since there is no indicial root between 0 and 1, σ admits an expansion with bounded leading term. Then σ can be lifted to $\tilde{X}_\infty$ of the form $\{a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2} + \text{decaying terms}\}$, where a, b are constants. If Γ is nontrivial, then $a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2}$ is not Γ-equivariant, so $a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2} = 0$, and σ vanishes by the argument above. From Remark 3.3, it is clear that $\dim(\mathcal{Z}_\tau(X, \Theta))$ is equal to 2 only in the case that X is biholomorphic to $\mathbb{C}^2$, and is zero otherwise.

□

Next, we consider harmonic $(0,2)$-forms with values in the holomorphic tangent bundle.

Proposition 3.5. *Let X be a Kähler ALE surface. Then $\mathcal{H}_\delta(X, \Lambda^{0,2} \otimes \Theta) = 0$, where $\delta \in (-2, -1)$.*

Proof. The proposition follows an argument in [LM08, Theorem 4.2], with minor modifications. For completeness, we give a proof here following their idea. Let $\sigma \in \mathcal{H}_\delta(X, \Lambda^{0,2} \otimes \Theta)$. Recall that the conjugated Hodge star operator $\bar{*}$ maps σ to a $\Box$-harmonic form $\sigma^* \in \mathcal{H}_\delta(X, \Lambda^{2,0} \otimes \Omega^1)$. Let $\tilde{\sigma}^*$ be its lifting on $\tilde{X}_\infty$. Following the same notion as in the proof of Proposition 3.4, we have

$$\tilde{\sigma}^* = \sum_j a_j dz_1 \wedge dz_2 \otimes dz_j + O(|z|^{-3}|\tilde{\sigma}^*|), \tag{3.13}$$

where a_j are $\bar{\partial}_{Euc}$-holomorphic functions and $a_j = O(|z|^\delta)$ as $z \to \infty$. As shown in the proof of Proposition 3.4, $a_j = 0$, so $\tilde{\sigma}^* = O(|z|^{\delta-3})$ as $z \to \infty$. Since the growth rate of $\tilde{\sigma}^*$ can be dropped by 3 iteratively, $\tilde{\sigma}^*$ can decay to any order under the coordinate of $\{z_1, z_2\}$, then $\tilde{\sigma}^*$ can be extended to C with $\tilde{\sigma}^* = 0$ on C. Since

$$\Omega^{2,0} \otimes \Omega^1|_C = (\Omega^1(C) \oplus N_C^*) \otimes (\Omega^{2,0}(C) \otimes N_C^*), \tag{3.14}$$

we have

$$\Omega^{2,0} \otimes \Omega^1|_C = (\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \otimes (\mathcal{O}(-2) \otimes \mathcal{O}(-1)) = \mathcal{O}(-5) \oplus \mathcal{O}(-4). \tag{3.15}$$

This implies that $\tilde{\sigma}^*$ vanishes on any rational curve with self-intersection $+1$ on $\tilde{X}_\infty$. By the same argument as in Proposition 3.4, there is an open neighborhood $U \subset S$ of C, such that $\tilde{\sigma}^*$ vanishes identically over U. Then σ vanishes on an open subset of X, which implies that σ vanishes identically on X. □

The next Proposition shows that $\mathcal{H}_\delta(X, \Lambda^{0,1} \otimes \Theta)$ automatically has a faster decaying rate.

Proposition 3.6. *Let X be an ALE Kähler surface, and $\hat{X}$ be its analytic compactification. Then for any $-3 < \tau < 0$ we have*

$$\mathcal{H}_\tau(X, \Lambda^{0,1} \otimes \Theta) = \mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta), \tag{3.16}$$

and if Γ is not cyclic or $\Gamma = \mathbb{Z}/2\mathbb{Z}$, then

$$\mathcal{H}_\tau(X, \Lambda^{0,1} \otimes \Theta) = \mathcal{H}_{-4}(X, \Lambda^{0,1} \otimes \Theta). \tag{3.17}$$

Proof. First, we will show that any decaying harmonic element $\phi \in H^0(X, \Lambda^{0,1} \otimes \Theta)$ has a decay rate of at least $O(r^{-3})$ at infinity. Since X is Kähler, by [Mor07, Part 5], the operator $\square$ admits an expansion near infinity of the form

$$\square = \frac{1}{2}\nabla^*\nabla + \mathfrak{R} \tag{3.18}$$

on $\Lambda^{0,1} \otimes \Theta$, where ∇ is the covariant derivative on $\Lambda^{0,1} \otimes \Theta$, and $\mathfrak{R}$ is an operator given by curvature forms acting on the same bundle. The leading term of $\nabla^*\nabla$ is the Euclidean Laplacian Δ_{Euc}, so we have

$$\square = \frac{1}{2}\Delta_{Euc} + Q, \tag{3.19}$$

where Q denotes lower order terms. The element ϕ admits an expansion of the form

$$\phi = f + O(r^{-3+\epsilon}), \tag{3.20}$$

as $r \to \infty$, where f is of the form

$$f = \sum_{i,j} \frac{f_{i,j}}{r^2} d\bar{z}_i \otimes \frac{\partial}{\partial z_j}, \tag{3.21}$$

and $f_{i,j}$ are constants. Since ϕ is $\square$-harmonic, both $\bar{\partial} f = 0$ and $\bar{\partial}^* f = 0$. It is easy to see that this implies that each $f_{i,j} = 0$. Consequently, $\phi = O(r^{-3+\epsilon})$ as $r \to \infty$. Since ϕ admits an expansion with harmonic leading term, we must have $\phi = O(r^{-3})$.

Furthermore, if Γ is not cyclic or $\Gamma = \mathbb{Z}/2\mathbb{Z}$, then Γ contains the element -1. The leading term of ϕ is of the form

$$f = \sum_{k,l} \frac{A_{kl}}{r^4} d\bar{z}_k \otimes \frac{\partial}{\partial z_l}, \tag{3.22}$$

where A_{kl} is a linear combination of $z_1, \bar{z}_1, z_2, \bar{z}_2$. Since any such nonzero f is not invariant under the action of -1, we must have $f = 0$, and then $\phi = O(r^{-4})$. This finishes the proof of (3.16) and (3.17).

□

The following Proposition will be used in Section 7.

Proposition 3.7. *Let (X, g, J) be an ALE Kähler surface. Then*

$$b_1(X) = 0, \tag{3.23}$$

where $b_1(X)$ denotes the first Betti number of X. Furthermore, for $\tau < 0$,

$$\dim(\mathcal{H}_\tau(X, \Lambda^{0,2})) = \dim(\mathcal{H}_\tau(X, \Lambda^{2,0})) = 0. \tag{3.24}$$

Proof. Let U be a tubular neighborhood of E_∞, $\hat{X} = X \cup E_\infty$ be the analytic compactification of X. By Proposition 3.1, $\hat{X}$ is either a rational surface or a ruled surface, then $b^1(\hat{X}) = 0$. Since E_∞ is a tree of rational curves, $b^1(U) = 0$ and $b^1(X \cap U) = 0$. Then by the Mayer-Vietoris theorem, $b^1(X) = 0$.

Since $\dim(\mathcal{H}_\tau(X, \Lambda^{0,2})) = \dim(\mathcal{H}_\tau(X, \Lambda^{2,0}))$, we just need to show the latter one equals to zero. The proof is similar to the proof of Proposition 3.5, so we will skip some details. For any $\sigma \in \mathcal{H}_\tau(X, \Lambda^{2,0})$, let $\tilde{\sigma}$ be its lifting on $\tilde{X}_\infty$. Then

$$\tilde{\sigma} = a \cdot dz_1 \wedge dz_2 + O(|z|^{-3}\tilde{\sigma}) \tag{3.25}$$

where $a = O(|z|^\tau)$ as $r \to \infty$ is a $\bar{\partial}_{Euc}$-holomorphic function. Then $a = 0$, and $\tilde{\sigma} = O(|z|^{-3+\tau})$, which implies that $\tilde{\sigma}$ can decay to any order in the ALE coordinates $\{z_1, z_2\}$. Consequently, $\tilde{\sigma}$ can be extended to C. Furthermore,

$$\Omega^{2,0}|_C = \Omega^{2,0}(C) \otimes N_C^* = \mathcal{O}(-3). \tag{3.26}$$

This implies that $\tilde{\sigma}$ vanishes on an open neighborhood of C, so σ vanishes identically on X.

□

4. Small deformations of complex structure

First we need a fixed point theorem for operators on Banach spaces (see, for example [BR15]).

Lemma 4.1. *Let $F : \mathcal{B}_1 \to \mathcal{B}_2$ be a bounded differentiable operator between Banach spaces. In a small neighborhood of $0 \in \mathcal{B}_1$, $F(x) = F(0) + F'(0)\,x + Q(x)$, where*

$$\|Q(x) - Q(x')\|_{\mathcal{B}_2} \leq C_0 \cdot (\|x\|_{\mathcal{B}_1} + \|x'\|_{\mathcal{B}_1}) \cdot \|x - x'\|_{\mathcal{B}_1}. \tag{4.1}$$

Assume that $\|F(0)\|_{\mathcal{B}_2} \ll 1$. Then
(i) If $F'(0)$ is an isomorphism with right inverse G bounded by C_1, then there exists a ball $B(0, s) \subset \mathcal{B}_1$ such that there exists a unique $x \in B(0, s)$ with $F(x) = 0$.
(ii) If $F'(0)$ is Fredholm and surjective, with right inverse bounded by C_1, then there is an $s > 0$, so that $F^{-1}(0) \cap B(0, s)$ is isomorphic to $U \subset \ker(F'(0))$, where U is a small neighborhood of the origin.

The following lemma gives the properties of the linearized operator we will need in order to invoke Lemma 4.1.

Lemma 4.2. *Let (X, g, J) be a Kähler ALE surface. The linear operator*

$$P : C^{k,\alpha}_{\delta-1}(X, \Lambda^{0,1} \otimes \Theta) \xrightarrow{(\bar\partial^*, \bar\partial)} C^{k-1,\alpha}_{\delta-2}(X, \Theta) \oplus C^{k-1,\alpha}_{\delta-2}(X, \Lambda^{0,2} \otimes \Theta) \tag{4.2}$$

is surjective and Fredholm, for some $k \geq 3$ and $\delta \in (-2, -1)$.

Proof. It is not hard to see that P is an elliptic operator, and that the indicial roots of P are integral. Consequently, by standard weighted space theory, P is Fredholm since δ is non-integral [LM85].

The cokernel is given by

$$\ker(P^*) = \{(\sigma_1, \sigma_2) \in C^{k-1,\alpha}_{-2-\delta}(\Theta) \oplus C^{k-1,\alpha}_{-2-\delta}(\Lambda^{0,2} \otimes \Theta) : \bar\partial^* \sigma_2 = \bar\partial \sigma_1\}. \tag{4.3}$$

Let $(\sigma_1, \sigma_2) \in \ker(P^*)$, then

$$\bar\partial^* \sigma_1 = 0,\ \bar\partial^* \bar\partial \sigma_1 = \bar\partial^* \bar\partial^* \sigma_2 = 0, \tag{4.4}$$

which implies that σ_1 is $\Box$-harmonic. From Proposition 3.4, $\sigma_1 = 0$. Then

$$\bar\partial \sigma_2 = 0,\ \bar\partial^* \sigma_2 = 0, \tag{4.5}$$

and the vanishing of σ_2 follows from Proposition 3.5. Since $\ker(P^*) = 0$, and P is Fredholm, P is surjective. □

Now we prove a weighted version of Kuranishi's theorem.

Theorem 4.3. *Let (X, J_0, g_0) be a Kähler ALE surface, and let B_{ϵ_1} be an ϵ_1-ball around 0 in $\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$. Then there is a differentiable family of complex structures J_t for $t \in B_{\epsilon_1}$ such that*

$$J_t = J_0 + \phi(t),\ \phi(t) \in C^{k,\alpha}_{\delta-1}(X, \Lambda^{0,1} \otimes \Theta),\ \delta \in (-2, -1), k \geq 3 \tag{4.6}$$

where $\phi(t)$ satisfies

$$\phi(t) = t + \phi(t)^{\perp}, \tag{4.7}$$

where $\phi(t)^{\perp}$ is L^2-orthogonal to $\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$, and

$$\bar\partial^*(\phi(t)) = 0. \tag{4.8}$$

Furthermore, there exists an $\epsilon' > 0$, such that if $J = J_0 + \phi$ is any integrable complex structure satisfying

$$\|\phi\|_{C^{k,\alpha}_{\delta-1}} < \epsilon', \quad \text{and} \quad \bar\partial^*(\phi) = 0, \tag{4.9}$$

then ϕ is in the family J_t.

Finally, for any $t \in B_{\epsilon_1}$, there exists a constant C such that $\|\phi(t)\|_{C^{k,\alpha}_{-3}} \leq C \cdot \epsilon_1$.

Proof. Define the operator

$$F : C^{k,\alpha}_{\delta-1}(X, \Lambda^{0,1} \otimes \Theta) \to C^{k-1,\alpha}_{\delta-2}(X, \Theta) \oplus C^{k-1,\alpha}_{\delta-2}(X, \Lambda^{0,2} \otimes \Theta)$$
$$\phi \mapsto (\bar{\partial}^* \phi, \bar{\partial}\phi + [\phi, \phi]), \tag{4.10}$$

where $[\phi, \phi]$ is a globally defined operator, which can be expressed locally as

$$[\phi, \phi] = \frac{1}{2}\Big[\sum_{i,j} \phi_{i,j} d\bar{z}_i \otimes \frac{\partial}{\partial z_j}, \sum_{k,l} \phi_{k,l} d\bar{z}_k \otimes \frac{\partial}{\partial z_l} \Big] \tag{4.11}$$

$$= \sum_{i,j,k,l} \phi_{i,j} \frac{\partial \phi_{k,l}}{\partial z_j} d\bar{z}_i \wedge d\bar{z}_k \otimes \frac{\partial}{\partial z_l}. \tag{4.12}$$

Formally, $[\phi, \phi]$ can be written as $\phi * \nabla\phi$, where $*$ means a linear combination of quadratic terms, each involving some contraction of ϕ with $\nabla\phi$.

Clearly, F is a bounded differentiable mapping, and $F'(0) = (\bar{\partial}^*, \bar{\partial})$. Obviously, in a small neighborhood of 0 in $C^{k,\alpha}_{\delta-1}(X, \Lambda^{0,1} \otimes \Theta)$, F admits an expansion $F = F(0) + F'(0) + Q$. We next prove the estimate (4.1). Let r denote the radius. For any two elements $\phi, \phi' \in C^{k,\alpha}_{\delta-1}(\Lambda^{0,1} \otimes \Theta)$, we have

$$\begin{aligned} r^{2-\delta}|[\phi, \phi] - [\phi', \phi']| &= r^{2-\delta}|[\phi * \nabla\phi] - [\phi' * \nabla\phi']| \\ &= r^{2-\delta}|\phi * (\nabla\phi - \nabla\phi') - (\phi - \phi') * \nabla\phi'| \\ &\leq r^{\delta-1}\{r^{1-\delta}|\phi| r^{2-\delta}|\nabla\phi - \nabla\phi'| + r^{1-\delta}|\phi - \phi'| r^{2-\delta}|\nabla\phi'|\} \\ &\leq r^{\delta-1}(r^{1-\delta}(|\phi| + |\phi'|))(r^{2-\delta}(|\phi - \phi'| + |\nabla\phi - \nabla\phi'|)). \end{aligned} \tag{4.13}$$

Next, let $x \neq y \in X$, with $r(x) < r(y)$. Similarly, we have

$$\begin{aligned} & r(x)^{2-\delta} \frac{|([\phi, \phi] - [\phi', \phi'])(x) - ([\phi, \phi] - [\phi', \phi'])(y)|}{d(x, y)^\alpha} \\ & \leq r(x)^{\delta-1}\Big\{ r(x)^{1-\delta} \frac{|\phi(x) - \phi(y)|}{d(x, y)^\alpha} |\nabla\phi(x) - \nabla\phi'(x)| r(x)^{2-\delta} \Big\} + \\ & r(y)^{\delta-1}\Big\{ r(x)^{2-\delta} \frac{|(\nabla\phi(x) - \nabla\phi'(x)) - (\nabla\phi(y) - \nabla\phi'(y))|}{d(x, y)^\alpha} |\phi(y)| r(y)^{1-\delta} \Big\} + \\ & r(x)^{\delta-1}\Big\{ r(x)^{1-\delta} \frac{|(\phi(x) - \phi'(x)) - (\phi(y) - \phi'(y))|}{d(x, y)^\alpha} |\nabla\phi'(x)| r(x)^{2-\delta} \Big\} + \\ & r(y)^{\delta-1}\Big\{ r(x)^{2-\delta} \frac{|\nabla\phi'(x) - \nabla\phi'(y)|}{d(x, y)^\alpha} |\phi(y) - \phi'(y)| r(y)^{1-\delta} \Big\}. \end{aligned} \tag{4.14}$$

This shows that

$$\|[\phi, \phi] - [\phi', \phi']\|_{C^{0,\alpha}_{\delta-2}} \leq C_0(\|\phi\|_{C^{1,\alpha}_{\delta-1}} + \|\phi'\|_{C^{1,\alpha}_{\delta-1}}) \|\phi - \phi'\|_{C^{1,\alpha}_{\delta-1}}. \tag{4.15}$$

The higher derivative terms can be handled similarly, to prove

$$\|[\phi, \phi] - [\phi', \phi']\|_{C^{j-1,\alpha}_{\delta-2}} \leq C_j(\|\phi\|_{C^{j,\alpha}_{\delta-1}} + \|\phi'\|_{C^{j,\alpha}_{\delta-1}}) \|\phi - \phi'\|_{C^{j,\alpha}_{\delta-1}}, \ 1 \leq j \leq k \tag{4.16}$$

which shows that Q satisfies (4.1).

