Random regularity of a nonlinear Landau Damping solution for the Vlasov-Poisson equations with random inputs

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September 18, 2018

Abstract

In this paper, we study the nonlinear Landau damping solution of the Vlasov-Poisson equations with random inputs from the initial data or equilibrium. For the solution studied in [H.J. Hwang and J. J.L.Velazquez, Indiana University Mathematics Journal, Vol. 58, No. 6, 2009], we prove that the solution depends smoothly on the random input, if the long-time limit distribution function has the same smoothness, under some smallness assumptions. We also establish the decay of the higher-order derivatives of the solution in the random variable, with the same decay rate as its deterministic counterpart.

1 Introduction

The problem that we will consider in this paper is the following form of one-dimensional Vlasov-Poisson equations:

$$\begin{cases} \partial_t f + v\partial_x f + E\partial_v f = 0,\ -\infty < v < \infty,\ 0 < x < 2\pi,\ t > 0, \\ \partial_x E = \int_{-\infty}^{\infty} f(x,v,t)dv - 1, E(0,t) = E(2\pi,t), \int_0^{2\pi} E(x)dx = 0, \\ f(x,v,0) = f_0(x,v), \\ f(0,v,t) = f(2\pi,v,t), \\ \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} f(x,v,0)dvdx = 1, \end{cases} \tag{1.1}$$

where x, t are the space and time variables respectively, v is the velocity and $f = f(x,v,t)$ is the particle distribution function of electrons. $E = E(x,t)$ is the electric field generated by electrons and

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ions, which depends on $f(x, v, t)$. The background charge is assumed to be constant 1 to make the system electrically neutral. In this paper, we only focus on the solution such that $E(x, t), f(x, v, t)$ are 2π-periodic function with respect to x, which is usually considered in the study of Landau damping.

The Landau damping phenomenon, first discovered by Laudau [9] in 1946 for the V-P equations, is a famous physical phenomenon for collisionless plasmas. Landau showed that if given an initial distribution close enough to some spatial homogeneous equilibrium $f_0(v)$, then the electric field corresponding to the solution of linearized V-P equation will decay exponentially in time, if $f_0(v)$ satisfies certain conditions. After some works at the linear level (for examples [4, 5]), Caglioti and Maffei [3] in 1998, used scattering approach proved the existence of a class of Landau damping solutions. This result was later improved by Hwang and Velazquz [6] with a larger class of possible asymptotic limits and more general condition. In 2011, Mouhot and Villani [13] gave a sufficient condition for the initial data under which the Landau damping phenomenon will happen. They established exponential Landau damping in analytic regularity. Bedrossian, Masmoudi and Mouhot [2] later gave a simpler and more general proof of nonlinear Landau damping with similar regularity of initial data.

In this paper, we will mainly focus on the general result proved by Hwang and Velazquz [6], which are concerned with deterministic initial data and limit distribution function. But in reality the initial data and the final result are observed by experiments, which might have uncertainty because of measure error. Therefore, we need to consider the solution dependence on such uncertainty and study the uncertainly propagation and how it affects the solution for large time. This is called uncertainly quantification (UQ) [18]. For such problems, one tries to understand how the uncertainty will propagate, and how the solution depends on the random inputs, which are important to validate and calibrate the kinetic models, and also help design a mechanism to control the uncertainty.

To model the uncertainty, we introduce a random variable z in a random space $I_z \subset R^{d_z}$ with prescribed probability density distribution $\pi(z)$, where d_z is the dimension of the random space. Then the uncertainty can enter the problem through the z-dependent initial data $f_0 = f_0(x, v, z)$. We may also assume that the equilibrium solution to depend on z. In addition, the uncertainty may also enter into the system through background density (say by randomly perturbing the background density 1), or boundary data. Then the solution for f and E will also depend on z. In this paper we will focus the case of random initial data and/or equilibrium state, while the analysis can certainly be mode more general for other sources of uncertainties.

While in conducting UQ one can ask many questions about the impact of random inputs to the problem or solution, in this paper we will focus on one aspect of the problem, namely, to understand the regularity of the solution (defined in the sense of [6]) in the random space. In [6], it was proved that it is possible to obtain solutions of the V-P system defined in $t \in (0, \infty)$, such that $f(x, v, t) \to f_\infty(x, v, t)$ and $E(x, t) \to 0$ as $t \to \infty$, where $f_\infty(x, v, t)$ is a free streaming function defined by $f_\infty(x, v, t) = f_e(v) + g_\infty(x - vt, v)$, and $f_e(v)$ is a spatial homogeneous equilibrium of the V-P equation satisfying $\int f_e(v)dv = 1$. In [6], it was proved that if $f_e(v)$ and $g_\infty(x, v)$ satisfy some more general stability condition, one can construct Landau damping solution that converges to f_∞ as $t \to \infty$. Now, by assuming that $f_e(v)$ and $g_\infty(x, v)$ smoothly depend on z (the high-order derivatives in z are smooth), we prove that the constructed solution $f(x, v, t, z)$ and electric field $E(x, t, z)$ will maintain the same smoothness in the random space, for all t. This shows that there exists a large class of initial data depending smoothly on z, and the solutions corresponding to the initial data will also smoothly depend on z. Such smoothness result is not only important for understanding how the uncertainty propagates in time but also important for understanding the accuracy of numerical approximations in the random space. See previous works on elliptic/parabilic problems [1, 14], and for hyperbolic problems [12, 19].