By Lemma 4.2, we know $P = F'(0)$ is Fredholm. We can choose a right inverse operator G such that the image of G is L^2-orthogonal to $\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$. By Lemma 4.1, we have that $F^{-1}(0)$ is locally isomorphic to a small neighborhood of 0 in $\mathcal{H}_{\delta-1}(X, \Lambda^{0,1} \otimes \Theta)$, such that for each $t \in B_{\epsilon_1}$, there is a $\phi(t) = t + \phi(t)^{\perp} \in F^{-1}(0)$, where $\phi(t)^{\perp}$ is L^2-orthogonal to $\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$, and $J_0 + \phi(t)$ is the corresponding complex structure. It is a straightforward consequence of the implicit function theorem that the mapping $\psi : t \mapsto \phi(t)$ is differentiable. This finishes the proof of the existence of the family of complex structures.

By the fixed point Lemma 4.1, near 0, the zero set of $(\bar{\partial}^*, \bar{\partial} + [\,,\,])$ is locally bijective to the kernel of $(\bar{\partial}^*, \bar{\partial})$, which is $\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$. Thus there exists an $\epsilon' > 0$ such that for any $\|\phi\|_{C^{k,\alpha}_{\delta-1}} < \epsilon'$, $\bar{\partial}^*\phi = 0$, $\bar{\partial}\phi + [\phi, \phi] = 0$, then $J = J_0 + \phi$ is in the family we just constructed.

Next we will show that $\phi(t) \in C^{k,\alpha}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$ for $t \in B_{\epsilon_1}$. In fact, we will show that for any $\phi \in C^{k,\alpha}_{\delta}(X, \Lambda^{0,1} \otimes \Theta)$ that satisfies the system

$$\bar{\partial}\phi = -[\phi, \phi], \;\; \bar{\partial}^*\phi = 0, \tag{4.17}$$

then $\phi \in C^{k,\alpha}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$. To see this, (4.17) implies that $\Box\phi \in C^{k-2,\alpha}_{2\delta-2}(X, \Lambda^{0,1} \otimes \Theta)$. Then outside of a compact subset,

$$\Delta_{Euc}\phi = \Box\phi + (\Delta_{Euc} - \Box)\phi = O(r^{2\delta-2}), \tag{4.18}$$

which implies that

$$\phi = \sum_{i,j} \frac{a_{i,j}}{r^2} d\bar{z}_i \otimes \frac{\partial}{\partial z_j} + O(r^{-3+\epsilon}), \tag{4.19}$$

where $a_{i,j}$ are constants and $\frac{a_{i,j}}{r^2}$ is Δ_{Euc}-harmonic. By the same argument as in the proof of (3.16), we have $a_{i,j} = 0$. Then a similar argument shows that $\phi \in C^{k,\alpha}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$. The last statement follows from finite-dimensionality of the kernel. □

5. Versality of Kuranishi family

The next result shows that any nearby complex structure can be brought into the family J_t by a suitable diffeomorphism. Of course, the implicit function theorem requires a mapping between Banach spaces to be differentiable. The following lemma, inspired by [Biq06], will be used below to this end.

Lemma 5.1. *Let (X, g_0, J_0) be a Kähler ALE surface with $g_0 \in C^{\infty}$. Then for any sufficiently small $\epsilon > 0$, for any smooth Kähler ALE metric $\|g - g_0\|_{C^{k,\alpha}_{\delta}} < \epsilon$, there exists an $\epsilon_1 > 0$, such that for an ϵ_1-ball $B_{\epsilon_1} \subset C^{k+1,\alpha}_{\delta+1}(TX)$, the map Ψ*

$$\begin{aligned} \Psi : B_{\epsilon_1} &\to C^{k,\alpha}_{\delta}(S^2(T^*X)) \\ Y &\mapsto \Phi_{Y*}g - g \end{aligned} \tag{5.1}$$

is smooth.

Proof. First, for any g such that $\|g - g_0\|_{C^{k,\alpha}_\delta} < \epsilon$ where ϵ is sufficiently small, there exist constants λ, i_0, such that $\|Ric(g)\|_{C^{k-2,\alpha}} \leq \lambda$, $\rho_{inj}(g) \geq i_0 > 0$. Then, following [And90], for any $\epsilon_2 > 0$, there exists a cover of harmonic coordinate charts $\{B_{r_s}(x_s)\}$, such that $r_s \geq r_{harm}$, where $r_{harm} = r_{harm}(\lambda, i_0, \epsilon_2, k, \alpha) > 0$, and for any $y \in B_{r_s}(x_s)$,

$$g_{ij}(x_s) = \delta_{ij},\ (1+\epsilon_2)^{-1}\delta_{ij} \leq g_{ij}(y) \leq (1+\epsilon_2)\delta_{ij},\ r_{harm}^{k+\alpha}|g_{ij}(y)|_{C^{k,\alpha}} \leq (1+\epsilon_2), \tag{5.2}$$

where $(u_1, \ldots, u_4)$ is the harmonic coordinate of $B_{r_s}(x_s)$, and $g_{ij}(y) = g(y)(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$. Now let $Y \in C^{k+1,\alpha}(TB_{r_s}(x_s))$, with $\|Y\|_{C^{k+1,\alpha}} < \epsilon_3$, where $\epsilon_3 > 0$. Note that Y has at least the same regularity in harmonic coordinates as in the original ALE coordinates. Then Y can be represented as $Y = \sum_{j=1}^4 Y_j \frac{\partial}{\partial u_j}$, where each $Y_j \in C^{k+1,\alpha}(B_{r_s}(x_s))$. Fix any $y \in B_{r_s}(x_s)$, and indices i,j, and define $G_{ij}(Y) = (\Phi_Y)_* g_{ij}(y)$. Then for any positive integer L, when ϵ_3 is sufficiently small, $G_{ij}(Y)$ has the Taylor expansion:

$$G_{ij}(Y) = \sum_{|\mathcal{L}| \leq L} D^{\mathcal{L}} G_{ij}(0) \frac{Y_1^{l_1} \ldots Y_n^{l_n}}{l_1! \cdot \ldots \cdot l_n!} + O(|Y|^{L+1}), \tag{5.3}$$

where $\mathcal{L}$ denotes a multi-index, as $|Y| \to 0$. There exist similar expansions for

$$G^{\mathcal{I}}_{ij}(Y) = (\nabla^{\mathcal{I}} \Phi_{Y*} g)_y \Big(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\Big), \tag{5.4}$$

with $|\mathcal{I}| \leq k + \alpha$. Note that

$$(\Phi_Y)_* g(y) = \sum_{i,j} G_{ij} d((\Phi_Y)_* u_i) \otimes d((\Phi_Y)_* u_j), \tag{5.5}$$

and similar Taylor expansions hold for $(\Phi_Y)_* u_i$. These expansions imply that Ψ is smooth over an ϵ_3-ball $B_{\epsilon_3} \subset C^{k+1,\alpha}(TB_{r_s}(x_s))$, i.e, for

$$\Psi|_{B_{r_s}(x_s)} : B_{\epsilon_3} \to C^{k,\alpha}(S^2(T^* B_{r_s}(x_s))), \tag{5.6}$$

the Lth-differential $D^L\Psi$ exists, and $\|D^L \Psi\| \leq C$, where $C = C(\lambda, i_0, \epsilon_2, k, \alpha, L)$ is independent of the choice of harmonic coordinate chart. Denote by $A_{1,2}$ a Euclidean annulus of radius between 1 and 2, with a metric satifying the boundeness assumptions at the beginning. Since $A_{1,2}$ can be obviously be covered by finitely many harmonic coordinate charts, by the argument above, $\Psi|_{A_{1,2}}$ is smooth. Then by a standard dilation argument, which dilates $A_{R,2R}$ to $A_{1,2}$, we have for $B_{\epsilon_3} \subset C^{k+1,\alpha}_{\delta+1}(TA_{R,2R})$,

$$\Psi|_{A_{R,2R}} : B_{\epsilon_3} \to C^{k,\alpha}_\delta(S^2(T^* A_{R,2R})) \tag{5.7}$$

is smooth, and there exists a constant $C = C(\lambda, i_0, \epsilon_2, k, \alpha, L)$, such that the estimate $\|D^L \Psi|_{A_{R,2R}}\| \leq C$ holds. Note that C is indepedent of the dilation factor R. Since we can choose R arbitarily large, then for $\epsilon_1 > 0$ sufficiently small, we have for $B_{\epsilon_1} \subset C^{k+1,\alpha}_{\delta+1}(TX)$,

$$\Psi : B_{\epsilon_1} \to C^{k,\alpha}_\delta(S^2(T^*X)) \tag{5.8}$$

is smooth as a mapping between Banach spaces. ∎

Lemma 5.2. *Let (X, J_0, g_0) be a Kähler ALE surface with $g_0 \in C^\infty$. There exists an $\epsilon_1' > 0$ such that for any complex structure $\|J_1 - J_0\|_{C^{k,\alpha}_\delta} < \epsilon_1'$, where $k \geq 3, \alpha \in (0,1), \delta \in (-2,-1)$, there exists a unique diffeomorphism Φ, of the form Φ_Y for $Y \in C^{k+1,\alpha}_{\delta+1}(TX)$ such that $\delta_0(\Phi^*(J_1)) = 0$. Consequently, $\Phi^*(J_1)$ is in the family J_t from Theorem 4.3.*

Proof. Let $J_1 = J_0 + \phi$, where $\|\phi\|_{C^{k,\alpha}_\delta} < \epsilon_1'$ for some small ϵ_1' to be determined later, and $\bar\partial\phi + [\phi,\phi] = 0$. Note that $\delta_0 J_0 = 0$ since $\omega_0 = \frac{\sqrt{-1}}{2} g_0(J_0\cdot,\cdot)$ is Kähler.

Define the operator $\mathfrak{N}$ as:

$$
\begin{aligned}
C^{k+1,\alpha}_{\delta+1}(TX) \times C^{k,\alpha}_\delta(X, \Lambda^{0,1}\otimes\Theta) &\to C^{k-1,\alpha}_{\delta-1}(T^*X) \\
(Y,\phi) &\mapsto \delta_{\Phi_{Y*}g_0}(J_0+\phi),
\end{aligned} \tag{5.9}
$$

where Φ_Y is defined in (1.18), and $\delta_{\Phi_{Y*}g_0}$ is the divergence operator associated to metric g_0. Note that

$$
\delta_{\Phi_{Y*}g_0}(J_0+\phi) = 0 \Leftrightarrow \delta_{g_0}\Phi_Y^*(J_0+\phi) = 0, \tag{5.10}
$$

so a zero of $\mathcal{N}$ satisfies the desired gauge condition. (The latter map is not differentiable, which is why we consider the former, see [Biq06].) By Lemma 5.1, $\Phi_{Y*}g_0$ acts smoothly on $C^{k+1,\alpha}_{\delta+1}(TX)$, so $\Phi_{Y*}g_0$ has the expansion:

$$
\Phi_{Y*}g_0 = g_0 + h \tag{5.11}
$$

where $\|h\|_{C^{k,\alpha}_\delta} \leq C\|Y\|_{C^{k+1,\alpha}_{\delta+1}}$, for Y sufficiently small. In particular, we have the estimate $|h| \leq Cr^\delta \|Y\|_{C^{k+1,\alpha}_{\delta+1}}$. We also have the expansion of $\nabla_{\Phi_{Y*}g_0}$

$$
\nabla_{\Phi_{Y*}g_0} = \nabla_{g_0} + A_1 + A_2, \tag{5.12}
$$

where A_1 is the linear order term, and by the expansion formula

$$
\nabla_{g_0+h} = \nabla_{g_0} + (g_0+h)^{-1} * \nabla_{g_0} h, \tag{5.13}
$$

[GV16, Formula 3.39], and the estimate of h, we have

$$
|A_1| \leq C \cdot r^{\delta-1}\|Y\|_{C^{k+1,\alpha}_{\delta+1}}; \tag{5.14}
$$

A_2 denotes the lower order terms, and $|A_2| \leq C \cdot r^{\delta-2}\|Y\|_{C^{k+1,\alpha}_{\delta+1}}$. Then the divergence operator $\delta_{\Phi_{Y*}g_0}$ admits an expansion

$$
\delta_{\Phi_{Y*}g_0} = \delta_{g_0} + \delta_1 + \delta_2, \tag{5.15}
$$

where

$$
\delta_1 = h * \nabla_{g_0} + g_0 * A_1,\ \delta_2 = \delta_{\Phi_{Y*}g_0} - \delta_{g_0} - \delta_1, \tag{5.16}
$$

and $*$ means a linear combination of tensor contractions. By the expansion of $\delta_{\Phi_{Y*}g_0}$, it is clear that $\delta_{\Phi_{Y*}g_0}(J_0+\phi) \in C^{k-1,\alpha}_{\delta-1}(T^*X)$, and $\mathfrak{N}$ is well-defined bounded map between corresponding Banach spaces. A similar argument shows that $\mathfrak{N}$ is smooth. Note that the exponential map and the one-parameter group of diffeomorphisms induced by a vector field both have the same linearization at time 0. Consequently, at $(0,0)$, the linearization of $\mathfrak{N}$ restricted to the tangent space of the first variable is

$$\begin{aligned} D\,\mathfrak{N}_0 : C^{k+1,\alpha}_{\delta+1}(TX) &\to C^{k-1,\alpha}_{\delta-1}(T^*X) \\ Y &\mapsto \delta_0 \mathfrak{L}_Y J_0 = \square_0 Y, \end{aligned} \tag{5.17}$$

where the adjoint operator of $D\,\mathfrak{N}_0$ is:

$$\begin{aligned} (D\,\mathfrak{N}_0)^* : C^{k+1,\alpha}_{-3-\delta}(T^*X) &\to C^{k-1,\alpha}_{-5-\delta}(TX) \\ \eta &\mapsto (\square_0 \eta)^{\#}. \end{aligned} \tag{5.18}$$

The kernel of $(D\,\mathfrak{N}_0)^*$ consists of those $\eta \in C^{k+1,\alpha}_{-3-\delta}(T^*X)$ such that $(\square_0\eta)^{\#} = 0$, which implies that $\eta^{\#}$ is the real part of a holomorphic vector field. From Proposition 3.4, $\eta = 0$. Since the cokernel of $D\,\mathfrak{N}_0$ is trivial, $D\,\mathfrak{N}_0$ is surjective. Note also that the kernel of $D\,\mathfrak{N}_0$ is trivial.

Of course, in a small neighborhood of $(0,0)$ in $C^{k+1,\alpha}_{\delta+1}(TX) \times C^{k,\alpha}_{\delta}(X, \Lambda^{0,1}\otimes\Theta)$, $D\,\mathfrak{N}$ is surjective. Since $\mathfrak{N}$ is smooth, condition (4.1) is satisfied, so by Lemma 4.1, there exists a unique diffeomorphism Φ_Y such that $\Phi_Y^* J_1 - J_0$ is divergence-free. Since $\Phi_Y^*(J_1) - J_0$ satisfies equations (4.17), following the last part of the proof in Theorem 4.3, $\Phi_Y^*(J_1) - J_0 \in C^{k,\alpha}_{-3}(X, \Lambda^{0,1}\otimes\Theta)$. Then by Theorem 4.3, $\Phi_Y^*(J_1)$ is in the family J_t. □

6. Essential deformations of complex structure

The aruments in this section are inspired by [CT94, Theorem 3.1], with some appropriate modifications for the smoothness arguments. Recall that the space $\mathbb{W}$ is defined as

$$\mathbb{W} = \{Y \in \mathcal{H}_1(X, TX) \mid \mathfrak{L}_Y g_0 = O(r^{-1}),\ \mathfrak{L}_Y J_0 = O(r^{-3}),\ \text{as } r\to\infty\}. \tag{6.1}$$

Clearly, $\mathbb{W}$ is finite-dimensional. Next, we prove two lemmas which will be important in the subsequent gauging theorem.

Lemma 6.1. *For $Z \in \mathbb{W}$ sufficiently small, the time-one flow for the one-parameter group of diffeomorphisms generated by Z exists. Furthermore, there exist a matrix $A \in \mathrm{U}(2)$ and vector $\hat{\epsilon}_0 \in \mathbb{C}^2$, such that in ALE coordinates,*

$$d(\Phi_Z(x), Ax + \hat{\epsilon}_0) = O(r^{-2+\epsilon}), \tag{6.2}$$

as $r \to \infty$, for any $\epsilon > 0$.

Proof. First, note that $Z \in \mathbb{W}$ admits an expansion

$$Z = Z_1 + Z_0 + Z_{-2+\epsilon}, \tag{6.3}$$

for any $\epsilon > 0$, where Z_1 is in the Lie algebra of $\mathrm{U}(2)$, Z_0 is vector field with constant coefficients, and $Z_{-2+\epsilon} = O(r^{-2+\epsilon})$ as $r \to \infty$. To see this, by standard harmonic expansion for kernel elements of $\square_0$, we have that any $Z \in \mathbb{W}$ satisfies

$$Z = Z_1 + Z_0 + O(r^{-1+\epsilon}), \tag{6.4}$$

where Z_1 is any vector field which is homogeneous of degree one, and Z_0 is a vector field with constant coefficients. The conditions that $\mathfrak{L}_Z g_0 = O(r^{-1})$ and $\mathfrak{L}_Z J_0 = O(r^{-3})$ as $r \to \infty$ imply that $\mathfrak{L}_{Z_1} g_{Euc} = 0$ and $\mathfrak{L}_{Z_1} J_{Euc} = 0$, that is, Z_1 is in the Lie algebra of U(2). Next,

$$\mathfrak{L}_{Z_1+Z_0} J_0 = \mathfrak{L}_{Z_1+Z_0}(J_0 - J_{Euc}) = \mathfrak{L}_{Z_1+Z_0}(O(r^{-3})), \tag{6.5}$$

which implies that

$$\mathfrak{L}_{Z_0+Z_1} J_0 = O(r^{-3}), \tag{6.6}$$

as $r \to \infty$. Consequently,

$$\square_0(Z - Z_1 - Z_0) = O(r^{-4}). \tag{6.7}$$

Since there is no harmonic term of degree -1, the claimed expansion (6.3) follows.

Clearly for Z sufficiently small in norm, the one-parameter group of diffeomorphisms which is defined by

$$x'(t) = Z(x(t)),\ x(0) = x, \tag{6.8}$$

exists by the standard short-time existence theorem for ODEs. By the expansion (6.3), $x(t)$ will be given by a family of rotations in U(2), and translations plus decaying terms. □

Next, for $Z \in \mathbb{W}$, we define $\Phi_Z : X \to X$ to be time-one flow for the one-parameter group of diffeomorphisms generated by Y. Note that this is different from the definition given in (1.18).

The next lemma states some useful properties of $\mathbb{W}$, which will be needed in the proof of the subsequent gauging theorem.

Lemma 6.2. *Let (X, J_0, g_0) be a Kähler ALE surface, the space $\mathbb{W}$ satisfies the following properties:*

- *There exists a small neighborhood $U \subset \mathbb{W}$ of 0, and an $\epsilon' > 0$ such that if g_1 is any metric satisfyfing $\|g_1 - g_0\|_{C^{k,\alpha}_\delta} < \epsilon'$, then there exists a constant C, such that for any $Z \in U$,*

$$\|\Phi_Z^* g_1 - g_0\|_{C^{k,\alpha}_\delta} \leq C(\|Z\|_{C^{k+1,\alpha}_1} + \epsilon'). \tag{6.9}$$

- *Given $\epsilon > 0$ sufficiently small, for any complex structure $J_1 = J_0 + \phi$ such that $\|\phi\|_{C^{k,\alpha}_{-3+\epsilon}} < \epsilon'$ for some sufficiently small ϵ', there exists a small neighborhood $U \subset \mathbb{W}$ of 0, and a constant C, such that for any $Z \in U$,*

$$\|\Phi_Z^* J_1 - \mathfrak{L}_Z J_1 - J_0\|_{C^{k,\alpha}_{-3+\epsilon}} \leq C \cdot (\|Z\|_{C^{k+1,\alpha}_1})^2 + \epsilon'. \tag{6.10}$$

Proof. First, we address the estimate (6.9). Let $\Phi_{A,\hat{\epsilon}_0}$ denote the mapping $x \mapsto Ax+\hat{\epsilon}_0$. Using Lemma 6.1,

$$\begin{aligned}|\Phi_Z^* g_1 - g_0| &= |\Phi_Z^* g_1 - g_1 + g_1 - g_0| \leq |\Phi_Z^* g_1 - g_1| + |g_1 - g_0| \\ &\leq |(\Phi_{A,\hat{\epsilon}_0} + O(r^{-2+\epsilon}))^*(\delta_{ij} + O(r^\delta)) - (\delta_{ij} + O(r^\delta))| + \epsilon' r^\delta \\ &\leq C\|Z\|_{C_1^{k+1,\alpha}} r^\delta + \epsilon' r^\delta + o(r^\delta) \\ &\leq C r^\delta (\|Z\|_{C_1^{k+1,\alpha}} + \epsilon').\end{aligned}$$

as $r \to \infty$. Since Φ_Z is the time one flow of $Z \in \mathbb{W}$, and $\mathbb{W}$ is finite-dimensional, a similar estimate holds on any compact subset of X, so the C^0_δ part of the norm is bounded by the right hand side of (6.9). Higher regularity estimates are similar, and are omitted.

Next we will discuss (6.10). As in the first part, since $\mathbb{W}$ is finite-dimensional, we only need to make estimates outside of a compact set $\overline{B_R(p_0)}$, where a global coordinate exists. Let $\gamma : [0,1] \to X$ be the path $\gamma(t) = \Phi_{tZ}(x)$. First, we estimate

$$\begin{aligned}r(x)^{3-\epsilon}|\Phi_Z^* J_0 - J_0 - \mathfrak{L}_Z J_0|(x) &\leq C r^{3-\epsilon} \int_0^1 |\Phi_{tZ}^*(\mathfrak{L}_Z J_0)(\gamma(0)) - \mathfrak{L}_Z J_0(\gamma(0))| dt \\ &\leq C r^{3-\epsilon} \int_0^1 \int_0^t |\mathfrak{L}_Z(\mathfrak{L}_Z J_0)(\gamma(s))| ds dt.\end{aligned} \tag{6.11}$$

By the expansion (6.3), given $c > 0$, the estimates

$$(1-c) r(\gamma(s)) \leq r(x) = r(\gamma(0)) \leq (1+c) r(\gamma(s)) \tag{6.12}$$

are satisfied for all Z sufficiently small in norm. Also,

$$|\mathfrak{L}_Z(\mathfrak{L}_Z J_0)|(\gamma(s)) \leq C r(\gamma(s))^{-5} |Z(\gamma(s))|^2, \tag{6.13}$$

so we have

$$\begin{aligned}r(x)^{3-\epsilon}|\Phi_Z^* J_0 - J_0 - \mathfrak{L}_Z J_0|(x) &\leq C \int_0^1 \int_0^t r(\gamma(s))^{3-\epsilon} r(\gamma(s))^{-5} |Z|^2(\gamma(s)) ds dt \\ &\leq C \int_0^1 \int_0^t r(\gamma(s))^{-2-\epsilon} |Z|^2(\gamma(s)) ds dt \leq C \|Z\|^2_{C_1^{k+1,\alpha}}.\end{aligned} \tag{6.14}$$

Next, we estimate

$$\begin{aligned}r(x)^{3-\epsilon}|\Phi_Z^* \phi - \phi - \mathfrak{L}_Z \phi|(x) &\leq C r^{3-\epsilon} \int_0^1 |\Phi_{tZ}^*(\mathfrak{L}_Z \phi)(\gamma(0)) - \mathfrak{L}_Z \phi(\gamma(0))| dt \\ &\leq C r^{3-\epsilon} \int_0^1 \int_0^t |\mathfrak{L}_Z(\mathfrak{L}_Z \phi)(\gamma(s))| ds dt.\end{aligned} \tag{6.15}$$

Since $|\mathfrak{L}_Z(\mathfrak{L}_Z\phi)|(\gamma(s)) \leq Cr(\gamma(s))^{-5+\epsilon}|Z|^2(\gamma(s))$, similarly, we have the estimate

$$
\begin{aligned}
r(x)^{3-\epsilon}|\Phi_Z^*\phi - \phi - \mathfrak{L}_Z\phi|(x) &\leq C\int_0^1\int_0^t r(\gamma(s))^{3-\epsilon}(r(\gamma(s))^{-5+\epsilon}|Z|^2(\gamma(s)))dsdt \\
&\leq C\int_0^1\int_0^t r(\gamma(s))^{-2}|Z|^2(\gamma(s))dsdt \leq C\|Z\|^2_{C_1^{k+1,\alpha}}.
\end{aligned}
\tag{6.16}
$$

It follows that

$$
\begin{aligned}
&r(x)^{3-\epsilon}|\Phi_Z^*J_1 - J_1 - \mathfrak{L}_Z J_1|(x) \\
&\leq r(x)^{3-\epsilon}\{|\Phi_Z^*J_0 - J_0 - \mathfrak{L}_Z J_0|(x) + |\Phi_Z^*\phi - \phi - \mathfrak{L}_Z\phi|(x)\} \leq C\|Z\|^2_{C_1^{k+1,\alpha}},
\end{aligned}
\tag{6.17}
$$

where C is uniform for all $Z \in \mathbb{W}$ of sufficiently small norm. Higher derivative estimates are similar, and are omitted. Finally, (6.10) follows from

$$
\begin{aligned}
\|\Phi_Z^*J_1 - J_0 - \mathfrak{L}_Z J_1\|_{C^{k,\alpha}_{-3+\epsilon}} &\leq \|\Phi_Z^*J_1 - J_1 - \mathfrak{L}_Z J_1\|_{C^{k,\alpha}_{-3+\epsilon}} + \|\phi\|_{C^{k,\alpha}_{-3+\epsilon}} \\
&\leq C\cdot\|Z\|^2_{C_1^{k+1,\alpha}} + \epsilon'.
\end{aligned}
\tag{6.18}
$$

□

Recall that

$$
\mathbb{V} = \{\theta \in \mathcal{H}_{-3}(X, End_a(TX)) \mid \theta = \mathfrak{L}_Y J,\ Y \in \mathbb{W}\}. \tag{6.19}
$$

The deformations of complex structure arising from elements of $\mathbb{V}$ become trivial if one takes into account changes of coordinates at infinity, as the following theorem shows.

Theorem 6.3. *Let (X, J_0, g_0) be a Kähler ALE surface with $g_0 \in C^\infty$. There exists an $\epsilon_1' > 0$ such that for any complex structure $\|J_1 - J_0\|_{C^{k,\alpha}_\delta} < \epsilon_1'$, where $k \geq 3, \alpha \in (0,1), \delta \in (-2,-1)$, there exists a diffeomorphism Φ, of the form $\Phi_{Y_1}\circ\Phi_Z\circ\Phi_{Y_2}$ for $Y_1, Y_2 \in C^{k+1,\alpha}_{\delta+1}(TX)$ and $Z \in \mathbb{W}$, such that $\delta_0(\Phi^*(J_1) - J_0) = 0$, and such that $\Phi^*(J_1) - J_0$ is L^2-orthogonal to $\mathbb{V}$. Furthermore, there exists a constant C so that*

$$
\|\Phi^*J_1 - J_0\|_{C^{k,\alpha}_{-3}} < C\epsilon_1'. \tag{6.20}
$$

Proof. Let $J_1 = J_0 + \phi$, where $\phi \in C^{k,\alpha}_\delta(X, \Lambda^{0,1}\otimes\Theta)$, and $\bar\partial\phi + [\phi,\phi] = 0$. By Lemma 5.2 and Theorem 4.3, without loss of generality, we may assume that $\phi(t) \in C^{k,\alpha}_{-3}(X, \Lambda^{0,1}\otimes\Theta)$.

Define the operator $\mathfrak{N}$ as:

$$
\begin{aligned}
&C^{k+1,\alpha}_{-2+\epsilon}(TX)\times C^{k,\alpha}_{-3+\epsilon}(\Lambda^{0,1}\otimes\Theta)\times\mathbb{W} \to C^{k-1,\alpha}_{-4+\epsilon}(T^*X)\times\mathbb{R}^m \\
&(Y,\phi,Z)\mapsto \\
&\Big\{\delta_{\Phi_{Y*}g_0}(\Phi_Z^*(J_0+\phi(t))), \int\langle\Phi_Z^*(J_0+\phi), \Phi_{Y*}v_1\rangle_{\Phi_{Y*}g_0}dV_{\Phi_{Y*}g_0}, \\
&\qquad\cdots, \int\langle\Phi_Z^*(J_0+\phi), \Phi_{Y*}v_m\rangle_{\Phi_{Y*}g_0}dV_{\Phi_{Y*}g_0}\Big\},
\end{aligned}
\tag{6.21}
$$

where $0 < \epsilon \ll 1$, $\delta_{\Phi_{Y*}g_0}$ is the divergence operator associated to the metric $\Phi_{Y*}g_0$, $m = \dim(\mathbb{V})$, $\{v_1, \ldots, v_m\}$ is an orthonormal basis of $\mathbb{V}$. Recall that Φ_Y is induced by the exponential map of the decaying field Y. By Lemma 5.1 with domain weight $-3+\epsilon$, the mapping $Y \mapsto \Phi_{Y*}g_0 - g_0$ is smooth, so we have the following expansions as shown in the proof of Lemma 5.2:

$$\Phi_{Y*}g_0 = g_0 + h, \tag{6.22}$$

where h satisfies $|h| \leq C \cdot r^{-3+\epsilon} \|Y\|_{C^{k+1,\alpha}_{-2+\epsilon}}$. Also,

$$\nabla_{\Phi_{Y*}g_0} = \nabla_{g_0} + A_1 + A_2 \tag{6.23}$$

where A_1 is the linear order term, satisfying $|A_1| \leq C \cdot r^{-4+\epsilon} \|Y\|_{C^{k+1,\alpha}_{-2+\epsilon}}$, and A_2 denotes the lower order terms, satisfying $|A_2| \leq C \cdot r^{-5+\epsilon} \|Y\|_{C^{k+1,\alpha}_{-2+\epsilon}}$.

The divergence operator $\delta_{\Phi_{Y*}g_0}$ admits an expansion

$$\delta_{\Phi_{Y*}g_0} = \delta_{g_0} + \delta_1 + \delta_2, \tag{6.24}$$

where

$$\delta_1 = h * \nabla_{g_0} + g_0 * A_1, \ \delta_2 = \delta_{\Phi_{Y*}g_0} - \delta_{g_0} - \delta_1. \tag{6.25}$$

We also have the expansion

$$\Phi_Z^*(J_0 + \phi(t)) = I + II + III, \tag{6.26}$$

where

$$\begin{aligned} I &= J_0 + \phi(t) \\ II &= \mathfrak{L}_Z(J_0 + \phi(t)) \\ III &= \Phi_Z^*(J_0 + \phi(t)) - I - II. \end{aligned} \tag{6.27}$$

with

$$|II| \leq |\bar{\partial}_0 Z| + |\mathfrak{L}_Z \phi(t)| \leq C \cdot r^{-3+\epsilon}(1 + \epsilon_1') \|Z\|_{C^{k+1,\alpha}_1} \tag{6.28}$$

and by (6.10)

$$|III| \leq C \cdot r^{-3+\epsilon} \cdot (\epsilon_1' + \|Z\|^2_{C^{k+1,\alpha}_1}). \tag{6.29}$$

By direct computation, we have

$$\begin{aligned} |\delta_{g_0} I| &= |\delta_{g_0} \phi(t)| \leq \epsilon_1' \cdot r^{-4+\epsilon} \\ |\delta_{g_0} II| &= |\Box_0 Z + \delta_{g_0}(\mathfrak{L}_Z \phi(g))| \leq (C + \epsilon_1') \cdot r^{-4+\epsilon} \|Z\|_{C^{k+1,\alpha}_1} \\ |\delta_{g_0} III| &\leq C \cdot r^{-4+\epsilon}(\|Z\|^2_{C^{k+1,\alpha}_1} + \epsilon_1'). \end{aligned} \tag{6.30}$$

Also by the above estimates we have

$$
\begin{aligned}
|\delta_1 I| &= |h * \nabla_{g_0}\phi(t) + g_0 * A_1(J_0+\phi(t))| \leq (C+\epsilon_1')\cdot r^{-4+\epsilon}\|Y\|_{C^{k+1,\alpha}_{-2+\epsilon}}\\
|\delta_1 II| &= |h * \nabla_{g_0}\mathfrak{L}_Z(J_0+\phi(t)) + g_0 * A_1\mathfrak{L}_Z(J_0+\phi(t))|\\
&\leq (C+\epsilon_1')\cdot r^{-4+\epsilon}\|Z\|_{C^{k+1,\alpha}_1}\|Y\|_{C^{k+1,\alpha}_{-2+\epsilon}}\\
|\delta_1 III| &\leq C\cdot r^{-4+\epsilon}(\|Z\|^2_{C^{k+1,\alpha}_1}+\epsilon_1')\|Y\|_{C^{k+1,\alpha}_{-2+\epsilon}}.
\end{aligned}
\tag{6.31}
$$

The terms $\delta_2 I, \delta_2 II$ and $\delta_2 III$ obey similar estimates. Consequently,

$$
|\delta_{\Phi_{Y*}g_0}(\Phi_Z^*(J_0+\phi(t)))| \leq (C+\epsilon_1')\cdot r^{-4+\epsilon}(\|Y\|_{C^{k+1,\alpha}_{-2+\epsilon}}+\|Z\|_{C^{k+1,\alpha}_1}).
\tag{6.32}
$$

Higher regularities can be estimated similarly, which shows that $\mathfrak{N}$ does indeed map into the claimed image space and it is bounded. Similar arguments show that $\mathfrak{N}$ is smooth, since $\mathbb{W}$ is finite-dimensional, and $\mathfrak{N}$ is linear over ϕ.

At $(0,0,0)$, the linearization of $\mathfrak{N}$ restricted on the tangent of the first and third variables is:

$$
\begin{aligned}
D\,\mathfrak{N}_0 &: C^{k+1,\alpha}_{-2+\epsilon}(TX)\times\mathbb{W}\to C^{k-1,\alpha}_{-4+\epsilon}(T^*X)\times\mathbb{R}^m\\
(Y,Z) &\mapsto \Big\{\delta_0\mathfrak{L}_Y J_0 = \Box_0 Y, \int\langle\mathfrak{L}_Z J_0-\mathfrak{L}_Y J_0, v_1\rangle_{g_0}dV_{g_0},\\
&\qquad\cdots, \int\langle\mathfrak{L}_Z J_0-\mathfrak{L}_Y J_0, v_m\rangle_{g_0}dV_{g_0}\Big\}.
\end{aligned}
\tag{6.33}
$$

Restricting the domain and image, consider $D\mathfrak{N}_0$ mapping from $C^{k+1,\alpha}_{-2+\epsilon}(TX)$ to $C^{k-1,\alpha}_{-4+\epsilon}(T^*X)$. The adjoint of this restricted mapping, which we again denote by $D\,\mathfrak{N}_0$, is:

$$
\begin{aligned}
(D\,\mathfrak{N}_0)^* : C^{k+1,\alpha}_{-\epsilon}(T^*X) &\to C^{k-1,\alpha}_{-2-\epsilon}(TX)\\
\eta &\mapsto (\Box_0\eta)^\#.
\end{aligned}
\tag{6.34}
$$

The kernel of $(D\,\mathfrak{N}_0)^*$ consists of those $\eta\in C^{k+1,\alpha}_{-\epsilon}(T^*X)$ such that $(\Box_0\eta)^\# = 0$, which implies that $\eta^\#$ is the real part of a holomorphic vector field. From Proposition 3.4, $\eta = 0$. Since the cokernel of $D\,\mathfrak{N}_0$ is trivial, $D\,\mathfrak{N}_0$ is surjective on the restricted domain and range.