Recently there has been a rapid progress in studying uncertain collisional kinetic equations, for both linear equations [7, 10] and nonlinear equations [8, 11, 15]. The regularity and local sensitivity analysis in the random space for all of these works are based on energy estimates, due to the hypocoercivity of the linearized kinetic operators. Nonlinear terms are controlled by the hypocoercive terms together with the assumption that the initial data is near the global equilibrium. The VP equations (1.1), however, is a Hamiltonian system with conservation of energy, thus is time reversible, and one cannot use dissipative energy estimate. In [16], the first attempt was made to study the random regularity of Landau damping solution with random uncertainty, using the solution constructed in [3]. This is a following-up work, based on the solution constructed in [6], which includes more general solution both initially and in long-time. More specifically, the solution constructed in [3] behaves asymptotically as free streaming solutions and are sufficiently flat in the velocity space, while the enlarged class of solution constructed in [6] replaces the flatness condition in [3] by a stability condition. Of course, it is of significant interest to study such problems for the solution of Mouhot and Villani [13], which is the goal of a forthcoming paper [17].

The paper is organized as follows. In section 2, we will present the main result and, by using the characteristic variable (to be denoted by u), transform the equation into a form more convenient for the stuty of long-time bahavior. We also give some relevant lemmas and previous results from [6] for the deterministic problem. In section 3, we give the exact form of the higher derivatives, for (mixed) physical and random variables. In section 4, we first deal with higher-order u-derivatives of relevant physical quantities, whose norms can be bounded, using mathematical induction with suitable smallness assumption on the initial data. Finally, in section 5, we will prove the main result using similar arguments and analysis as in sections 2 and 4. Our results show exponential decay of the higher-order derivatives in z with the same rate as the original deterministic solution.

2 Main result

2.1 Laudau damping solution with uncertainty

Now, If we construct a solution f that depends on the random uncertain variable z, the electric filed E will also have uncertainty. (The uncertainty comes from f_e and g_∞ as we said before, and we will make f_e, g_∞ clear later in Theorem 2.5). For each fixed z we introduce the characteristic variable.

$$u = x - vt,$$

$$\tilde{f}(u,v,t,z)=\tilde{f}(x-vt,v,t,z)=f(x,v,t,z).$$

Then one has the following three relations

$$\partial_t f=\partial_t\tilde{f}-v\partial_u\tilde{f},\ \partial_x f=\partial_u\tilde{f},\ \partial_v f=\partial_v\tilde{f}-t\partial_u\tilde{f}.$$

For each fixed z, we note $\tilde{f}(u,v,t,z)$ by $f(u,v,t,z)$ for convenience in the sequel. Hence, (1.1) finally becomes:

$$\begin{cases}\partial_t f-tE(u+vt,t,z)\partial_u f+E(u+vt,t,z)\partial_v f=0,\ -\infty<v<\infty,\ 0<u<2\pi,\ t>0,\\ \partial_u E=\displaystyle\int_{-\infty}^{\infty}f(u-wt,w,t,z)dw-1, E(0,t,z)=E(2\pi,t,z), \int_0^{2\pi}E(u,z)du=0,\\ f(u,v,0,z)=f_0(u,v,z),\\ f(0,v,t,z)=f(2\pi,v,t,z),\\ \displaystyle\frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}f(u,v,t,z)dvdu=1,\end{cases}\tag{2.1}$$

where $E(u,t,z), f(u,v,t,z)$ are still 2π-periodic functions with respect to u.

We now introduce a norm that will be used in the subsequent sections.

Fix $r>0$. For a function $F=F(u,t)$, denote

$$||F||_r=\sup_{t>0}e^{rt}||F(\cdot,t)||_{L^\infty},\tag{2.2}$$

and the corresponding space is

$$L_r^\infty=\left\{F|F\in L^\infty(\mathbb{R}\times\mathbb{R}^+),\ ||F||_r<\infty\right\},$$

which is obviously a complete space. In this paper, we will mainly consider the space of continuous functions in L_r^∞, denoted by $\mathcal{C}_r$, which is a closed subset of L_r^∞. We should notice that for a function with uncertainty, we can consider its $||\cdot||_r$ with fixed z.

The theorem [6] below shows the existence of Laudau damping solution to (2.1) for each fixed z. For convenience, we omit z in the rest of this section, but we should notice all constants and coefficients (like $\epsilon_1,\epsilon_2,\alpha,A,C$) given below depend on z.

Theorem 2.1 *If $f_e(v)$ and $g_\infty(u,v)$ satisfy the following assumptions:*

(a) The function $f_e(v)$ is analytic in the strip $|\mathrm{Im}(v)|\leqslant A$ and satisfies

$$|f_e(v)|\leqslant\frac{\varepsilon_1}{1+|v|^\alpha},\quad \alpha>1,\quad \alpha\neq 2,\qquad \int f_e(v)dv=1.\tag{2.3}$$

(b) The Landau function defined by

$$\Phi(\eta;n)=\int_{\mathbb{R}}\frac{\partial_v f_e(w)}{w-\eta}dw-n^2,\ \eta\in\mathbb{C}\backslash\mathbb{R},\ n\in\mathbb{Z}\backslash\{0\}.\tag{2.4}$$

does not have zeros in the half-plane $\{Im(\eta) \geqslant 0\}$ *for any* $n \in \mathbb{Z}\backslash\{0\}$.

(c) The function $g_\infty(u,v)$ *is analytic in the set* $|\mathrm{Im}(u)| \leqslant A$, $|\mathrm{Im}(v)| \leqslant A$, *and satisfies*

$$|g_\infty(u,v)| \leqslant \frac{\varepsilon_2}{1+|v|^\alpha}, \quad \alpha > 1, \quad \alpha \neq 2. \tag{2.5}$$

(d) The function $g_\infty(u,v)$ *is periodic in the u variable with period* 2π *and it satisfies*

$$\int_0^{2\pi} g_\infty(u,v)du = 0. \tag{2.6}$$

Then there exists $\varepsilon_0 = \varepsilon_0(\alpha, A) > 0,\ r = r(\alpha, A) > 0$ *such that for any* $f_e,\ g_\infty$ *satisfying the above assumptions with* $\varepsilon_2 \leqslant \varepsilon_0$, *there exists a solution* $f(u,v,t)$ *of (2.1) defined for* $x \in [0, 2\pi]\,,\ v \in \mathbb{R}$ *and* $0 \leqslant t < \infty$ *that satisfies*

$$|E(u,t)| \leqslant C\varepsilon_2 e^{-rt},\ |\partial_u E(u,t)| \leqslant Ce^{-rt},\ \lim_{t\to\infty} |f(u,v,t) - f_e(v) - g_\infty(u,v)| = 0. \tag{2.7}$$

Remark 2.1 *The condition that* $\alpha \neq 2$ *in (2.3),(2.5) seems a bit artificial, however this is needed in order to avoid the onset of logarithmic terms that would introduce nonessential technical difficulties (see [6]). Actually one can transfer the* $\alpha = 2$ *case to the* $1 < \alpha < 2$ *case by multiplying a constant* C *to* $\varepsilon_1, \varepsilon_2$, *and this theorem is still true for* $\alpha = 2$.

Corollary 2.2 *There exists a function* $f_0 \in C^1(\mathbb{R}\times\mathbb{R})$ *with* $f_0(u+2\pi, v) = f_0(u,v)$ *such that the corresponding solution of system (2.1) is defined for* $0 < t < \infty$ *and satisfies*

$$|E(u,t)| < C\varepsilon_2 e^{-rt} \qquad 0 \leqslant t < \infty, \tag{2.8}$$

where ε_2 is from (2.5) in Theorem 2.1.

To prove the above theorem, [6] uses the following lemma and proposition, we pose it here because it will also be useful in our proof.

Lemma 2.3 *Suppose that conditions (c) and (d) are satisfied. Then the function*

$$H(u,t) = \int_{-\infty}^{\infty} g_\infty(u - wt, w)dw \tag{2.9}$$

satisfies

$$|H(u,t)| \leqslant C\varepsilon_2 e^{-rt}, \quad u \in [0, 2\pi], \tag{2.10}$$

where $r > 0$ *can be chosen arbitrarily close to* α, $0 < r < \alpha$ *and* $C > 0$ *depends on* α, r, A.

Proposition 2.4 *Suppose that* $f_e(v)$ *satisfies the assumptions of Theorem 2.1 and* $h(u,t) \in C(\mathbb{R}\times\mathbb{R}^+)$ *is a function satisfying condition* (d) *in Theorem 2.1, as well as the estimate:*

$$|h(u,t)| \leqslant Be^{-rt}, \quad u \in \mathbb{R}, \quad t \geqslant 0, \tag{2.11}$$

for some $0 < r < a$. *Then there exists a funtion* $E(u,t) \in C(\mathbb{R} \times \mathbb{R}^+)$ *solving*

$$\partial_u E(u,t) = \int_t^\infty ds \int_{-\infty}^{\infty} \partial_v f_e(w) E(u - wt + ws, s) dw + h(u,t), \tag{2.12}$$

satisfying $\int_0^{2\pi} E(u,t)du = 0$, $E(u + 2\pi, t) = E(u,t)$ *for* $z \in \mathbb{R}$, $t \geqslant 0$ *and*

$$|E(u,t)| + |\partial_u E(u,t)| \leqslant CBe^{-rt} \tag{2.13}$$

with some $C > 0$ *which depends only on* A, α, r.

2.2 The main result

As we said before, the uncertainty comes from f_e and g_∞ constructed in Theorem 2.1. Then fixing a positive integer K, with the following assumption, we will provide an estimate on $\partial_z^K(E)$, which gives regularity of E on the random variable z.

Assumptions:

(i) $f_e(v,z)$ *satisfies (a) and (b) for some constant* ε_1, A, α, *independent of* z.

(ii) $g_\infty(u,v,z), \partial_u g_\infty(u,v,z), \partial_v g_\infty(u,v,z)$ *satisfies (c) and (d) for some constant* ε_2, A, α, *independent of* z.

(iii) All the u, v, z*-derivatives of* $f_e(v,z)$ *satisfy (a) for the same constant* ε_1, A, α *as (i), independent of* z.

(iv) All the u, v, z*-derivatives of* g_∞ *up to order* K, *except those mentioned in (ii), satisfy (c) for constant* C^*, A, α, *which means they are analytic in the same area and satisfy (2.5) with* ε_2 *replaced by a constant* $C*$ *that is not necessarily small as* ε_2 *is, independent of* z.

Theorem 2.5 *Under the aforementioned assumption, there exists* $\varepsilon_0(A, K, \alpha) > 0$ *such that for* $\varepsilon_2 < \varepsilon_0(A, K, \alpha)$, *the solution given by Theorem 2.1 satisfies*

$$||\partial_z^k E||_r \leqslant C, \ 0 \leqslant k \leqslant K \tag{2.14}$$

for all $0 < r < \alpha$ *and* C *depends on* $K, \varepsilon_1, \varepsilon_2, C^*, A, r, \alpha$, *independent of* z.

Corollary 2.6 *Under the aforementioned assumptions and conditions in Theorem 2.5, the solution* $f(u,v,t,z)$ *constructed in Theorem 2.1 satisfies the following estimate:*

$$||\partial_z^k \left[f(u,v,t,z) - f_e(v,z) - g_\infty(u,v,z)\right]||_{L^\infty(x,v)} \leqslant C(1+t)e^{-rt}, \ \forall t \geqslant 0, \ 0 \leqslant k \leqslant K \tag{2.15}$$

for all $0 < r < \alpha$ *and* C *depends on* $K, \varepsilon_1, \varepsilon_2, C^*, A, r, \alpha$, *and is independent of* z.