It follows that the full mapping $D\mathfrak{N}_0$ is surjective, since $\mathbb{V}$ is generated by $\mathfrak{L}_Z J_0$ for $Z\in\mathbb{W}$. Clearly then, in a small neighborhood of $(0,0,0)$, $D\,\mathfrak{N}$ is surjective. Since the mapping $\mathfrak{N}$ is smooth, Condition (4.1) is satisfied, so the existence of diffeomorphism Φ then follows from Lemma 4.1. Finally, it is easy to see that there exists a constant C so that

$$
\|\mathfrak{L}_Z J_1\|_{C^{k,\alpha}_{-3}} \leq C\epsilon_1',
\tag{6.35}
$$

so the estimate (6.20) follows from (6.10). $\square$

7. Stability of Kähler metrics

In this section, we want to generalize Kodaira-Spencer's stability theory of Kähler metrics to ALE surfaces to the family of complex structures found above [KS60]. Specifically, we want to find a smooth family of Kähler forms on (X, J_t) for t small, where J_t for $t \in B_{\epsilon_1} \subset \mathcal{H}_{ess}(X, \Lambda^{0,1} \otimes \Theta)$ is the family of complex structures constructed in Section 4. In a slight abuse of notation, the notation X_t will stand for (X, J_t). All the norms below in this section are based on the central metric g_0.

Lemma 7.1. *The integer-valued function* $\dim(\mathcal{H}_\tau(X_t, \Lambda^{p,q}))$ *is upper semicontinuous at* $t = 0$*, for any* $\tau \in (-2, 0)$*.*

Proof. The Hodge Laplacian $\Delta_d = dd^* + d^*d$ is a Fredholm operator on $C^{k,\alpha}_\tau$ $(-2 < \tau < 0, k \geq 2)$. The following is proved in [Bar86]: If P is a pseudo-differential operator such that $\|P - \frac{1}{2}\Delta_d\|_{op} < \epsilon$ for some ϵ sufficiently small, then $\ker(P) \leq \ker(\Delta_d)$, where $\|\cdot\|_{op}$ is the norm of operators from $C^{k,\alpha}_\tau$ to $C^{k-2,\alpha}_{\tau-2}$, which is defined as:

$$\|P\|_{op} = sup\{\|P(u)\|_{C^{k-2,\alpha}_{\tau-2}} : \|u\|_{C^{k,\alpha}_\tau} = 1\}. \tag{7.1}$$

We may apply this to the operator $P = \square_t = \bar{\partial}_t^* \bar{\partial}_t + \bar{\partial}_t \bar{\partial}_t^*$, since we have $\|\square_t - \frac{1}{2}\Delta_d\|_{op} < \epsilon$ when $t \in \Delta^{d_1}_\gamma$ is small enough (when $t = 0$, $\square_0 = \frac{1}{2}\Delta_d$ by the Kähler identities). Then

$$\dim(\mathcal{H}_\tau(X_t, \Lambda^{p,q})) \leq \dim(\mathcal{H}_\tau(X_0, \Lambda^{p,q})). \tag{7.2}$$

□

Let $e_t^{\cdot,\cdot}$ denote the dimension of the space of $\square_t$-harmonic forms with decay rate of τ. Note that by a similar argument as in Proposition 3.6, any such harmonic form must decay like $O(r^{-3})$ as $r \to \infty$. So by [Joy00, Sections 8.4 and 8.9], and Proposition 3.7, we have

$$b_0^{2,0} = b_0^{0,2} = b_0^{1,0} = b_0^{0,1} = 0. \tag{7.3}$$

From semi-continuity, it follows that

$$e_t^{2,0} = e_t^{0,2} = e_t^{1,0} = e_t^{0,1} = 0. \tag{7.4}$$

The proof of the following theorem is inspired by [BR15].

Theorem 7.2. *By choosing* ϵ_1 *small enough, for the family of deformation* (X, J_t)*,* $t \in B_{\epsilon_1}$*, there exists a smooth family of Kähler forms* ω_t *on* X_t*, and*

$$\|\omega_t - \omega_0\|_{C^{k,\alpha}_\delta} \leq C \cdot \|t\|_{C^{k,\alpha}_{-3}} \leq C \cdot \epsilon_1 \tag{7.5}$$

for some constant C*, and some* $k \geq 3, \delta \in (-2, -1)$*.*

Furthermore, letting $\mathfrak{G}$ *denote the group of holomorphic isometries of* (X, g_0, J_0)*, the mapping*

$$\begin{aligned} \aleph : B_{\epsilon_1} &\to \Lambda^2(T^*X) \\ t &\mapsto \omega_t \end{aligned} \tag{7.6}$$

can be chosen equivariantly with respect to the action of $\mathfrak{G}$. *That is,* $\aleph(\iota^* t) = \iota^*\aleph(t)$, *for all* $\iota \in \mathfrak{G}$.

Proof. Let $g_t = \frac{1}{2}(g + g\circ J_t)$ be the Hermitian metric on X_t, where g is the Riemannian metric of the central fiber and J_t is the complex structure of X_t. Let $\Box_t = \bar{\partial}_t{}^*\bar{\partial}_t + \bar{\partial}_t\bar{\partial}_t{}^*$ be the $\bar{\partial}_t-$Laplacian defined with respect to g_t. Let $\omega_t' = \frac{\sqrt{-1}}{2} g_t(J_t\,,\,)$ be the corresponding $(1,1)$-form. We want to perturb ω_t' to acquire a Kähler form ω_t. We have $\omega_t' = \omega_t^h + \phi_t$, where ω_t^h is the $\Box_t$-harmonic part of ω_t', which is asymptotic to $dz_t \wedge d\bar{z}_t$ at the rate of $-\delta$, with $\|\phi_t\|_{C^{k,\alpha}_{\delta}} \le C\|\Box_t\omega_t'\|_{C^{k-2,\alpha}_{\delta-2}}$. We next show that this claimed decomposition is valid. Consider the following Fredholm operator

$$\Box_t : C^{k,\alpha}_{\delta}(X_t, \Lambda^{1,1}) \to C^{k-2,\alpha}_{\delta-2}(X_t,\Lambda^{1,1}). \tag{7.7}$$

The kernel $\ker\Box_t^* \subset C^{k,\alpha}_{-2-\delta}(X_t,\Lambda^{1,1})$ is of finite dimension. For any $\sigma \in \ker\Box_t^*$, σ admits an expansion

$$\sigma = \sum_{k=1}^{2} \frac{a_k}{r^2} dz_t \wedge d\bar{z}_t + O(r^{-3}) \tag{7.8}$$

as $r \to \infty$, where a_k are constants. Since

$$\begin{aligned} 0 = \int_X \langle \sigma, \Box_t\sigma\rangle_{g_t} dV_{g_t} &= \lim_{R\to\infty}\Big\{ \int_{B(R)} \big(\langle\bar{\partial}\sigma,\bar{\partial}\sigma\rangle_{g_t} + \langle\bar{\partial}^*\sigma,\bar{\partial}^*\sigma\rangle_{g_t}\big) dV_{g_t} \\ &\quad + \int_{S(R)} \big(\langle\sigma, \bar{\partial}r\lrcorner\bar{\partial}\sigma\rangle_{g_t} + \langle\bar{\partial}^*\sigma, \bar{\partial}r\lrcorner\sigma\rangle_{g_t}\big) dA_{g_t}\Big\} \\ &= \int_X \big(\langle\bar{\partial}\sigma,\bar{\partial}\sigma\rangle_{g_t} + \langle\bar{\partial}^*\sigma,\bar{\partial}^*\sigma\rangle_{g_t}\big) dV_{g_t}. \end{aligned} \tag{7.9}$$

This shows that $\bar{\partial}\sigma = 0$, $\bar{\partial}^*\sigma = 0$. Since also $\bar{\partial}_t - \bar{\partial}_{Euc} = O(r^{-3})$, we have $\bar{\partial}_{Euc}\sigma = O(r^{-5})$. Then $a_k = 0$ and $\sigma = O(r^{-3})$. This implies that $\Box_t\omega_t' \in (\ker\Box_t^*)^{\perp} \cap C^{k-2,\alpha}_{\delta-2}(X_t,\Lambda^{1,1})$, which is the L^2-complement of $\ker\Box_t^*$ in $C^{k-2,\alpha}_{\delta-2}(X_t,\Lambda^{1,1})$. Then there exists a $\phi_t \in C^{k,\alpha}_{\delta}(X_t,\Lambda^{1,1})$ such that $\Box_t\phi_t = \Box_t\omega_t'$ and $\|\phi_t\|_{C^{k,\alpha}_{\delta}} \le C\|\Box_t\omega_t'\|_{C^{k-2,\alpha}_{\delta-2}}$. Then we can let $\omega_t^h = \omega_t' - \phi_t$ which is $\Box_t$-harmonic.

Proposition 7.3. *If* $\Box_t\omega_t^h = 0$, *then* $\bar{\partial}_t\omega_t^h = 0$.

Proof. $\bar{\partial}_t\Box_t\omega_t^h = \Box_t\bar{\partial}_t\omega_t^h = 0$, then $\bar{\partial}_t\omega_t^h$ is $\Box_t$-harmonic, with decay rate of $-(\delta-1)$. However, $e_t^{2,1} = e_t^{0,1} = 0$ by the upper semi-continuity and since the conjugate Hodge-star operator maps harmonic forms to harmonic forms. Then $\bar{\partial}_t\omega_t^h = 0$. □

We want to find ω_t such that $d\omega_t = 0$; for this we need a lemma.

Lemma 7.4. *We have* $\partial_t\omega_t^h = \bar{\partial}_t a_t$, *where* $a_t \in C^{k,\alpha}_{\delta}(X_t,\Lambda^{2,0})$, $k \ge 3$. *Furthermore,*

$$\|a_t\|_{C^{k,\alpha}_{\delta}} \le C\|\partial_t\omega_t^h\|_{C^{k-1,\alpha}_{\delta-1}} \le C'\|t\|_{C^{k,\alpha}_{\delta}} \tag{7.10}$$

Proof. First note that

$$\bar{\partial}_t\partial_t\omega_t^h = -\partial_t\bar{\partial}_t\omega_t^h = 0. \tag{7.11}$$

Then

$$\partial_t\omega_t^h \in C^{k-1,\alpha}_{\delta-1}(X_t,\Lambda^{2,1}) = \mathcal{H}_{\delta-1}(X_t,\Lambda^{2,1}) \oplus \Box_t C^{k+1,\alpha}_{\delta+1}(X_t,\Lambda^{2,1}). \tag{7.12}$$

(This decomposition follows since $\Box_t$ is Fredholm on $C^{k,\alpha}_{\delta+1}(X_t,\Lambda^{2,1})$.) Then

$$\partial_t\omega_t^h = h_t + (\bar\partial_t\bar\partial_t^* + \bar\partial_t^*\bar\partial_t)f_t \tag{7.13}$$

with $f_t \in C^{k+1,\alpha}_{\delta+1}(X_t,\Lambda^{2,1})$, where h_t is the harmonic part. Since $\bar\partial_t\partial_t\omega_t^H = 0$, it follows that $\Box_t\bar\partial_t f_t = 0$. From injectivity of $\Box_t$ on $C^{k,\alpha}_\delta$, we have $\bar\partial_t f_t = 0$. Then

$$\partial_t\omega_t^h = h_t + \bar\partial_t\bar\partial_t^* f_t = h_t + \bar\partial_t a_t, \tag{7.14}$$

where $a_t = \bar\partial_t^* f_t \in C^{k,\alpha}_\delta(X_t,\Lambda^{2,0})$. Since $e_t^{2,1} = 0$, then $h_t = 0$, and $\partial_t\omega_t^h = \bar\partial_t a_t$. Since $\dim(\mathcal{H}_{\delta+1}(X_t,\Lambda^{2,1})) = 0$,

$$\Box_t : C^{k+1,\alpha}_{\delta+1}(X_t,\Lambda^{2,1}) \to C^{k-1,\alpha}_{\delta-1}(X_t,\Lambda^{2,1}) \tag{7.15}$$

is an isomorphism. Then $\|f_t\|_{C^{k+1,\alpha}_{\delta+1}} \le C\|\partial_t\omega_t^h\|_{C^{k-1,\alpha}_{\delta-1}}$ implies that

$$\|a\|_{C^{k,\alpha}_\delta} \le C\|\partial_t\omega_t^h\|_{C^{k-1,\alpha}_{\delta-1}} \le C'\|t\|_{C^{k,\alpha}_\delta}, \tag{7.16}$$

□

Now we have $a_t \in C^{k,\alpha}_\delta(X_t,\Lambda^{2,0})$ and $\bar\partial_t\bar a_t = 0$. Let $\delta+2$ not be a indicial root of $\Box_t$, then

$$\Box_t : C^{k+2,\alpha}_{\delta+2}(X_t,\Lambda^{0,2}) \to C^{k,\alpha}_\delta(X_t,\Lambda^{0,2}) \tag{7.17}$$

is Fredholm. Since $\dim(\mathcal{H}_{-4-\delta}(X_t,\Lambda^{0,2})) = 0$, it follows that $\Box_t$ is surjective. We can therefore choose $f_t \in C^{k+2,\alpha}_{\delta+2}(X_t,\Lambda^{0,2})$ such that $\bar a_t = \Box_t f_t = \bar\partial_t\bar\partial_t^* f_t$ with

$$\|f_t\|_{C^{k+2,\alpha}_{\delta+2}} \le C\|\bar a_t\|_{C^{k,\alpha}_\delta} \le C'\|t\|_{C^{k,\alpha}_\delta}. \tag{7.18}$$

Let $\bar\beta_t = \bar\partial_t^* f_t$, and let $\omega_t = \omega_t^h + \bar\partial_t\beta_t$. Then $d\omega_t = \partial_t\omega_t^h - \bar\partial_t a_t = 0$. Choose ϵ_1 to be small enough, then ω_t is a closed positive (1,1)-form. □

8. Deformations of scalar-flat Kähler metrics

Notice that the previous sections did not use the scalar-flat assumption; this section is where we start using it.

8.1. **The linearized operator.** In previous sections, we have shown that there exists a family of decaying deformation of complex structure with dimension equal to $\dim(\mathcal{H}_{ess}(X,\Lambda^{0,1}\otimes\Theta))$, and there exists a smooth family of Kähler forms along the deformation. In this section, we will show that near an initial scalar-flat Kähler ALE metric ω_0, there is a smooth family of scalar-flat Kähler ALE metrics under these deformations. The arguments in this section are inspired by [LS93, LS94].

Denote $S(\omega_0 + \sqrt{-1}\partial\bar{\partial} f)$ as the scalar curvature of X with metric $\omega_0 + \sqrt{-1}\partial\bar{\partial} f$. We now consider S as mapping between weighted Hölder spaces, with some $0 < \epsilon \ll 1$

$$\begin{aligned} S : C^{k,\alpha}_{\epsilon}(X) &\to C^{k-4,\alpha}_{\epsilon-4}(X) \\ f &\mapsto S(\omega_0 + \sqrt{-1}\partial\bar{\partial} f). \end{aligned} \tag{8.1}$$

Its linearization and adjoint are maps between:

$$L : C^{k,\alpha}_{\epsilon}(X) \to C^{k-4,\alpha}_{\epsilon-4}(X) \tag{8.2}$$

$$L^* : C^{k,\alpha}_{-\epsilon}(X) \to C^{k-4,\alpha}_{-4-\epsilon}(X). \tag{8.3}$$

By direct calculation,

$$L(f) := D\,S(\omega_0 + \sqrt{-1}\partial\bar{\partial} f) = -(\Delta^2 f + R_{i,\bar{j}} \nabla^i \nabla^{\bar{j}} f). \tag{8.4}$$

Define $\bar{\partial}^\# f = g_0^{i,\bar{j}} \bar{\partial}_j f$. Also by direct calculation,

$$(\bar{\partial}\bar{\partial}^\#)^* (\bar{\partial}\bar{\partial}^\#) f = \Delta^2 f + R_{i,\bar{j}} \nabla^i \nabla^{\bar{j}} f + \nabla_k S \nabla^k f. \tag{8.5}$$

Clearly then, when ω_0 is scalar-flat, $L(f) = -(\bar{\partial}\bar{\partial}^\#)^*(\bar{\partial}\bar{\partial}^\#)(f)$, see [LS94].

The following lemma shows that the linearized mapping is surjective.

Lemma 8.1. *For $0 < \epsilon \ll 1$, $k \geq 4$, L is surjective.*

Proof. Obviously, L is an elliptic operator, and the indicial roots of L are the integers. Consequently, by standard weighted space theory, L is Fredholm since δ is non-integral [LM85].

Let $f \in \ker(L^*)$, then $f = O(r^{-\epsilon})$ and $(\bar{\partial}\bar{\partial}^\#)^*(\bar{\partial}\bar{\partial}^\#) f = 0$. Since

$$f (\bar{\partial}\bar{\partial}^\#)^*(\bar{\partial}\bar{\partial}^\#) f = o(r^{-4}), \tag{8.6}$$

integrating by parts, It follows that $\bar{\partial}\bar{\partial}^\# f = 0$, so $\bar{\partial}^\# f$ is a holomorphic vector field on X which has a decay rate of $-(\epsilon - 1)$, and Proposition 3.4 then implies that $\bar{\partial}^\# f = 0$. Since f is a real-valued function, f must be constant; but since $f = O(r^{-\epsilon})$, we have $f \equiv 0$. We have shown that $\ker(L^*) = \{0\}$, so L is surjective. □

Proposition 8.2. *For $0 < \epsilon \ll 1$ and $k \geq 4$, $\ker(L)$ is spanned by the constants.*

Proof. The operator L admits an expansion near infinity of the form $L = \Delta^2_{Euc} + Q$, where Q represents lower order terms. Let

$$N(L, \tau) = \text{Index}(L : C^{k,\alpha}_{\tau}(X) \to C^{k-4,\alpha}_{\tau-4}(X)). \tag{8.7}$$

Then, by the relative index theorem of Lockhart-McOwen,

$$N(L, \epsilon) - N(L, -\epsilon) = N(\Delta^2_{Euc}, \epsilon) - N(\Delta^2_{Euc}, -\epsilon) = 2. \tag{8.8}$$

see [LM85]. The above argument shows that $Ker(L) \cap C^{k,\alpha}_{-\epsilon}(X) = \{0\}$, which implies that

$$\ker(L) \cap C^{k,\alpha}_{\epsilon}(X) = \{\text{constants}\}. \tag{8.9}$$

□

8.2. **Construction of $\mathfrak{F}$.** Now we start the construction of the class $\mathfrak{F}$. First, we show a weighted version $\partial\bar{\partial}-$lemma on ALE surfaces. This is similar to [Joy00, Theorem 8.4.4], but with slightly different assumptions.