Remark 2.2 *The above theorem implies if we want similar Landau damping phenomenon on* $\partial_z^k E$, g_∞ *and its first* u, v*-derivative should be small enough, and we only need its higher order* u, v*-derivatives and* z*-derivatives have similar form of boundedness as in (c) with a larger constant* C^* *replacing* ε_2. *Besides, by the Residual Theorem, since* g_∞ *is analytic in* $|\mathrm{Im}(u)| \leqslant A$, $|\mathrm{Im}(v)| \leqslant A$, *the results also imply that its* u, v*-derivatives are small if* g_∞ *is small enough.*

Remark 2.3 *The conditions in Theorem 2.5 are not so restrictive. In fact, from the proof in the following section, one can see that for g_∞, if the u, v, z-derivatives of g_∞ up to order K is L^∞ and L_1 bounded uniformly in z, then we can still have the same conclusion. Since the conditions (2.3),(2.4) in the Theorem 2.1 are stability condition for the linearized problem, we prefer to use condition (2.5) for g_∞, which has a similar form to (2.3).*

3 Preliminaries

In this section, we use the characteristic method to give an exact nonlinear formula of the solution and also use the Faa di Bruno formula to take partial derivatives on both sides of the equations.

For each z, the characteristic equations associated with (2.1) satisfy:

$$\begin{cases} \dfrac{dU(s;u,v,t,z)}{ds} = -sE(U+Vs,s,z) \\ U(t;u,v,t,z) = u \end{cases}, \quad \begin{cases} \dfrac{dV(s;u,v,t,z)}{ds} = E(U+Vs,s,z) \\ V(t;u,v,t,z) = v \end{cases}. \tag{3.1}$$

We define the functions

$$U_\infty(u,v,t,z) = U(\infty;u,v,t,z), \quad V_\infty(u,v,t,z) = V(\infty;u,v,t,z). \tag{3.2}$$

The solution defined in Theorem 2.1 can then be written as

$$\begin{aligned} f(u,v,t,z) &= \lim_{s\to\infty} f(U(s;u,v,t,z), V(s;u,v,t,z), s, z) \\ &= f_e(V_\infty(u,v,t),z) + g_\infty(U_\infty(u,v,t), V_\infty(u,v,t), z), \end{aligned} \tag{3.3}$$

where $u = x - vt$.

Combine with (2.1), we get

$$\begin{aligned} \partial_u E(u,t,z) = &\int_{-\infty}^{\infty} f_e(V_\infty(u-wt,w,t,z),z)dw \\ &+ \int_{-\infty}^{\infty} g_\infty(U_\infty(u-wt,w,t,z), V_\infty(u-wt,w,t,z),z)dw - 1, \end{aligned} \tag{3.4}$$

Taking partial derivatives z and u on both sides of (3.1) and (3.4). Then (3.1) will give:

$$\begin{aligned} &\frac{d}{ds}(\partial_u^n \partial_z^m U(s;u,v,t,z)) = -s\partial_u^n\partial_z^m(E(U+sV,s,z)), \\ &\partial_u^n\partial_z^m U(t;u,v,t,z) = \begin{cases} 1 & n=1, m=0 \\ 0 & otherwise \end{cases}; \\ &\frac{d}{ds}(\partial_u^n \partial_z^m V(s;u,v,t,z)) = \partial_u^n\partial_z^m(E(U+Vs,s,z)), \\ &\partial_u^n\partial_z^m V(t;u,v,t,z) = 0, \end{aligned} \tag{3.5}$$

where by the Faa di Bruno formula,

$$\partial_u^n\partial_z^m(E(U+sV,s,z)) =\partial_u E\partial_u^n\partial_z^m(U+sV)+\partial_u^n\partial_z^m E(\partial_u(U+sV))^n + C_{\alpha,\beta,\delta,\zeta}\sum_{\substack{\alpha+\beta\leqslant n+m,\\ \alpha\geqslant 1, 0\leqslant\beta\leqslant m}}\partial_u^\alpha\partial_z^\beta E\prod_{\substack{\sum_i^\alpha\delta_i=n,\sum_i^\alpha\zeta_i=m-\beta,\\ \delta_i\geqslant 0,\zeta_i\geqslant 0,\delta_i+\zeta_i\geqslant 1}}\partial_u^{\delta_i}\partial_z^{\zeta_i}(U+Vs). \tag{3.6}$$

Furthermore, (3.4) gives:

$$\begin{aligned}\partial_u^{n+1}\partial_z^m E(u,t,z) = \int_{-\infty}^{\infty} & \partial_u^n\partial_z^m(f_e(V_\infty(u-wt,w,t,z),z)) \\ &+\partial_u^n\partial_z^m(g_\infty(U_\infty(u-wt,w,t,z),V_\infty(u-wt,w,t,z),z))dw,\end{aligned} \tag{3.7}$$

where

$$\partial_u^n\partial_z^m(f_e(V_\infty,z)) =\partial_v f_e\partial_u^n\partial_z^m(V_\infty) + C_{\alpha,\beta,\delta,\zeta}\sum_{\substack{\alpha+\beta\leqslant n+m,\\ \alpha\geqslant 1, 0\leqslant\beta\leqslant m}}\partial_v^\alpha\partial_z^\beta f_e\prod_{\substack{\sum_i^\alpha\delta_i=n,\sum_i^\alpha\zeta_i=m-\beta,\\ \delta_i\geqslant 0,\zeta_i\geqslant 0,\delta_i+\zeta_i\geqslant 1}}\partial_u^{\delta_i}\partial_z^{\zeta_i}(V_\infty) \tag{3.8}$$

and

$$\begin{aligned}&\partial_u^n\partial_z^m(g(U_\infty,V_\infty,z)) = \partial_u g_\infty\partial_u^n\partial_z^m(U_\infty)+\partial_v g_\infty\partial_u^n\partial_z^m(V_\infty)\\ &+C_{\alpha,\beta,\delta,\zeta}\sum_{\substack{\alpha\geqslant 0,\beta\geqslant 0,\gamma\geqslant 0\\ M_1+M_2=n\\ N_1+N_2+\gamma=m\\ 1\leqslant\alpha+\beta\leqslant n+m-\gamma,\\ m\geqslant\gamma}}\partial_u^\alpha\partial_v^\beta\partial_z^\gamma g_\infty\prod_{\substack{\sum_i^\alpha\delta_i^1=M_1,\\ \sum_i^\alpha\zeta_i^1=N_1,\\ \delta_i^1\geqslant 0,\\ \zeta_i^1\geqslant 0,\delta_i^1+\zeta_i^1\geqslant 1}}\partial_u^{\delta_i^1}\partial_z^{\zeta_i^1}(U_\infty)\prod_{\substack{\sum_i^\beta\delta_i^2=M_2,\\ \sum_i^\beta\zeta_i^2=N_2,\\ \delta_i^2\geqslant 0,\zeta_i^2\geqslant 0\\ \delta_i^2+\zeta_i^2\geqslant 1}}\partial_u^{\delta_i^2}\partial_z^{\zeta_i^2}(V_\infty).\end{aligned} \tag{3.9}$$