Lemma 8.3. *Consider the Kähler ALE surface (X, J_0, g_0). Let $\zeta \in C^{k,\alpha}_{\delta}(X, \Lambda^{1,1})$ be a real $(1,1)$ form, for some $k \geq 2$ and $\delta \in (-2,-1)$, be a real form, satisfying $d\zeta = 0$. Then there exists $\theta \in C^{k+2,\alpha}_{\delta+2}(X)$ such that $\zeta = \zeta_h + \sqrt{-1}\partial\bar{\partial}\theta$, and $\|\theta\|_{C^{k+2,\alpha}_{\delta+2}} \leq C \cdot \|\zeta - \zeta_h\|_{C^{k,\alpha}_{\delta}}$, where $\zeta_h \in \mathcal{H}_{\delta}(X, \Lambda^{1,1})$.*

Proof. Since ζ is closed, can be decomposed as $\zeta = \zeta_h + \partial\eta + \bar{\partial}\xi$, where $\zeta_h \in \mathcal{H}_{\delta}(X, \Lambda^{1,1})$, $\eta \in C^{k+1,\alpha}_{\delta+1}(X, \Lambda^{0,1})$, and $\xi \in C^{k+1,\alpha}_{\delta+1}(X, \Lambda^{1,0})$. Since ζ is a real form, we can assume that $\eta = \bar{\xi}$. Consider the operator $\Box$ and its Fredholm adjoint $\Box^*$:

$$\begin{aligned} \Box &: C^{k+3,\alpha}_{\delta+3}(X, \Lambda^{0,1}) \to C^{k+1,\alpha}_{\delta+1}(X, \Lambda^{0,1}) \\ \Box^* &: C^{k+3,\alpha}_{-3-\delta}(X, \Lambda^{0,1}) \to C^{k+1,\alpha}_{-\delta-5}(X, \Lambda^{0,1}) \end{aligned} \tag{8.10}$$

This implies that $\Box$ has finite-dimensional cokernel. Then we have

$$\eta = \eta_h + \Box\gamma = \eta_h + \bar{\partial}\bar{\partial}^*\gamma + \bar{\partial}^*\bar{\partial}\gamma, \tag{8.11}$$

where $\gamma \in C^{k+3,\alpha}_{\delta+3}(X, \Lambda^{0,1})$ and η_h is the $\Box$-harmonic part. Without loss of generality, assume $\eta_h = 0$. Since g_0 is Kähler, $\partial\bar{\partial}^*(\bar{\partial}\gamma) = -\bar{\partial}\partial^*(\bar{\partial}\gamma) = 0$, so we can assume that $\bar{\partial}\gamma = 0$. Then $\zeta = \zeta_h + \sqrt{-1}\partial\bar{\partial}\theta'$, where $\theta' = 2Im(\bar{\partial}^*\gamma) \in C^{k+2,\alpha}_{\delta+2}(X)$.

Now consider the Fredholm operator

$$F = \sqrt{-1}\partial\bar{\partial} : C^{k+2,\alpha}_{\delta+2}(X) \to C^{k,\alpha}_{\delta}(X, \Lambda^{1,1}) \tag{8.12}$$

Let $(\ker F^*)^{\perp}$ be the L^2-complement of $\ker F^*$ in $C^{k,\alpha}_{\delta}(X, \Lambda^{1,1})$. The argument above shows that $\ker F^* = \ker \Box^*$. Then $\zeta - \zeta_h \in (\ker F^*)^{\perp}$. By the same argument as in the proof of Theorem 7.2, we can show that there exists a $\theta \in C^{k+2,\alpha}_{\delta+2}(X)$, such that $\sqrt{-1}\partial\bar{\partial}\theta = \zeta - \zeta_h$, and $\|\theta\|_{C^{k+2,\alpha}_{\delta+2}} \leq C\|\zeta - \zeta_h\|_{C^{k,\alpha}_{\delta}}$ for some constant C. ☐

Theorem 8.4. *Let (X, J_0, g_0) be a scalar-flat Kähler ALE metric on a surface X. Then there is family $\mathfrak{F}$ of scalar-flat Kähler metrics near g_0, parametrized by B, that is, there is a differentiable mapping*

$$F : B^1 \times B^2 \to \mathcal{M}(X), \tag{8.13}$$

into the space of smooth Riemannian metrics $\mathcal{M}(X)$ with $\mathfrak{F} = F(B^1 \times B^2)$. Furthermore, letting $\mathfrak{G}$ denote the group of holomorphic isometries of (X, J_0, g_0), the mapping F is equivariant with respect to the action of $\mathfrak{G}$. That is, for $\iota \in \mathfrak{G}$, and $(t, \rho) \in B^1 \times B^2$, we have $F(\iota^* t, \iota^* \rho) = \iota^* F(t, \rho)$.

Proof. Recall the parameter space $B = B^1 \times B^2$, where

(8.14)

$$B^1 = B_{\epsilon_1}(0) \subset \mathcal{H}_{ess}(X_0, End_a(TX_0)); \; B^2 = B_{\epsilon_2}(0) \subset \mathcal{H}_{\delta}(X_0, \Lambda^{1,1}) \simeq H^{1,1}(X_0).$$

Let t denote the parameter of B^1. By Section 7, $\dim(\mathcal{H}_\delta(X_t,\Lambda^{p,q}))$ (using the metric determined by ω_t) is upper semicontinuous, and

$$\begin{aligned}\dim(\mathcal{H}_\delta(X_0,\Lambda^{1,1})) &= \dim(H^{1,1}(X_0))\\ \dim(\mathcal{H}_\delta(X_0,\Lambda^{2,0})) &= \dim(\mathcal{H}_\delta(X_0,\Lambda^{0,2})) = 0.\end{aligned} \tag{8.15}$$

So for ϵ_1 sufficiently small,

$$\dim(\mathcal{H}_\delta(X_t,\Lambda^{2,0})) = \dim(\mathcal{H}_\delta(X_t,\Lambda^{0,2})) = 0, \tag{8.16}$$

and also

$$\begin{aligned}\dim(\mathcal{H}_\delta(X_t,\Lambda^{2,0})) &\geq \dim(H^{2,0}(X_t))\\ \dim(\mathcal{H}_\delta(X_t,\Lambda^{0,2})) &\geq \dim(H^{0,2}(X_t)).\end{aligned} \tag{8.17}$$

This implies that $\dim(H^2(X_t)) = \dim(H^{1,1}(X_t))$ which is a topological invariant. Then

$$\begin{aligned}\dim(H^2(X_t)) \leq \dim(\mathcal{H}_\delta(X_t,\Lambda^{1,1})) \leq \dim(\mathcal{H}_\delta(X_0,\Lambda^{1,1})) =\\ = \dim(H^2(X_0)) = \dim(H^2(X_t)).\end{aligned} \tag{8.18}$$

This implies that $\dim(\mathcal{H}_\delta(X_t,\Lambda^{1,1})) = \dim(H^2(X_t))$ is constant, so choose a smooth family of isomorphisms ψ_t which map $\mathcal{H}_\delta(X_0,\Lambda^{1,1})$ to $\mathcal{H}_\delta(X_t,\Lambda^{1,1})$ for t sufficiently small. Note that $\mathfrak{G}$ acts on $\mathcal{H}_\delta(X_0,\Lambda^{1,1})$, and from Theorem 7.2, $\mathfrak{G}$ also acts on $\mathcal{H}_\delta(X_t,\Lambda^{1,1})$. Clearly we can choose ψ_t to be equivariant with respect to these actions.

It has been shown in Section 7 that there is a smooth family of Kähler forms ω_t. Let $\rho \in B^2$, define

$$\omega(t,\rho) = \omega_t + \psi_t(\rho) \tag{8.19}$$

Now consider the mapping

$$\mathcal{S} : B^1 \times B^2 \times C^{k,\alpha}_{\epsilon}(X) \to C^{k-4,\alpha}_{\epsilon-4}(X), \tag{8.20}$$

defined by

$$\mathcal{S} : (t,\rho,f) \mapsto S(\omega(t,\rho) + \sqrt{-1}\partial_t\bar{\partial}_t f). \tag{8.21}$$

We endow the domain with the product norm, where B^1 and B^2 have the L^2-norm. By direct calculation, the linearization of $\mathcal{S}$ at 0 is:

$$D\mathcal{S} = \begin{pmatrix} *, & -Ric_h, & L \end{pmatrix} \tag{8.22}$$

where Ric_h is the harmonic part of Ricci form. Since L is surjective as shown in Lemma 8.3, $D\mathcal{S}$ is surjective by the lemma above. Next, we recall the expansion of the curvature tensor

$$Rm(g_0+h) = Rm(g_0) + (g_0+h)^{-1} * \nabla^2 h + (g_0+h)^{-2} * \nabla h * \nabla h, \tag{8.23}$$

where $*$ denotes a various tensor contractions, and h is a symmetric tensor such that $g_0 + h$ is a Riemannian metric [GV16, Formula 3.40]. In our case, h can be written as

$$h(,) = (\omega(t,\rho) + \sqrt{-1}\partial_t\bar{\partial}_t f)(-J_t,) - \omega(0,0)(-J_0,). \tag{8.24}$$

Next, by (7.5), we have the estimates

$$
\begin{aligned}
|h| &\le C(|t|+|\rho|+|\nabla^2 f|) \le C(r^{\delta}\|t\|_{C^{k,\alpha}_{\delta}} + r^{\delta}\|\rho\|_{C^{k,\alpha}_{\delta}} + r^{-2+\epsilon}\|f\|_{C^{k,\alpha}_{\epsilon}})\\
|\nabla h| &\le C(|\nabla t|+|\nabla\rho|+|\nabla^3 f|) \le C(r^{-1+\delta}\|t\|_{C^{k,\alpha}_{\delta}} + r^{-1+\delta}\|\rho\|_{C^{k,\alpha}_{\delta}} + r^{-3+\epsilon}\|f\|_{C^{k,\alpha}_{\epsilon}})\\
|\nabla^2 h| &\le C(|\nabla^2 t|+|\nabla^2\rho|+|\nabla^4 f|) \le C(r^{-2+\delta}\|t\|_{C^{k,\alpha}_{\delta}} + r^{-2+\delta}\|\rho\|_{C^{k,\alpha}_{\delta}} + r^{-4+\epsilon}\|f\|_{C^{k,\alpha}_{\epsilon}}).
\end{aligned} \tag{8.25}
$$

Then by (8.23), we have

$$
\begin{aligned}
S(g_0+h) - DS\cdot h = {}& C_0(g_0,h) * Rm(g_0) * h * h\\
&+ C_1(g_0,h) * h * \nabla^2 h + C_2(g_0,h) * \nabla h * \nabla h,
\end{aligned} \tag{8.26}
$$

where $C_0(g_0,h), C_1(g_0,h)$, and $C_2(g_0,h)$ are bounded functions, when h is sufficiently small. Without loss of generality, we can assume that $-2+\epsilon<\delta$. Then

$$
\begin{aligned}
|S(g_0+h) - DS\cdot h| &\le C\{r^{-2}|h|^2 + |h|\cdot|\nabla^2 h| + |\nabla h|\cdot|\nabla h|\}\\
&\le C\cdot r^{-2+2\delta}\cdot(\|t\|_{C^{k,\alpha}_{\delta}} + \|\rho\|_{C^{k,\alpha}_{\delta}} + \|f\|_{C^{k,\alpha}_{\epsilon}})^2
\end{aligned} \tag{8.27}
$$

which implies that

$$
\|S(g_0+h) - DS\cdot h\|_{C^{k,\alpha}_{-4+\epsilon}} \le C(\|t\|_{C^{k,\alpha}_{\delta}} + \|\rho\|_{C^{k,\alpha}_{\delta}} + \|f\|_{C^{k,\alpha}_{\epsilon}})^2. \tag{8.28}
$$

Furthermore,

$$
\begin{aligned}
&\|S(g_0+h) - S(g_0+h') - DS\cdot(h-h')\|_{C^{k,\alpha}_{-4+\epsilon}} \le\\
&\qquad C\cdot\big(\|t-t'\|_{C^{k,\alpha}_{\delta}} + \|\rho-\rho'\|_{C^{k,\alpha}_{\delta}} + \|f-f'\|_{C^{k,\alpha}_{\epsilon}}\big)\\
&\qquad\quad \cdot\big(\|t\|_{C^{k,\alpha}_{\delta}} + \|\rho\|_{C^{k,\alpha}_{\delta}} + \|f\|_{C^{k,\alpha}_{1}} + \|t'\|_{C^{k,\alpha}_{\delta}} + \|\rho'\|_{C^{k,\alpha}_{\delta}} + \|f'\|_{C^{k,\alpha}_{\epsilon}}\big).
\end{aligned} \tag{8.29}
$$

Since B^1 and B^2 are finite-dimensional, we can replace the corresponding norms by the L^2-norm. This shows that $\mathcal{S}$ satisfies the condition (4.1). Then by Lemma 4.1, the zero set of $\mathcal{S}$ is in one-to-one correspondence with the kernel of $\mathcal{S}'(0)$. Since any solution of the nonlinear equation can be written as a kernel element plus a unique element in the image of a bounded right inverse for the linearized operator, the zero set of $\mathcal{S}$ can also be written as a graph over $B^1\times B^2$. That is, for any $(t,\rho)\in B^1\times B^2$, there exists a unique $f(t,\rho)$, up to constants, of

$$
\mathcal{S}(t,\rho,f(t,\rho)) = 0, \tag{8.30}
$$

so a family of scalar-flat Kähler metrics over B can be constructed as

$$
F:(t,\rho)\mapsto g(t,\rho) = \Big(\omega(t,\rho) + \sqrt{-1}\partial_t\bar{\partial}_t f(t,\rho)\Big)(-J_t\cdot,\cdot), \tag{8.31}
$$

where $g(0,0)=g_0$. It is a straightforward consequence of the implicit function theorem that the mapping F is differentiable. Then the image of F gives us the family of scalar-flat Kähler metrics $\mathfrak{F}$, and the construction is clearly equivariant with respect to the action of $\mathfrak{G}$.

To finish the proof, we show that metrics in $\mathfrak{F}$ are smooth. By the result of [TV05a], any Kähler constant scalar curvature metric satisfies an equation of the form

$$\Delta Ric = Rm * Ric, \tag{8.32}$$

where the right hand side denotes quadratic curvature contractions involving the full curvature tensor Rm and the Ricci tensor. By a Moser iteration method and regularity bootstrap argument in harmonic coordinates, it follows that g is smooth, see [TV05b, Theorem 6.4]. □

9. Versality and uniqueness of the moduli space

In this section, all norms are defined based on the initial Kähler metric g_0. We let (g_1, J_1) be any scalar-flat Kähler metric satisfying $\|g_1 - g_0\|_{C^{k,\alpha}_\delta} < \epsilon_3$, (for some $k \geq 4$, $\delta \in (-2, -1)$, and ϵ_3 will be determined later). We begin with the following lemma.

Lemma 9.1. *For ϵ_3 sufficiently small, if $\|g_1 - g_0\|_{C^{k,\alpha}_\delta} < \epsilon_3$, then there exists a diffeomorphism $\Phi_0 : X \to X$ and constants C_1, C_2 so that*

$$\|\Phi_0^* g_1 - g_0\|_{C^{k,\alpha}_\delta} < C_1 \epsilon_3 \tag{9.1}$$

$$\|\Phi_0^* J_1 - J_0\|_{C^{k,\alpha}_\delta} < C_2 \epsilon_3. \tag{9.2}$$

Proof. Let ∇_0 denote the covariant derivative of g_0, and Γ_0, Γ_1 denote the Christoffel symbols of g_0, g_1, respectively. Then $\|\Gamma_1 - \Gamma_0\|_{C^{k-1,\alpha}_{\delta-1}} < C\epsilon_3$, so in particular $|\Gamma_1 - \Gamma_0| < \frac{C\epsilon_3}{(1+r)^{-\delta+1}}$ as $r \to \infty$.

Since (g_i, J_i) is Kähler, $\nabla_i J_i = 0$ for $i = 0, 1$. We now estimate $|J_1 - J_0|(p)$ along any geodesic ray γ starting at p_0. From [HL15, Lemma 1.1], both J_1 and J_0 have a finite limit along γ as $t \to \infty$, and furthermore

$$J_i = (J_i)_0 + O(r^{-2}), \tag{9.3}$$

where $(J_i)_0$ is a constant complex structure on $\mathbb{R}^4$ for $i = 0, 1$. We are assuming that $(J_0)_0 = J_{Euc}$. Clearly, there exists a diffeomorphism $\Phi_0 : X \to X$ so that

$$\Phi_0^* J_1 = J_{Euc} + O(r^{-2}), \tag{9.4}$$

as $r \to \infty$ such that (9.1) is satisfied. This can be done, for example, by connecting a rotation defined on the complement of a large ball $X \setminus B(p_0, 2R)$ to the identity transformation on $B(p_0, R)$ by smooth path of rotations of each sphere on the annulus $A(R, 2R)$. We then have

$$\begin{aligned} |\Phi_0^* J_1 - J_0|(p) &= |\Phi_0^* J_1 - J_0|(p) - \lim_{t\to\infty} |\Phi_0^* J_1 - J_0|(\gamma(t)) \\ &\leq \int_{s=r}^{\infty} |\nabla_0(\Phi_0^* J_1 - J_0)(\gamma(s))| ds \\ &\leq \int_{s=r}^{\infty} |\nabla_0 \Phi_0^* J_1(\gamma(s))| ds \\ &< \int_{s=r}^{\infty} \frac{C \cdot \epsilon_3}{(1+s)^{-\delta+1}} ds \leq \frac{C' \cdot \epsilon_3}{(1+r)^{-\delta}}. \end{aligned} \tag{9.5}$$

Since this estimate is true along any ray, it follows that $\|\Phi_0^* J_1 - J_0\|_{C^0_\delta} < C'\epsilon_3$. We can estimate the higher regularities in the same way, and (9.2) follows. □

We next prove the "versality" of our family $\mathfrak{F}$. That is, for g_1 as above, there exists a diffeomorphism Φ such that $\Phi^* g_1$ is in the class $\mathfrak{F}$.