Remark 3.4 *Although equations (3.5)-(3.9) above are complicated, we will see in the next section that only the first or the second term will matter when we do induction.*

4 The higher-order u-derivatives of U, V and E

In this section, we will establish the relation between the higher-order u-derivatives of U, V and E, and also prove by induction that the u-derivatives of E up to K are in $\mathcal{C}_r$, where $0 < r < \alpha$ with α given in (2.3)-(2.6).

4.1 The higher-order u-derivatives of U, V

In this subsection, by assuming that $\|\partial_u^k E\|_r < \infty$ for all z (this will be proved in next subsection), we estimate the higher-order derivatives of U and V in u. Actually, if we didn't take ∂_z derivative on both sides of (3.5), for each fixed z, we can treat them like they are deterministic, therefore we can use the same procedure as (6.15)-(6.19) in [6]. Consider equation (3.5) in the case of $n = 1, m = 0$

$$\begin{aligned}
&\frac{d}{ds}(\partial_u U(s; u, v, t, z) - 1) = -s[\partial_u E(\partial_u U - 1) + s\partial_u E \partial_u V + \partial_u E],\\
&\partial_u U(t; u, v, t, z) - 1 = 0;\\
&\frac{d}{ds}(\partial_u V(s; u, v, t, z)) = [\partial_u E(\partial_u U - 1) + s\partial_u E \partial_u V + \partial_u E],\\
&\partial_u V(t; u, v, t, z) = 0.
\end{aligned} \tag{4.1}$$

If we assume $||\partial_u E||_r < \infty$ for all z, then using the Gronwall type argument (see (6.15)-(6.19) in [6]) gives:

$$\begin{aligned}
&|\partial_u U(s; u, v, t, z) - 1| \leqslant C_1 ||\partial_u E||_r (t+1)e^{-rt},\\
&|\partial_u V(s; u, v, t, z)| \leqslant C_1 ||\partial_u E||_r e^{-rt},
\end{aligned} \tag{4.2}$$

hence

$$\begin{aligned}
&|\partial_u U(\infty; u, v, t, z) - 1| \leqslant C_1 ||\partial_u E||_r (t+1)e^{-rt},\\
&|\partial_u V(\infty; u, v, t, z)| \leqslant C_1 ||\partial_u E||_r e^{-rt}.
\end{aligned} \tag{4.3}$$

Consider the case $n = 2, m = 0$:

$$\begin{aligned}
&\frac{d}{ds}(\partial_u^2 U(s; u, v, t, z)) = -s[\partial_u E(\partial_u^2 U) + s\partial_u E \partial_u^2 V + \partial_u^2 E(\partial_u U + s\partial_u V)^2 + 2s\partial_u^2 E \partial_u U \partial_u V],\\
&\partial_u^2 U(t; u, v, t, z) = 0;\\
&\frac{d}{ds}(\partial_u^2 V(s; u, v, t, z)) = [\partial_u E(\partial_u^2 U) + s\partial_u E \partial_u^2 V + \partial_u^2 E(\partial_u U + s\partial_u V)^2 + 2s\partial_u^2 E \partial_u U \partial_u V],\\
&\partial_u^2 V(t; u, v, t, z) = 0.
\end{aligned} \tag{4.4}$$

If we assume $||\partial_u^2 E||_r < \infty$ for all z, by the similar argument as before we will further have

$$\begin{aligned}
&|\partial_u^2 U(s; u, v, t, z)| \leqslant C_2(||\partial_u^2 E||_r + 1)(t+1)e^{-rt},\\
&|\partial_u^2 V(s; u, v, t, z)| \leqslant C_2(||\partial_u^2 E||_r + 1)e^{-rt}.
\end{aligned} \tag{4.5}$$

Furthermore, if for $1 < n \leqslant k - 1$, we have

$$\begin{aligned}
&|\partial_u^n U(s; u, v, t, z)| \leqslant C_n(||\partial_u^n E||_r + 1)(t+1)e^{-rt},\\
&|\partial_u^n V(s; u, v, t, z)| \leqslant C_n(||\partial_u^n E||_r + 1)e^{-rt}.
\end{aligned} \tag{4.6}$$

Then for $n = k$, using (3.5) and (3.6), we obtain

$$\begin{aligned}
&\frac{d}{ds}(\partial_u^n U(s; u, v, t, z)) = -s(\partial_u E \partial_u^n (U + Vs) + \partial_u^n E(\partial_u (U + Vs))^n + R(s; u, v, t, z)),\\
&\partial_u^n U(t; u, v, t, z) = 0;\\
&\frac{d}{ds}(\partial_u^n V(s; u, v, t, z)) = \partial_u E \partial_u^n (U + Vs) + \partial_u^n E(\partial_u (U + Vs))^n + R(s; u, v, t, z),\\
&\partial_u^n V(t; u, v, t, z) = 0,
\end{aligned} \tag{4.7}$$

where $R(s;u,v,t,z)$ is the remaining part of the expansion.