Theorem 9.2. *Let $-2 < \delta < -1$ be fixed. There exists an $\epsilon_3 > 0$ such that for any scalar-flat Kähler metric (g_1, J_1) satisfying $\|g_1 - g_0\|_{C^{k,\alpha}_\delta} \leq \epsilon_3$, there exists a diffeomorphism $\Phi : X \to X$ of the form $\Phi_0 \circ \Phi_{Y_1} \circ \Phi_Z \circ \Phi_{Y_2}$ where Φ_0 is as in Lemma 9.1, $Y_1, Y_2 \in C^{k+1,\alpha}_{\delta+1}(TX)$, and $Z \in \mathbb{W}$, such that $\Phi^* g_1 \in \mathfrak{F}$. Furthermore, there exists a constant C so that*

$$\|\Phi^* g_1 - g_0\|_{C^{k,\alpha}_\delta} \leq C\epsilon_3. \tag{9.6}$$

Proof. Let Φ_0 denote the diffeomorphism from Lemma 9.1. Then $\Phi_0^* J_1$ satisfies the assumptions of Theorem 6.3, so there exists a diffeomorphism $\tilde{\Phi}$ satisyfing the properties stated in that theorem. In particular, $\tilde{\Phi}$ is of the form $\tilde{\Phi} = \Phi_{Y_1} \circ \Phi_Z \circ \Phi_{Y_2}$, where $Y_1, Y_2 \in C^{k+1,\alpha}_{\delta+1}(TX)$, and $Z \in \mathbb{W}$. The estimate (9.6) for $\Phi_0 \circ \tilde{\Phi}$ is then proved as follows. For Φ_{Y_1}, we estimate

$$\|\Phi_{Y_1}^* \Phi_0^* g_1 - g_0\|_{C^{k,\alpha}_\delta} \leq \|\Phi_{Y_1}^* \Phi_0^* g_1 - \Phi_0^* g_1\|_{C^{k,\alpha}_\delta} + \|\Phi_0^* g_1 - g_0\|_{C^{k,\alpha}_\delta} \leq C\epsilon_3. \tag{9.7}$$

This estimate holds since $\Phi_0^* g_1$ is also a scalar-flat Kähler metric which is smooth by the last observation in the proof of Theorem 8.4, so when ϵ_3 is sufficiently small, Lemma 5.1 holds for g_1. The same argument applies to Φ_{Y_2}. The estimate (9.6) then follows from (9.7), (6.9) and Lemma 9.1.

Let ω_1 denote the Kähler form of $\Phi^* g_1$. It was shown in Section 8 that $\dim(H^{1,1}(X_t))$ is locally constant, which implies $H^{1,1}(X_0) \simeq H^{1,1}(X_t)$ for any $t \in B^1$. Then there exists a $\rho \in \mathcal{H}_\delta(X_0, \Lambda^{1,1})$ such that $[\omega(\phi_{ess}, \rho)] = [\omega_1] \in H^{1,1}(X_t)$, where $\omega(\phi_{ess}, \rho)$ is defined in (8.19). Let $g' = F(\phi_{ess}, \rho)$, and let ω' denote the corresponding Kähler form. Then by Lemma 8.3, $\omega' - \omega_1 = \sqrt{-1}\bar{\partial}_t \bar{\partial}_t^* f$ for some potential function $f \in C^{k+2,\alpha}_{\delta+2}(X)$, and $\|f\|_{C^{k+2,\alpha}_{\delta+2}} \leq C\|\omega' - \omega_1\|_{C^{k,\alpha}_\delta}$.

As shown in the proof of Theorem 8.4, in a neighborhood of g_0, a scalar-flat Kähler metric is uniquely determined by the Kähler class and complex structure. This implies that $g' = g_1$. □

This completes the proof of Theorem 1.4. Next, we will complete the proof of Theorem 1.7. From Theorem 8.4, the mapping F is $\mathfrak{G}$-equivariant. That is, for $\iota \in \mathfrak{G}$, and $(t, \rho) \in B^1 \times B^2$, we have $F(\iota^* t, \iota^* \rho) = \iota^* F(t, \rho)$. This clearly implies that two elements in the same orbit of $\mathfrak{G}$ are isometric, which proves the first statement in Theorem 1.7.

For the second statement in Theorem 1.7, we need the following lemma, which says that in the non-hyperkähler case, the dimension of $\mathfrak{F}$ is the same as the dimension of the parameter space.

Lemma 9.3. *Let (X, g, J) be as above. If g is not hyperkähler then F is injective, so $\dim(\mathfrak{F}) = d = \dim(B^1 \times B^2)$.*

Proof. If $F(\phi_1,\rho_1) = F(\phi_2,\rho_2) = \tilde{g}$, then the metric $\tilde{g}$ is Kähler with respect to the two complex structures J_1 and J_2, corresponding to ϕ_1 and ϕ_2, respectively.

We have $J_i \in End(TX)$ $(i = 1, 2)$ such that $J_i^2 = -1$. Locally, J_i can be considered as an purely imaginary Hamiltonian number. Define

$$J_3 = \frac{J_1J_2 - J_2J_1}{\|J_1J_2 - J_2J_1\|}, \tag{9.8}$$

where for $p \in X$,

$$\|J_i(p)\| = \sup_{0\neq v\in T_pX}\Big\{\frac{\|J_i(v)\|_h}{\|v\|_h}\Big\}. \tag{9.9}$$

Then $J_3 \in End(TX)$. We claim that J_3 is well-defined, $J_3^2 = -1$, J_1, J_2, J_3 are linearly independent. Clearly,

$$(J_1J_2 - J_2J_1)(J_1J_2 - J_2J_1) = -2 + (J_1J_2J_1J_2 - J_2J_1J_2J_1). \tag{9.10}$$

Since $J_1J_2J_1J_2 - J_2J_1J_2J_1$ is real and has norm which is < 2, we have that

$$(J_1J_2 - J_2J_1)(J_1J_2 - J_2J_1) \neq 0, \tag{9.11}$$

so J_3 is well-defined, and $J_3 \cdot J_3 = -1$. If we write J_1, J_2 locally as Hamiltonian numbers $a_1I + b_1J + c_1K$, $a_2I + b_2J + c_2K$ (where $a_i, b_i, c_i \in \mathbb{R}$, $I^2 = J^2 = K^2 = -1$), then

$$J_1J_2 - J_2J_1 = (b_0c_1 - b_1c_0)I + (c_0a_1 - c_1a_0)J + (a_0c_1 - a_1c_0)K, \tag{9.12}$$

which is linearly independent with J_1 and J_2. Then J_1, J_2, J_3 are linearly independent. As a result, $\{J_1, J_2, J_3\}$ gives a hyperkähler structure. Then we have proved that, if $\Gamma \not\subset \mathrm{SU}(2)$, then $J_1 = J_2$, then by the proof of Theorem 8.4, F is injective. □

The next proposition immediately implies the second part of Theorem 1.7.

Proposition 9.4. *For any two element $g_1, g_2 \in \mathfrak{F}$ associated to divergence-free complex structures J_1, J_2, if there exists a small diffeomorphism Φ_Y which is induced by the exponential map of a vector field $Y \in C^{k+1,\alpha}_{\delta+1}(TX)$, such that $g_1 = \Phi_Y^* g_2$, then g_1, g_2 are the same.*

Proof. First, assume X is non-hyperkähler. Since $\Phi_Y^* g_2 = g_1$, by Lemma 9.3, $J_1 = \Phi_Y^* J_2$. However by Lemma 5.2, there is a unique small diffeomorphism that gauges J_2 to be divergence-free. Then $\Phi_Y = Id$, $g_1 = g_2$.

When X is hyperkähler, then by rotating the hyperkähler sphere, there exists a J_1' such that J_1' is compatible with g_1 and $(X, \Phi_Y^* J_2)$ is biholomorphic to (X, J_1'). Then by the same argument above, $g_1 = g_2$. □

Since $\mathfrak{F}$ is of finite dimension, and $\mathfrak{G}$ is a compact group action on $\mathfrak{F}$, the dimension m of $\mathfrak{M} = \mathfrak{F}/\mathfrak{G}$ is well-defined. In the non-hyperkähler case, by Lemma 9.3,

$$m = d - (\text{the dimension of a maximal orbit of } \mathfrak{G}). \tag{9.13}$$

(For the hyperkähler case, recall Remark 1.11.)

10. Deformations of the minimal resolution

In this section, we prove Theorems 1.10 and 1.12.

10.1. **Harmonic representation of $H^1(X,\Theta)$.** Let X denote the minimal resolution of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of U(2) without complex reflections. In the following, we want to construct a weighted version of Hodge theory, that links the sheaf cohomology with the decaying harmonic forms.

The divisor $E = \cup_i E_i$ is a union of irreducible components which are rational curves, with only normal crossing singularities. Let $Der_E(X)$ denote the sheaf dual to logarithmic 1-forms along E (see [Kaw78]). We note that $Der_E(X)$ is a locally free sheaf of rank 2, see [Wah75]. Away from E, this is clear. If $p \in E_i$, we can choose a holomorphic coordinate chart $\{z_1, z_2\}$ such that near p, $E_i = \{z_1 = 0\}$. Then local sections of $Der_E(X)$ are generated by $\{z_1\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\}$.

The short exact sequence

$$0 \to Der_E(X) \to \Theta_X \to \mathcal{O}_E(E) \to 0, \tag{10.1}$$

induces an exact sequence of cohomologies

$$0 \to H^1(X, Der_E(X)) \to H^1(X,\Theta) \to H^1(E, \mathcal{O}_E(E)) \to H^2(X, Der_E(X)). \tag{10.2}$$

Since E is composed of rational curves whose self-intersection numbers are negative, we have $H^0(E, \mathcal{O}_E(E)) = 0$.

Proposition 10.1. *We have the vanishing result: $H^1(X, Der_E(X)) = 0$.*

Proof. This is a result which is implicit in work of [BKR88, Bri68, Lau73, Wah75]. However, since the result is not exactly stated there, we give an analytic proof in the Appendix, see Proposition 12.3. □

By Siu's vanishing theorem ([Siu69]), since X is a non-compact σ-compact complex manifold, for any coherent analytic sheaf $\mathscr{F}$ on X, the top degree sheaf cohomology $H^2(X,\mathscr{F})$ is trivial. Consequently, $H^2(X, Der_E(X)) = 0$, which gives us an isomorphism $H^1(\Theta_X) = H^1(O_E(E))$. Let $-e_j$ be the self-intersection number of each rational curve $E_j \subset E$, and let k_Γ be the number of rational curves (which is equal to b_2). Then

$$\dim(H^1(X,\Theta)) = \sum_{j=1}^{k_\Gamma}(e_j - 1). \tag{10.3}$$

Theorem 10.2. *Let (X, g, J) denote the minimal resolution of $\mathbb{C}^2/\Gamma$ with any ALE Kähler metric g of order $\tau > 1$. Then*

$$H^1(X,\Theta) \cong \mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta) \cong \mathcal{H}_{ess}(X, \Lambda^{0,1} \otimes \Theta) \tag{10.4}$$

Proof. First, we consider the case on $\mathbb{C}^2 \setminus \{0\}$, and we compute $H^1(\mathbb{C}^2 \setminus \{0\}, \Theta)$. The domain $\mathbb{C}^2 \setminus \{0\}$ can be covered by two charts: $U_1 = \{z_1 \neq 0\}$ and $U_2 = \{z_2 \neq 0\}$.

Note that U_1, U_2 are each isomorphic to $\mathbb{C}\times\mathbb{C}^*$, and $U_1\cap U_2$ is isomorphic to $\mathbb{C}^*\times\mathbb{C}^*$. Then $\theta_1 \in \Gamma(U_1,\Theta)$ can be expanded into a Laurent series

$$\theta_1 = \sum_{i\in\mathbb{Z}_{\geq 0}, j\in\mathbb{Z}} a^1_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_1} + b^1_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_2}, \tag{10.5}$$

$\theta_2 \in \Gamma(U_2,\Theta)$ can be expanded into

$$\theta_2 = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_{\geq 0}} a^2_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_1} + b^2_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_2}, \tag{10.6}$$

$\theta_{1,2} \in \Gamma(U_1\cap U_2,\Theta)$ can be expanded into

$$\theta_{1,2} = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}} a^{1,2}_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_1} + b^{1,2}_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_2}. \tag{10.7}$$

Then any $\theta_{1,2} \in \Gamma(U_1\cap U_2,\Theta)$ which cannot be represented as $\theta_1-\theta_2$ must be of the form

$$\theta_{1,2} = \sum_{i\in\mathbb{Z}_-, j\in\mathbb{Z}_-} a^{1,2}_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_1} + b^{1,2}_{i,j} z_1^i z_2^j \frac{\partial}{\partial z_2}. \tag{10.8}$$

By Dolbeault's lemma, we have $H^1(X,\Theta)\simeq H^0(X,\Lambda^{0,1}\otimes\Theta)$ ([GH94]). Using this, we transform the Čech cohomology element (10.8) to an element of Dolbeault cohomology $\phi\in\Gamma(\mathbb{C}^2\setminus\{0\},\Lambda^{0,1}\otimes\Theta)$. Let ρ_1,ρ_2 be a partition of unity, where ρ_1 is supported in U_1 and $\rho_1=1$ for $|z_1|>1$; ρ_2 is supported in U_2 and $\rho_2=1$ for $|z_2|>1$; $\rho_1+\rho_2=1$. Then define $\phi\in\Gamma(\mathbb{C}^2\setminus\{0\},\Lambda^{0,1}\otimes\Theta)$ as

$$\phi = \begin{cases} \bar\partial(\rho_2\cdot\theta_{1,2}) & \text{on } U_1 \\ -\bar\partial(\rho_1\cdot\theta_{1,2}) & \text{on } U_2 \end{cases}. \tag{10.9}$$

The form ϕ is well-defined since $\bar\partial(\rho_2\cdot\theta_{1,2})+\bar\partial(\rho_1\cdot\theta_{1,2})=\bar\partial(\theta_{1,2})=0$ on $U_1\cap U_2$. Furthermore, ϕ is decaying at infinity, since the degrees appearing in (10.8) are all negative.

Now for any closed form $\phi\in H^0(X,\Lambda^{0,1}\otimes\Theta)$, since $X-E$ is biholomorphic to $\mathbb{C}^2\setminus\{0\}/\Gamma$, ϕ corresponds with a Γ-equivariant form $\phi'\in H^0(\mathbb{C}^2\setminus\{0\},\Lambda^{0,1}\otimes\Theta)$. Then $\phi'=\bar\partial\tau+\psi$, with ψ a closed form and decaying at infinity by the argument above. Since Γ is a finite group, by averaging τ by the group action, we can assume that τ is Γ-invariant. Let ρ be a cutoff function on X which equals to 0 in $B(R)$ and 1 outside of $B(2R)$. Then $\phi-\bar\partial(\rho\cdot\tau)$ is a closed decaying form, which is in the same class as ϕ.

This shows that the natural mapping

$$\mathcal{H}_{-3}(X,\Lambda^{0,1}\otimes\Theta)\to H^1(X,\Theta) \tag{10.10}$$

is surjective. We next show that this mapping is injective.

Let $\phi=\bar\partial\eta\in\mathcal{H}_{-3}(X,\Lambda^{0,1}\otimes\Theta)$, where $\eta\in\Gamma(X,\Theta)$. Then $[\phi]=0\in H^1(X,\Theta)$. We want to prove $\phi=0$ by showing that $\phi=\bar\partial\xi$ for some $\xi=O(r^{-2})$. Let χ be a smooth cutoff function defined on X with compact support that contains E. Then

$\phi = \bar{\partial}(\chi\eta) + \bar{\partial}((1-\chi)\eta)$. Since π is biholomorphic on $X \setminus E$, $(1-\chi)\eta$ can be pushed down by π and extended to ζ on $\mathbb{C}^2$ such that $\zeta = 0$ in a neighborhood of the origin. And also $\bar{\partial}\zeta = O(r^{-3})$ on $\mathbb{C}^2$ by the decaying rate of ϕ. Exactly as in (3.11) above, by the Poincaré lemma, there exists a $\sigma \in C^{\infty}_{-2}(\mathbb{C}^2, \Theta)$ such that $\bar{\partial}\sigma = \bar{\partial}\zeta$. Average σ by the group action of Γ such that σ is Γ-invariant, then $\sigma(0) = 0$. Note that σ is holomorphic in a neighborhood of 0. By the argument in the Appendix, we can lift up σ to $\tilde{\sigma} \in \Gamma(X, \Theta)$, where $\tilde{\sigma} = O(|z|^{-2})$. Then we have $\phi = \bar{\partial}(\chi\eta + \tilde{\sigma})$. Since $\chi\eta + \tilde{\sigma} = O(r^{-2})$, $\phi = 0$.

Finally, we are going to prove that

$$\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta) \simeq \mathcal{H}_{ess}(X, \Lambda^{0,1} \otimes \Theta), \tag{10.11}$$

It is clear that $\mathcal{H}_{ess}(X, \Lambda^{0,1} \otimes \Theta) \subset \mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta)$. To show the isomorphism, we need to show that $\mathbb{V} = \{0\}$. Let $Y \in \mathbb{W}$, by definition, $Y \in \mathcal{H}_1(X, \Theta)$. Then Y has the following expansion as $r \to \infty$:

$$Y = \sum_{i,j} a_{i,j} z_i \frac{\partial}{\partial z_j} + \sum_k b_k \frac{\partial}{\partial z_k} + O(r^{-3+\epsilon}) \tag{10.12}$$

where $a_{i,j}, b_k$ are constants. The decaying rate of $O(r^{-3+\epsilon})$ comes from the fact that $\Box$ has no indicial roots between -2 and 0, and there is no $\bar{\partial}$-closed kernel element corresponding to the root of -2.