By induction

$$||R(s;u,v,t,z)||_{L^\infty(u,v)} \leqslant C(t+1)e^{-2rt}$$

for all z and C depending on z,k. Therefore, by similar argument as before, if $||\partial_u^n E||_r < \infty$ for all $0 \leqslant n \leqslant k$, then

$$\begin{aligned} |\partial_u^k U(s;u,v,t,z)| &\leqslant C_k(||\partial_u^k E||_r + 1)(t+1)e^{-rt}, \\ |\partial_u^k V(s;u,v,t,z)| &\leqslant C_k(||\partial_u^k E||_r + 1)e^{-rt}. \end{aligned} \tag{4.8}$$

4.2 The higher-order u-derivatives of E

In this section, we estimate the higher-order u-derivatives of E. Since in (2.7), we have got $|E(u,t,z)| + |\partial_u E(u,t,z)| \leqslant Ce^{-rt}$, we want to estimate $||\partial_u^k E(u,t,z)||_r$ by induction for $k \geqslant 2$. Also from the assumption of Theorem 2.5, one can see the constant C is independent of the random variable z.

For convenience, in this section we use the same symbol C to represent the constant that only depends on $n, r, \alpha, A, \varepsilon_1, \varepsilon_2, C^*$ but might be different in different equations.

Assume $0 < n \leqslant k-1$, $||\partial_u^n E||_r < C$, then by (4.8)

$$\begin{aligned} |\partial_u^n U(s;u,v,t,z)| &\leqslant C(||\partial_u^n E||_r + 1)(t+1)e^{-rt}, \\ |\partial_u^n V(s;u,v,t,z)| &\leqslant C(||\partial_u^n E||_r + 1)e^{-rt} \end{aligned}$$

for all $0 < n \leqslant k-1$.

We use (3.7)-(3.9) in the case of $n = k-1, m = 0$. From (3.7),

$$\begin{aligned} \partial_u^k E(u,t,z) = &\int_{-\infty}^{\infty} \partial_u^{k-1}(f_e(V_\infty(u-wt,w,t,z),z))dw \\ &+ \int_{-\infty}^{\infty} \partial_u^{k-1}(g_\infty(U_\infty(u-wt,w,t,z),V_\infty(u-wt,w,t,z),z))dw. \end{aligned} \tag{4.9}$$

Let

$$I = \int_{-\infty}^{\infty} \partial_u^{k-1}(f_e(V_\infty,z))dw, \ J = \int_{-\infty}^{\infty} \partial_u^{k-1}(g_\infty(U_\infty,V_\infty,z))dw,$$

we then get

$$I = \int_{-\infty}^{\infty} \partial_v f_e(V_\infty)\partial_u^{k-1}(V_\infty) + C_{\alpha,\delta,\zeta} \sum_{1\leqslant\alpha\leqslant k-1} \partial_v^\alpha f_e \prod_{\sum_i^\alpha \delta_i = k-1-\alpha, \delta_i\geqslant 0} \partial_u^{\delta_i+1}(V_\infty)dw. \tag{4.10}$$

Let

$$I_1 = \int_{-\infty}^{\infty} \partial_v f_e(V_\infty)\partial_u^{k-1}(V_\infty)dw,$$

$$I_2 = \int_{-\infty}^{\infty} C_{\alpha,\delta,\zeta} \sum_{1\leqslant\alpha\leqslant k-1} \partial_v^\alpha f_e \prod_{\sum_i^\alpha \delta_i = k-1-\alpha, \delta_i\geqslant 0} \partial_u^{\delta_i+1}(V_\infty)dw.$$

First, we estimate I_1. Expanding $\partial_v f_e(V_\infty(u,w,z,t))$ by Taylor series expansion of $\partial_v f(v)$ at point $v=w$,

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} (\partial_v f_e(V_\infty) - \partial_v f_e(w))\partial_u^{k-1}(V_\infty) dw + \int_{-\infty}^{\infty} \partial_v f_e(w)\partial_u^{k-1}(V_\infty) dw \\
&= \int_{-\infty}^{\infty} \partial_v^2 f_e(w)(V_\infty - w)\partial_u^{k-1}(V_\infty) dw + \int_{-\infty}^{\infty} \partial_v f_e(w)\partial_u^{k-1}(V_\infty) dw + \int_{-\infty}^{\infty} R(w,t) dw,
\end{aligned}
\tag{4.11}
$$

where $R(w,t)$ is the Lagrange remainder term of the Taylor series expansion.

Besides, integrating (**??**) on both sides, we can easily get

$$
\begin{aligned}
&|U_\infty(u,w,t,z) - u| \leqslant C||E||_r(t+1)e^{-rt}, \\
&|V_\infty(u,w,t,z) - w| \leqslant C||E||_r e^{-rt},
\end{aligned}
\tag{4.12}
$$

where $||\cdot||_r$ is from (2.2).

One can see that three parts of the (4.11) RHS can be respectively bounded by,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |\partial_v^2 f_e(w)(V_\infty - w)\partial_u^{k-1}(V_\infty)| dw \leqslant C\varepsilon_1 ||E||_r ||\partial_u^{k-1} E||_r e^{-2rt}, \\
&\int_{-\infty}^{\infty} |\partial_v f_e(w)\partial_u^{k-1}(V_\infty)| dw \leqslant C\varepsilon_1(||\partial_u^{k-1}E||_r + 1)e^{-rt}, \\
&\int_{-\infty}^{\infty} |R(w,t)| dw \leqslant C\varepsilon_1 e^{-2rt},
\end{aligned}
\tag{4.13}
$$

where the three inequalities is because of the assumption of Theorem 2.5.