If Γ is nontrivial, then $b_k = 0$. Denote

$$\pi : X \mapsto \mathbb{C}^2/\Gamma \tag{10.13}$$

as the minimal resolution of $\mathbb{C}^2/\Gamma$. Since $Z = \sum_{i,j} a_{i,j} z_i \frac{\partial}{\partial z_j}$ is a holomorphic vector field on $\mathbb{C}^2/\Gamma \setminus \{0\}$ and $\lim_{r\to 0} Z = 0$, by Proposition 12.30, Z can be lifted up to a holomorphic vector field $\tilde{Z}$ on X. Then $Y - \tilde{Z} \in \mathcal{H}_{-3}(X, \Theta)$. By using integration by parts, $Y - \tilde{Z}$ is holomorphic, then Y is holomorphic, $\bar{\partial}Y = 0$, and $V = 0$. Then $\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta) \simeq \mathcal{H}_{ess}(X, \Lambda^{0,1} \otimes \Theta)$.

Finally, if Γ is trivial, then X is biholomorphic to $\mathbb{C}^2$, and the coordinate vector fields extend to global holomorphic vector fields, so these do not give any non-trivial elements of $\mathbb{V}$. $\Box$

10.2. **Discussion of Table 1.1.** Cases 1 & 2: $\Gamma = \frac{1}{p}(1,1)$. This case has been studied in [Hon13]. The group of biholomorphic automorphisms is $\mathrm{GL}(2,\mathbb{C})$, and the identity component of the holomorphic isometry group is $\mathrm{U}(2)$. When $p = 3$, the action of $\mathrm{U}(2)$ coincides with the action of $\mathrm{SU}(2)$, and the dimension of generic orbits is 3. Then $\dim(\mathfrak{M}) = 5 - 3 = 2$. When $p > 3$, the dimension of each orbit is 4, and $\dim(\mathfrak{M}) = 2p - 5$.

Case 3: $\Gamma = \frac{1}{p}(1,q)$, where $q \neq 1, p-1$. In this case, by direct calculation, the subgroup in $\mathrm{U}(2)$ that commutes with Γ is isomorphic to $S^1 \times S^1$. By [ALM15b, Proposition 3.3] the identity component of the holomorphic isometry group must be $S^1 \times S^1$. Using the fact that the cyclic quotient singularity is characterized by the invariant polynomials in (12.3) below, it is easy to show that $S^1 \times S^1$ acts faithfully on $\mathfrak{F}$. Then $\mathfrak{G} = S^1 \times S^1$, the dimension of the generic orbit is 2, and $\dim(\mathfrak{M}) = j_\Gamma + k_\Gamma - 2$.

Case 4: Γ is non-cyclic and not in SU(2). In this case, the subgroup in U(2) that commutes with Γ is isomorphic to S^1, so by [ALM15b, Proposition 3.3] the identity component of the holomorphic isometry group must be S^1. The dimension of the generic orbit is 1, and $\dim(\mathfrak{M}) = j_\Gamma + k_\Gamma - 1$.

10.3. **Proof of Theorem 1.12.** A path of ALE Kähler metrics is defined as follows.

Definition 10.3. *A smooth path connecting two ALE Kähler surfaces* (X, g_0, J_0) *and* (X, g_1, J_1) *is a smooth family of complex structures* J_t *and a smooth family of ALE Kähler metrics* g_t *for* $t \in [0, 1]$*, such that*

$$(X, g(0), J(0)) = (X, g_0, J_0), \ (X, g(1), J(1)) = (X, g_1, J_1). \tag{10.14}$$

Proof of Theorem 1.12. In the following, X_t will stand for the triple (X, g_t, J_t). Since the index of a strongly continuous family of Fredholm operators is constant, the index of the operator P_t defined in Lemma 4.2 is locally constant along a path of ALE Kähler metrics. Consequently, $P(t)$ is constant along the path X_t. By the same argument as in Lemma 4.2, the cokernel of $P(t)$ is trivial for all $t \in [0, 1]$. Therefore $P(t)$ are all surjective Fredholm operators, with $\dim(\ker P(t)) = \dim(\ker P_0) = \dim(\ker P_1)$. Therefore if there exists a path that connects X_1 with a minimal resolution X_0, then $\dim(\mathcal{H}_{-3}(X_1, \Lambda^{0,1} \otimes \Theta)) = \dim(\mathcal{H}_{-3}(X_0, \Lambda^{0,1} \otimes \Theta))$. Since $\mathfrak{G}(X_1) = \{e\}$,

$$\dim_{\mathbb{R}}(\mathcal{H}_{ess}(X_1, \Lambda^{0,1} \otimes \Theta)) = \dim_{\mathbb{R}}(\mathcal{H}_{-3}(X_0, \Lambda^{0,1} \otimes \Theta)) - \dim(\mathfrak{G}(X_0)) = j_\Gamma. \tag{10.15}$$

In the non-hyperkähler case, Lemma 9.3 implies that

$$m(X_1) = j_\Gamma + b_2(X) = m_\Gamma, \tag{10.16}$$

and Theorem 1.7 implies that the local moduli space of scalar-flat Kähler ALE metrics near g_1 is a smooth manifold of dimension m_Γ.

As remarked above, the hyperkähler moduli space is constructed globally by Kronheimer, and $m_\Gamma = 3k - 3$; see Section 11 for some further remarks. □

11. Examples

In this section, we will discuss several examples to illustrate the theory.

11.1. **Hyperkähler case.** Recall that a hyperkähler metric is Kähler with respect to a 2-sphere of complex structures $S^2 = \{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}$.

Example 1 (A_1-type): the Eguchi-Hanson metric is an ALE Ricci-flat Kähler metric on $X = T^*S^2$. Since $k = 1$, $B^1 \subset \mathbb{R}^2$, $B^2 \subset \mathbb{R}$, $d_\Gamma = 2 + 1 = 3$. With respect to the complex structure I (the complex structure arising as the total space of a holomorphic line bundle), the biholomorphic isometry group is U(2). The quotient $\mathfrak{F}/\mathfrak{G}$ has two orbit-types. The orbit of $(0, \rho)$ is 1-dimensional. The orbit of (t, ρ) where t is non-zero, is also 1-dimensional. Consequently, $\mathfrak{M}$ is isomorphic to the 2-dimensional upper half space. The remaining parameter of complex structures just corresponds to a hyperkähler rotation, so the metrics obtained are all just scalings of the Eguchi-Hanson metric.

Instead, consider the complex structure J. The biholomorphic isometry group is SU(2). The subspace $\mathbb{V}$ is now of dimension 1, so our parameter space is now $\mathbb{R} \times \mathbb{R}$. The group $\mathfrak{G}$ now acts trivially, so our parameter space is a ball in $\mathbb{R}^2$. The remaining parameter of complex structures again just corresponds to a hyperkähler rotation, so the metrics obtained are all just scalings of the Eguchi-Hanson metric.

Example 2 (ADE-type): For the general A_k, D_k, E_k ($k \geq 2$) type ALE minimal resolution, the dimension of local moduli space of Ricci-flat Kähler metrics is $3k - 3$.

For A_k ($k \geq 2$), this is the case of Gibbons-Hawking ALE hyperkähler surface. $Aut(X) = \mathbb{C}^* \times S^1 \subset \mathbb{R}_+ \times \mathrm{U}(2)$. Recall that $\mathrm{U}(2) = \mathrm{U}(1) \times_{\mathbb{Z}_2} \mathrm{SU}(2)$, which acts on $\vec{v} \in \mathbb{C}^2$ as $g_L \cdot \vec{v} \cdot g_R$, where $g_L \in \mathrm{U}(1)$ is the left action and $g_R \in \mathrm{SU}(2)$ is the right action. The $\mathbb{C}^*$-action is generated by $(v_1, v_2) \to (\lambda v_1, \lambda v_2)$ where $\lambda \in \mathbb{C}^*$; the S^1-action is generated by $(v_1, v_2) \to (\lambda v_1, \lambda^{-1} v_2))$ where $|\lambda| = 1$. The $\mathbb{C}^*$ action induces a 2-dimensional action on the hyperkähler sphere, while the S^1 action preserves the hyperkähler structure. Then $m_\Gamma = d_\Gamma - 3 = 3k - 3$.

For the case of D_k, E_k, $Aut(X) = \mathbb{C}^*$. The $\mathbb{C}^*$ action can be interpreted as follows: let $\mathfrak{g}_{\mathbb{C}^*}$ denote the set of real vector fields which correspond to the Lie algebra of $\mathbb{C}^*$. For any $Y \in \mathfrak{g}_{\mathbb{C}^*}$, Φ_Y^* acts on the complex structures which gives an action on B^1. Since Y is a real vector field, Φ_Y^* is transverse to the action on the hyperkähler sphere (it is not transverse only in the A_k case). Then the dimension of the maximal orbit generated by the $\mathbb{C}^*$ action and the action on hyperkähler sphere is 3, so $m_\Gamma = d_\Gamma - 3$.

11.2. **Cyclic case.** Any cyclic action without complex reflections is conjugate to the action generated by

$$(z_1, z_2) \mapsto (\xi_p z_1, \xi_p^q z_2), \tag{11.1}$$

where ξ_p is a pth root of unity, and q is relatively prime to p, which we will call a $\frac{1}{p}(1, q)$ action. Define the integers $e_i \geq 2$, and k by the continued fraction expansion

$$\frac{p}{q} = e_1 - \cfrac{1}{e_2 - \cdots \cfrac{1}{e_k}} \equiv [e_1, \ldots, e_k]. \tag{11.2}$$

The singularity of $\mathbb{C}^2/\Gamma$ is known as a Hirzebruch-Jung singularity, and the exceptional divisor is a string of rational curves with normal crossing singularities. (More details of the Hirzebruch-Jung resolution can be found in the Appendix.)

If $1 \leq q < p$, then let $q' = p - q$. Let $e_i' \geq 2$, and k' denote integers arising in the the Hirzebruch-Jung algorithm for the $\frac{1}{p}(1, q')$-action. In [Rie74], Riemenschneider proved the formulas

$$\sum_{i=1}^{k}(e_i - 1) = \sum_{i=1}^{k'}(e'_i - 1), \tag{11.3}$$

$$k' = e - 2, \tag{11.4}$$

$$e = 3 + \sum_{i=1}^{k}(e_i - 2), \tag{11.5}$$

where e is the embedding dimension. In particular, these formulas give that

$$\sum_{i=1}^{k}(e_i - 1) = e + k - 3. \tag{11.6}$$

From Subsection 10.2 above, for $q \neq 1, p-1$, it follows that

$$m_\Gamma = 2e + 3k - 8. \tag{11.7}$$

11.3. **Non-cyclic cases.** The non-cyclic finite subgroups of U(2) without complex reflections are given in Table 11.1, where the binary polyhedral groups (dihedral, tetrahedral, octahedral, icosahedral) are respectively denoted by D^*_{4n}, T^*, O^*, I^*, and the map $\phi : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$ denotes the usual double cover, see [Bri68, BKR88, LV14] for more details.

Table 11.1. Non-cyclic finite subgroups of U(2) containing no complex reflections

$\Gamma \subset \mathrm{U}(2)$	Conditions	Order
$\phi(L(1,2l) \times D^*_{4n})$	$(l,2n) = 1$	$4ln$
$\phi(L(1,2l) \times T^*)$	$(l,6) = 1$	$24l$
$\phi(L(1,2l) \times O^*)$	$(l,6) = 1$	$48l$
$\phi(L(1,2l) \times I^*)$	$(l,30) = 1$	$120l$
Index–2 diagonal $\subset \phi(L(1,4l) \times D^*_{4n})$	$(l,2) = 2, (l,n) = 1$	$4ln$
Index–3 diagonal $\subset \phi(L(1,6l) \times T^*)$	$(l,6) = 3$	$24l$.

Table 11.2 lists the dimension of the moduli space for subgroups of U(2) for finite subgroups involving T^*, O^*, I^*. Computations are omitted, and we refer the reader to the papers of [Bri68, BKR88, LV14] for a complete description of the exceptional divisors.

Next, consider the case of $\phi(L(1,2l) \times D^*_{4n})$, where $(l,2n) = 1$. A computation, which is omitted, shows that

$$m_\Gamma = 3k + 2k' + \frac{2}{n}(l+q) + 4, \tag{11.8}$$

where $1 \leq q \leq n-1$ satisfies $q \equiv -l \mod n$, k is the length of the Hirzebruch-Jung string for $\frac{1}{n}(1,q)$, and k' is the length of the dual string.

The last case is that of the Index 2 subgroup contained in $\phi(L(1,4l)\times D^*_{4n})$, where $(l,2)=2,(l,n)=1$. A computation, which is omitted, shows that the same formula (11.8) holds in this case.

TABLE 11.2. Cases with T^*, O^*, I^* for $l>1$

$\Gamma\subset \mathrm{U}(2)$	m_Γ
$\phi(L(1,2l)\times T^*)$	
$l\equiv 1 \mod 6$	$\frac{1}{3}(l-1)+17$
$l\equiv 5 \mod 6$	$\frac{1}{3}(l-5)+15$
Index–3 diagonal $\subset \phi(L(1,6l)\times T^*)$	
$(l,6)=3$	$\frac{1}{3}(l-3)+16$
$\phi(L(1,2l)\times O^*)$	
$l\equiv 1 \mod 12$	$\frac{1}{6}(l-1)+20$
$l\equiv 5 \mod 12$	$\frac{1}{6}(l-5)+19$
$l\equiv 7 \mod 12$	$\frac{1}{6}(l-7)+18$
$l\equiv 11 \mod 12$	$\frac{1}{6}(l-11)+17$
$\phi(L(1,2l)\times I^*)$	
$l\equiv 1 \mod 30$	$\frac{1}{15}(l-1)+23$
$l\equiv 7 \mod 30$	$\frac{1}{15}(l-7)+19$
$l\equiv 11 \mod 30$	$\frac{1}{15}(l-11)+22$
$l\equiv 13 \mod 30$	$\frac{1}{15}(l-13)+19$
$l\equiv 17 \mod 30$	$\frac{1}{15}(l-17)+18$
$l\equiv 19 \mod 30$	$\frac{1}{15}(l-19)+20$
$l\equiv 23 \mod 30$	$\frac{1}{15}(l-23)+18$
$l\equiv 29 \mod 30$	$\frac{1}{15}(l-29)+19$

12. Appendix

In this Appendix, we will show the proof of two propositions regarding the minimal resolution $X\mapsto \mathbb{C}^2/\Gamma$. The first is a vanishing theorem for $H^1(X, Der_E(X))$. The second is a lifting property of Γ-invariant holomorphic vector fields on $\mathbb{C}^2/\Gamma$.

12.1. **Cyclic quotient singularity.** Consider a cyclic quotient singularity of the form $\Gamma=\frac{1}{p}(1,q)$ $(p\geq q)$. We will first give some additional detail regarding the Hirzebruch-Jung resolutions. Details can be found in [Rei, Kol07].

The continued fraction described in formula (11.2), can also be represented by lattice points

$$c_0=(1,0), c_1=\frac{1}{p}(1,q),\dots,c_{m+1}=(0,1), \tag{12.1}$$

with iterative relation

$$\begin{pmatrix} c_i \\ c_{i+1}\end{pmatrix}=\begin{pmatrix} 0 & 1\\ -1 & e_i\end{pmatrix}\cdot\begin{pmatrix} c_{i-1}\\ c_i\end{pmatrix}. \tag{12.2}$$

Meanwhile, the dual continued fraction $\frac{p}{p-q} = [a_1, \ldots, a_k]$ can be used to give the invariant polynomials:

$$u_0 = x^p, u_1 = x^{p-q}y, u_2, \ldots, u_k, u_{k+1} = y^p, \tag{12.3}$$

which satisfy the relation $u_{i-1}u_{i+1} = u_i^{a_i}$.

The polynomials $\{u_0, \ldots, u_{k+1}\}$ give an embedding of the cone in $\mathbb{C}^{k+2}$. Let

$$c_0 = (s_0, t_0), \ldots, c_{m+1} = (s_{m+1}, t_{m+1}) \tag{12.4}$$

be lattice points, where $s_0 = 0, t_0 = 1, s_m = 1, t_m = 0, s_{i+1} > s_i, t_{i+1} < t_i$. Let $\{\eta_i, \xi_i\}$ $(0 \leq i \leq m+1)$ be monomials forming the dual basis to $\{c_i, c_{i+1}\}$, i.e.,

$$c_i(\eta_i) = 1, c_i(\xi_i) = 0, c_{i+1}(\eta_i) = 0, c_{i+1}(\xi_i) = 1. \tag{12.5}$$

Proposition 12.1. *The numbers s_i, t_i satisfy the relation*

$$t_i s_{i+1} - t_{i+1} s_i = \frac{1}{p}. \tag{12.6}$$

Proof. We prove it by induction. First, note that $c_0 = (0,1), c_1 = \frac{1}{p}(1,q)$. Then $t_0 s_1 - t_1 s_0 = \frac{1}{p}$. Next, assume that $t_{i-1}s_i - t_i s_{i-1} = \frac{1}{p}$. By the recursive formula $c_{i+1} + c_{i-1} = a_i c_i$, it follows that

$$(s_{i+1}, t_{i+1}) + (s_{i-1}, t_{i-1}) = a_i(s_i, t_i). \tag{12.7}$$

Then we have

$$s_{i+1} = a_i s_i - s_{i-1},\ t_{i+1} = a_i t_i - t_{i-1}. \tag{12.8}$$

So finally,

$$t_i s_{i+1} - t_{i+1} s_i = t_i(a_i s_i - s_{i-1}) - (a_i t_i - t_{i-1})s_i = t_{i-1}s_i - t_i s_{i-1} = \frac{1}{p}. \tag{12.9}$$

□

By the formula (12.6), we have that $\eta_i = p \cdot (-t_{i+1}, s_{i+1}), \xi_i = p \cdot (t_i, -s_i)$. Then

$$\xi_i = \frac{x^{pt_i}}{y^{ps_i}}, \eta_i = \frac{y^{ps_{i+1}}}{x^{pt_{i+1}}}, \xi_{i+1} = \frac{x^{pt_{i+1}}}{y^{ps_{i+1}}}, \eta_{i+1} = \frac{y^{ps_{i+2}}}{x^{pt_{i+2}}}. \tag{12.10}$$

It follows that the coordinate transition from $\{\eta_i, \xi_i\}$ to $\{\eta_{i+1}, \xi_{i+1}\}$ for $\xi_{i+1} \neq 0$, is given by

$$\eta_i = \xi_{i+1}^{-1},\ \eta_{i+1} = \eta_i^{e_{i+1}}\xi_i,\ (0 \leq i \leq m-1) \tag{12.11}$$

which defines an acyclic cover $Y = Y_0 \cup Y_1 \ldots \cup Y_m$ of X satisfying

$$Y_i \cap Y_{i+1} \simeq \mathbb{C} \times \mathbb{C}^*,\ Y_i \cap Y_{i+k} = Y_i \cap Y_{i+1} \ldots \cap Y_{i+k}, \tag{12.12}$$

see [Rei, Theorem 3.2]. For use below, we record the following formulae:

$$
\begin{aligned}
\frac{\partial}{\partial \eta_i} &= \frac{1}{\eta_i}(s_i x \frac{\partial}{\partial x} + t_i y \frac{\partial}{\partial y}) \\
\frac{\partial}{\partial \xi_i} &= \frac{1}{\xi_i}(s_{i+1} x \frac{\partial}{\partial x} + t_{i+1} y \frac{\partial}{\partial y}) \\
\frac{\partial}{\partial x} &= \frac{p}{x}(t_i \xi_i \frac{\partial}{\partial \xi_i} - t_{i+1}\eta_i \frac{\partial}{\partial \eta_i}) \\
\frac{\partial}{\partial y} &= \frac{p}{y}(-s_i \xi_i \frac{\partial}{\partial \xi_i} + s_{i+1}\eta_i \frac{\partial}{\partial \eta_i}).
\end{aligned} \tag{12.13}
$$

12.2. **Vanishing property.** Next we will show the vanishing of $H^1(X, Der_E(X))$ for the minimal resolution X of $\mathbb{C}^2/\Gamma$.

Lemma 12.2. *When Γ is cyclic, $H^1(X, Der_E(X)) = 0$ for the minimal resolution X of $\mathbb{C}^2/\Gamma$.*

Proof. From (12.11) above,

$$
\frac{\partial}{\partial \xi_{i+1}} = -\eta_i^2 \frac{\partial}{\partial \eta_i} + e_{i+1}\eta_i\xi_i \frac{\partial}{\partial \xi_i}, \tag{12.14}
$$

$$
\frac{\partial}{\partial \eta_{i+1}} = \eta_i^{-e_{i+1}} \frac{\partial}{\partial \xi_i}. \tag{12.15}
$$

The sections of $Der_E(X)$ are generated by

$$
\Big\{\frac{\partial}{\partial \eta_i}, \xi_i \frac{\partial}{\partial \xi_i}\Big\}, \ \Big\{\frac{\partial}{\partial \xi_{i+1}}, \eta_{i+1}\frac{\partial}{\partial \eta_{i+1}}\Big\} \tag{12.16}
$$

on Y_i, Y_{i+1} respectively. For $\theta_i \in \Gamma(Y_i, Der_E(X))$, θ_i can be expanded as a Laurent series:

$$
\theta_i = \sum_{k\geq 0, l\geq 0} a^i_{k,l}\eta_i^k\xi_i^l \frac{\partial}{\partial \eta_i} + b^i_{k,l}\eta_i^k\xi_i^{l+1}\frac{\partial}{\partial \xi_i}. \tag{12.17}
$$

For $\theta_{i+1} \in \Gamma(Y_{i+1}, Der_E(X))$,

$$
\theta_{i+1} = \sum_{k\geq 0, l\geq 0} a^{i+1}_{k,l}\xi_{i+1}^k\eta_{i+1}^l \frac{\partial}{\partial \xi_{i+1}} + b^{i+1}_{k,l}\xi_{i+1}^k\eta_{i+1}^{l+1}\frac{\partial}{\partial \eta_{i+1}}. \tag{12.18}
$$

For $\theta_{i,i+1} \in \Gamma(Y_i \cap Y_{i+1}, Der_E(X))$ on the intersection $Y_i \cap Y_{i+1}$ where $\eta_i \neq 0$,

$$
\theta_{i,i+1} = \sum_{k\in\mathbb{Z}, l\geq 0} a^{i,i+1}_{k,l}\eta_i^k\xi_i^l \frac{\partial}{\partial \eta_i} + b^{i,i+1}_{k,l}\eta_i^k\xi_i^{l+1}\frac{\partial}{\partial \xi_i}. \tag{12.19}
$$

By the transition formula (12.11),

$$
\begin{aligned}
\theta_{i+1} = \sum_{k\geq 0, l\geq 0} \Big\{ &- a^{i+1}_{k,l}\eta_i^{-k+le_{i+1}+2}\xi_i^l \frac{\partial}{\partial \eta_i} \\
&+ (a^{i+1}_{k,l}e_{i+1}\eta_i^{-k+le_{i+1}+1}\xi_i^{l+1} + b^{i+1}_{k,l}\eta_i^{-k+le_{i+1}}\xi_i^{l+1})\frac{\partial}{\partial \xi_i}\Big\},
\end{aligned} \tag{12.20}
$$

which shows that the exponents of η_i in θ_{i+1} can be any negative integers. Then it is clear that for any $\theta_{i,i+1}$, there exist θ_i, θ_{i+1} such that on $Y_i \cap Y_{i+1}$, $\theta_{i,i+1} = \theta_{i+1} - \theta_i$. Furthermore, if $\{\theta_{k,l}\}$ $(k < l)$ is closed, then $\theta_{k,l} = \theta_{k,l+1} + \ldots + \theta_{l-1,l}$. Then $\{\theta_{k,l}\}$ is determined if and only if the set of consecutive elements $\{\theta_{i,i+1}\}$ is determined. These arguments imply that any closed $\{\theta_{k,l}\}$ is exact, so $H^1(X, Der_E(X)) = 0$. □

Proposition 12.3. *Let $X \mapsto \mathbb{C}^2/\Gamma$ be a minimal resolution, where $\Gamma \subset \mathrm{U}(2)$ with no complex reflections. Then $H^1(X, \Theta) \simeq H^1(E, \mathcal{O}_E(E))$ and $H^1(X, Der_E(X)) = 0$.*

Proof. In Lemma 12.2, we have shown that when Γ is cyclic, $H^1(X, Der_E(X)) = 0$, which implies that $H^1(X, \Theta) \simeq H^1(E, \mathcal{O}_E(E))$. In the following, we will use a relative index theorem to show this also holds for the general case.

Assume Γ is non-cyclic. We will first construct a Kähler form on the minimal resolution X of $\mathbb{C}^2/\Gamma$ by gluing Calderbank-Singer ALE surfaces Y_j $(j = 1, 2, 3)$ to a LeBrun orbifold X_0, which has three cyclic quotient singularities on a rational curve. The following sketches the gluing procedure.

Let x_i, $i = 1, 2, 3$ be the cyclic quotient singularities of X_0 with group Γ_i. Let (z_i^1, z_i^2) be local holomorphic coordinates on $U_i \setminus \{x_i\}$. Let ω_{X_0} be the Kähler form of the LeBrun metric. Then ω_{X_0} admits an expansion

$$\omega_{X_0} = \frac{\sqrt{-1}}{2}(\partial\bar{\partial}|z_i|^2 + \partial\bar{\partial}\xi_i) \tag{12.21}$$

on $U_i \setminus \{x_i\}$, where ξ_i is a potential function satisfying $\xi_i = O(|z_i|^4)$. For the LeBrun orbifold X_0, outside of a compact subset, it admits a holomorphic coordinate (v_1, v_2). Let Y_i denote the minimal resolution of $\mathbb{C}^2/\Gamma_i$. Outside of a compact subset of Y_i, there exist holomorphic coordinates (u_i^1, u_i^2). Let ω_{Y_i} be a Kähler form on Y_i corresponding to any Calderbank-Singer metric on Y_i. From [RS09], the Kähler form admits an expansion

$$\omega_{Y_i} = \frac{\sqrt{-1}}{2}(\partial\bar{\partial}|u_i|^2 + \partial\bar{\partial}\eta_i), \tag{12.22}$$

where $\eta_i - c\log(|u|^2) = O(|u_i|^{-1})$, for some constant c. In fact, a similar expansion holds for any scalar-flat Kähler ALE surface [ALM15b].

Next, we construct a Kähler form on X. Choose two small positive numbers a, b, we glue the regions $\frac{1}{a} \leq |u_i| \leq \frac{4}{a}$ and $b \leq |z_i| \leq 4b$, by letting $z_i = ab \cdot u_i$. This mapping is biholomorphic in the intersection. Let ρ be a smooth cutoff function satisfying $\rho(t) = 1$ when $t \leq 1$, $\rho = 0$ when $t \geq 2$. Let

$$\begin{aligned} \omega_b &= \begin{cases} \frac{\sqrt{-1}}{2}(\partial\bar{\partial}|z_i|^2 + \partial\bar{\partial}((1 - \rho(\frac{|z_i|}{2b}))\xi_i(z_i))) & \text{if } |z_i| \leq b \\ \omega_{X_0} & \text{if } |z_i| \geq 4b \end{cases} \\ \omega_a &= \begin{cases} \frac{\sqrt{-1}}{2}(\partial\bar{\partial}|u_i|^2 + \partial\bar{\partial}(\rho(a|u_i|)\eta_i(u_i))) & \text{if } |u_i| \geq 4a^{-1} \\ \omega_{Y_i} & \text{if } |u_i| \leq a^{-1} \end{cases}. \end{aligned} \tag{12.23}$$

Then we define

$$\omega_{a,b} = \begin{cases} a^{-2}b^{-2}\omega_b & \text{if } |z_i| \geq 2b \\ \omega_a & \text{if } |u_i| \leq 2a^{-1} \end{cases}. \tag{12.24}$$

For a, b sufficiently small, $\omega_{a,b}$ is a Kähler form on X. Since ω_{X_0} was ALE of order 2, the Kähler metric $\omega_{a,b}$ is also ALE of order 2.

Next, choose R_1, R_2, R_3, such that $0 < 2R_1 < R_2, R_3 > 0$, and define smooth functions r_1, r_2, r_3 as:

$$\begin{aligned} r_1(x) &= \begin{cases} |z_i| & \text{if } |z_i| \leq R_1 \\ 1 & \text{if } |z_i| \geq 2R_1 \end{cases} \\ r_2(x) &= \begin{cases} 1 & \text{if } |v| \leq R_2 \\ |v| & \text{if } |v| \geq 2R_2 \end{cases} \\ r_3(x) &= \begin{cases} 1 & \text{if } |u_i| \leq R_3 \\ |u| & \text{if } |u_i| \geq 2R_3 \end{cases} \end{aligned} \tag{12.25}$$

Define $\gamma : X \to \mathbb{R}_+$ by the following:

$$\gamma = \begin{cases} a^{-1}b^{-1}r_1 r_2 & \text{if } |z_i| \geq 2b \\ r_3 & \text{if } |u_i| \leq 2a^{-1} \end{cases}. \tag{12.26}$$

For $\delta \in \mathbb{R}$, define the weighted Hölder space $C^{k,\alpha}_{\delta,\gamma}(M, T)$ of sections of any vector bundle T over M as the closure of the space of C^∞-sections in the norm

$$\begin{aligned} \|\sigma\|_{C^{k,\alpha}_{\delta,\gamma}(M,T)} &= \sum_{|\mathcal{I}| \leq k} |\gamma^{-\delta+|\mathcal{I}|}\nabla^{\mathcal{I}}\sigma| \\ &+ \sum_{|\mathcal{I}|=k} \sup_{0<d(x,y)<\rho_{inj}} \Big(\min\{\gamma(x), \gamma(y)\}^{-\delta+k+\alpha} \frac{|\nabla^{\mathcal{I}}\sigma(x) - \nabla^{\mathcal{I}}\sigma(y)|}{d(x,y)^\alpha} \Big). \end{aligned} \tag{12.27}$$

Lemma 12.4. *Let X_0 be the LeBrun orbifold with quotient singularities x_1, x_2, x_3. The elliptic operator P:*

$$C^{k,\alpha}_{\delta,r_1r_2}(X_0,\Lambda^{0,1} \otimes \Theta) \xrightarrow{(\bar{\partial}^*,\bar{\partial})} C^{k-1,\alpha}_{\delta-1,r_1r_2}(X_0, \Theta) \oplus C^{k-1,\alpha}_{\delta-1,r_1r_2}(X_0, \Lambda^{0,2} \otimes \Theta) \tag{12.28}$$

is Fredholm and surjective, where $\delta \in (-2, -1), k \geq 3$.

Proof. An argument similar to that in Lemma 4.2 shows that any element of the kernel and cokernel is $\square$-harmonic. By the standard theory of harmonic functions, any $\square$-harmonic element which is $O(r_1^\delta)$ as $r_1 \to 0$ has a removable singularity. The remainder of the proof is almost the same as the proof of Lemma 4.2, and is omitted. □

Define the elliptic operator P as $(\bar{\partial}^*, \bar{\partial})$ with respect to the glued metric $\omega_{a,b}$, on the weighted space $C^{k,\alpha}_{\delta,\gamma}(X, \Lambda^{0,1} \otimes \Theta)$. Because $\omega_{a,b}$ is a Kähler ALE metric, by Lemma 4.2, P is Fredholm and surjective. Since each P_{Y_i} has a bounded right inverse for $i = 1, 2, 3$, and P_{X_0} has a bounded right inverse, a standard argument (see for example [RS05]) shows that there is a uniformly bounded right inverse of P, for a, b sufficiently small.

In [LV14, Proposition 6.1], it was shown that $\dim(\ker(P_{X_0})) = b_\Gamma - 1$, where $-b_\Gamma$ is the self-intersection number of the central divisor in X. For each Calderbank-Singer ALE space Y_j, by Lemma 12.2, we have $\dim(\ker(P_{Y_j})) = \sum_{i=1}^{k_j}(e_i^j - 1)$, where $-e_i^j$ is the self-intersection number of each irreducible exceptional divisor in Y_j. It was shown above in (10.10) that $\dim(H^1(X, \Theta)) \leq \dim(\mathcal{H}_{-3}(X, \Lambda^{0,1}\Theta))$. Also, from (10.2) above, we have $\dim(H^1(E, \mathcal{O}_E(E))) \leq \dim(H^1(X, \Theta))$. Combining these, we have that

(12.29)

$$\dim(H^1(X, \Theta)) \leq \dim(\mathcal{H}_{-3}(X, \Lambda^{0,1} \otimes \Theta)) = \dim(\ker(P_X)) =$$

$$= b_\Gamma - 1 + \sum_{j=1}^{3}\sum_{i=1}^{k_j}(e_i^j - 1) = \dim(H^1(E, \mathcal{O}_E(E))) \leq \dim(H^1(X, \Theta)).$$

This implies the isomorphism $H^1(X, \Theta) \simeq H^1(E, \mathcal{O}_E(E))$. Then by the exact sequence (10.2), $H^1(X, Der_E(X)) = 0$. □

12.3. **Lifting of holomorphic vector fields.** The following proposition was used above in the proof of Theorem 10.2. This is a standard result, but we provide a proof here for completeness.

Proposition 12.5. *Let $\pi : X \mapsto \mathbb{C}^2/\Gamma$ be a minimal resolution where Γ is as above. Then for any Γ-invariant holomorphic vector field ξ on $\mathbb{C}^2$ with $\xi = 0$ at the origin, there exists a lifting $\tilde{\xi}$ on X, such that, $\tilde{\xi}$ is holomorphic on X, and $\pi_*(\tilde{\xi})$ equals to ξ (modulo the action of Γ).*

Proof. First, consider the case of cyclic Γ. By the assumption, ξ is holomorphic and $\xi(0) = 0$. Then, by the formula (12.13),

$$\xi = f \cdot (t_i\xi_i\frac{\partial}{\partial \xi_i} - t_{i+1}\eta_i\frac{\partial}{\partial \eta_i}) + g \cdot (-s_i\xi_i\frac{\partial}{\partial \xi_i} + s_{i+1}\eta_i\frac{\partial}{\partial \eta_i}) \tag{12.30}$$

near the origin, where f, g are bounded holomorphic functions. Since $\pi : X \setminus E \mapsto \mathbb{C}^2/\Gamma\setminus\{0\}$ is biholomorphic, there is a natural holomorphic lifting $\tilde{\xi}$ of ξ to $X\setminus E$, and by formula 12.30, $\tilde{\xi}$ is bounded in a neighborhood of E. Then by Hartog's principle, $\tilde{\xi}$ can be extended to a holomorphic vector field on X.

For the general case when $\Gamma \subset \mathrm{U}(2)$ with no complex reflections, if Γ is not cyclic, then $\mathbb{C}^2/\Gamma$ has a partial resolution, which is a quotient of $\mathcal{O}(-2m)$, with exactly three cyclic orbifold singularities. For a Γ-invariant holomorphic vector field ξ on $\mathbb{C}^2$, ξ can be first lifted to a Γ-invariant holomorphic vector field on $\mathcal{O}(-2m)$ via the resolution of a $\frac{1}{2m}(1,1)$-type cyclic action, and ξ descends to the partial resolution. On the partial resolution, the lifting is Γ-invariant and equals to zero at the cyclic singularities. Then from the above paragraph, it can be lifted to a holomorphic vector field $\tilde{\xi}$ on the minimal resolution X. □

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