Now consider I_2. By (4.8) one can see all terms in I_2 have at least multiple of two $\partial_z^l V_\infty$ terms with $l > 0$, therefore, it is obvious

$$
|I_2| \leqslant \int_{-\infty}^{\infty} |C_{\alpha,\delta,\zeta} \sum_{1\leqslant \alpha \leqslant k-1} \partial_v^\alpha f_e \prod_{\sum_i^\alpha \delta_i = k-1-\alpha, \delta_i \geqslant 0} \partial_z^{\delta_i+1}(V_\infty)| dw \leqslant C\varepsilon_1 e^{-2rt}. \tag{4.14}
$$

Combine I_1 and I_2, and by the induction condition, we can obtain

$$
|I| \leqslant |I_1| + |I_2| \leqslant C\varepsilon_1(||\partial_u^{k-1}E||_r + 1)e^{-rt}. \tag{4.15}
$$

The estimation on J will be a little more complicated than the estimation of I. Note

$$
\begin{aligned}
J &= \int_{-\infty}^{\infty} \partial_u^{k-1}(g_\infty(U_\infty(u-wt,w,t,z), V_\infty(u-wt,w,t,z),z)) dw \\
&= \int_{-\infty}^{\infty} \partial_u g_\infty \partial_u^{k-1}(U_\infty) dw + \int_{-\infty}^{\infty} \partial_v g_\infty \partial_u^{k-1}(V_\infty) dw + R_1(w,t) dw \\
&= J_1 + J_2 + J_3,
\end{aligned}
\tag{4.16}
$$

where

$$
J_1 = \int_{-\infty}^{\infty} \partial_u g_\infty \partial_u^{k-1}(U_\infty) dw, \;\; J_2 = \int_{-\infty}^{\infty} \partial_v g_\infty \partial_u^{k-1}(V_\infty) dw, \;\; J_3 = \int_{-\infty}^{\infty} R_1(w,t) dw.
$$

By the similar method from (4.10)-(4.15), we can get

$$|J_2| \leqslant C\varepsilon_2(||\partial_u^{k-1}E||_r + 1)e^{-rt}. \tag{4.17}$$

Then one can use (4.7) to have a good estimate of J_1.

$$J_1 = \int_{-\infty}^{\infty}(\partial_u g_\infty(U_\infty, V_\infty) - \partial_u g_\infty(u-wt, w))\partial_u^{k-1}(U_\infty)dw + \int_{-\infty}^{\infty}\partial_u g_\infty(u-wt, w)\partial_u^{k-1}(U_\infty)dw. \tag{4.18}$$

Expand $\partial_u g_\infty(U_\infty, V_\infty)$ by the Taylor series expansion like the method dealing with $\partial_v f(V_\infty)$, we can show that the first part of J_1 is very small

$$\int_{-\infty}^{\infty}|(\partial_u g_\infty(U_\infty, V_\infty) - \partial_u g_\infty(u-wt, w))\partial_u^{k-1}(U_\infty)|dw \sim O((t+1)e^{-2rt}). \tag{4.19}$$

The remaining part, by equation (4.7),

$$\begin{aligned}
&\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt, w)\partial_u^{k-1}U_\infty dw \\
&= -\int_t^\infty s\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt, w)[\partial_u^{k-1}E(U+sV)(\partial_u(U+sV))^{k-1}]dwds + O(e^{-2rt}) \\
&= -\int_t^\infty s\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt, w)[\partial_u^{k-1}E(U+sV)]dwds + O(e^{-2rt}) \\
&= -\int_t^\infty s\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt, w)[\partial_u^{k-1}E(u-wt+ws, s)]dwds + O(||\partial_u^k E||_r||E||_r e^{-2rt}).
\end{aligned} \tag{4.20}$$

To deal with term J_4,

$$J_4 = -\int_t^\infty s\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt, w)[\partial_u^{k-1}E(u-wt+ws, s)]dwds,$$

notice that

$$\frac{Dg_\infty}{Dw}(u-wt, w) = -t\partial_u g_\infty(u-wt, w) + \partial_v g_\infty(u-wt, w). \tag{4.21}$$

Then

$$\begin{aligned}
&\int_t^\infty\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt, w)[-s\partial_u^{k-1}E(u-wt+ws, s)]dwds \\
&= \int_t^\infty\int_{-\infty}^{\infty}(-s\partial_u g_\infty(u-wt, w) + \partial_v g_\infty(u-wt, w))[\partial_u^{k-1}E(u-wt+ws, s)]dwds \\
&\quad - \int_t^\infty\int_{-\infty}^{\infty}\partial_v g_\infty(u-wt, w)[\partial_u^{k-1}E(u-wt+ws, s)]dwds.
\end{aligned} \tag{4.22}$$

Plugging (4.21) in and using integration by parts on the first part of (4.22) give

$$\begin{aligned}
&\int_t^\infty\int_{-\infty}^{\infty}\left[\frac{s}{t}\frac{Dg_\infty}{Dw}(u-wt, w) + (1-\frac{s}{t})\partial_v g_\infty(u-wt, w)\right]\left[\partial_u^{k-1}E(u-wt+ws, s)\right]dwds \\
&= \int_t^\infty(s-t)\frac{s}{t}\int_{-\infty}^{\infty}g_\infty(u-wt, w)\partial_u^k E(u-wt+ws, s)dwds \\
&\quad + \int_t^\infty\int_{-\infty}^{\infty}(1-\frac{s}{t})\partial_v g_\infty(u-wt, w)[\partial_u^{k-1}E(u-wt+ws, s)]dwds.
\end{aligned} \tag{4.23}$$

If $||\partial_u^k E||_r < \infty$, a crucial estimate is the following:

$$\begin{aligned}
&\left|\int_t^\infty (s-t)\frac{s}{t}\int_{-\infty}^\infty g_\infty(u-wt,w)\partial_z^k E(u-wt+ws,s)dwds\right| \\
&\leqslant C\varepsilon_2||\partial_u^k E||_r e^{-rt}\int_t^\infty (s-t)\frac{s}{t}e^{-r(s-t)}ds \leqslant C\varepsilon_2||\partial_u^k E||_r e^{-rt}.
\end{aligned} \tag{4.24}$$

By the same estimation on the second part of (4.23), then we can conclude the estimate of (4.21)

$$\begin{aligned}
&\int_t^\infty\int_{-\infty}^\infty |\partial_u g_\infty(u-wt,w)[-s\partial_u^{k-1}E(u-wt+ws,s)]|dwds \\
\leqslant &\int_t^\infty\int_{-\infty}^\infty |(-s\partial_u g_\infty(u-wt,w)+\partial_v g_\infty(u-wt,w))[\partial_u^{k-1}E(u-wt+ws,s)]|dwds \\
&+\int_t^\infty\int_{-\infty}^\infty |\partial_v g_\infty(u-wt,w)[\partial_u^{k-1}E(u-wt+ws,s)]|dwds \\
\leqslant &C\varepsilon_2(||\partial_u^k E||_r e^{-rt}+||\partial_u^{k-1}E||_r)e^{-rt}.
\end{aligned} \tag{4.25}$$

Remark 4.5 *The method used in (4.18)-(4.25) is crucial in this paper, as will be seen again in the sequel. If one directly puts the estimate of $\partial_u^k U_\infty$ into the equation, one will only get $O((t+1)e^{-rt})$, but if we put the characteristic function in it and omit other small terms and analyze J_1 carefully, we can get a better estimate.*

To deal with the last term J_3, one only needs to consider two special terms, because $\partial_u U_\infty \sim O(1)$ and if one directly applies (4.8), one will only get $O((t+1)e^{-rt})$ and other remaining term must be of $O(e^{-rt})$ because they will have $\partial_v^l V_\infty$ or $\partial_u^l U_\infty \partial_u^m U_\infty (l,m\geqslant 2)$ components.

By (3.9), (4.16), J_3 can be split into following three parts:

$$\begin{aligned}
J_3 = &\int_{-\infty}^\infty \partial_u^{k-1}g_\infty(\partial_u U_\infty)^{k-1}dw + \int_{-\infty}^\infty \sum_{m+l=k} C(l,m)\partial_u^l g_\infty(\partial_u U_\infty)^{l-1}\partial_u^m U_\infty dw \\
&+\int_{-\infty}^\infty R_2(w,t)dw(\leqslant Ce^{-rt}),
\end{aligned} \tag{4.26}$$

where

$$R_2(w,t) = R_1(w,t) - \partial_u^{k-1}g_\infty(\partial_u U_\infty)^{k-1} - \sum_{m+l=k} C(l,m)\partial_u^l g_\infty(\partial_u U_\infty)^{l-1}\partial_u^m U_\infty,$$

which can be well bounded as given in the second line of (4.26).

For the first term, using a similar method as in (4.18)-(4.25),

$$\begin{aligned}
&\left|\int_{-\infty}^\infty \partial_u^{k-1}g_\infty(\partial_u U_\infty)^{k-1}dw\right| = \left|\int_{-\infty}^\infty \partial_u^{k-1}g_\infty(\partial_u U_\infty - 1 + 1)^{k-1}dw\right| \\
&\leqslant C\left|\int_{-\infty}^\infty \partial_u^{k-1}g_\infty(\partial_u U_\infty - 1)dw\right| + C\left|\int_{-\infty}^\infty \partial_u^{k-1}g_\infty dw\right| + O((t+1)^{k-1}e^{-2rt}) \\
&\leqslant Ce^{-rt}.
\end{aligned} \tag{4.27}$$

As for the second term, for each l,m, using a similar method as in (4.18)-(4.27),

$$\begin{aligned}
&|\int_{-\infty}^{\infty}\partial_u^l g_\infty(\partial_u U_\infty-1+1)^{l-1}\partial_u^m U_\infty dw|\\
&\leqslant |\int_{-\infty}^{\infty}\partial_u^l g_\infty\partial_u^m U_\infty dw|+\int_{-\infty}^{\infty}R_1(w,t)dt(\leqslant Ce^{-rt})\\
&\leqslant Ce^{-rt}.
\end{aligned}\tag{4.28}$$

where $R_1(w,t)$ is the remainder of the expansion that can be bounded as given by the second inequality of (4.28).

Then combining (4.16)-(4.28) yields

$$|J|\leqslant C\varepsilon_2||\partial_u^k E||_r e^{-rt}+Ce^{-rt}.\tag{4.29}$$

In conclusion, combine all these together, one has

$$|\partial_u^k E|\leqslant |I|+|J|\leqslant C\varepsilon_2||\partial_u^k E||e^{-rt}+C(\varepsilon_1+\varepsilon_2+1)r^{-rt}.\tag{4.30}$$

Therefore by choosing ε_2 small enough($C\varepsilon_2<1$), we get

$$||\partial_u^k E||_r\leqslant C.\tag{4.31}$$

Remark 4.6 *Explicitly, the constant C in (4.31) will become large when k becomes large, but it can be seen in the following section that one only needs to consider the case $k\leqslant K$, which means we only have finite number (at most K) of constant C, therefore, we can always let ε_2 small enough to make $C\varepsilon_2<1$. This is also the reason why we need to choose ϵ_2 depending on K in Theorem 2.5.*

Remark 4.7 *In the above proof, we need to assume that $||\partial_u^k E||_r<\infty$ for each k, then we can obtain a uniform bound on it. Actually if we construct E as in [6], we can find that condition $||\partial_u^k E||_r<\infty$ is automatically satisfied, because we can use similar contraction map to prove $\partial_u^k E$ exists and $||\partial_u^k E||_r<\infty$. However, this analysis is too tedious and is not our main focus here, so we omit this proof.*

5 The mixed u,z-derivatives of U,V and E

In this section, we will use some previous results and the induction method to estimate the mixed u,z-derivatives of U,V and E and prove Theorem 2.5 and Corollary 2.6. Since the main estimate idea is obtained similarly as former section, we will omit some repeating arguments. For convenience all the constant C only depends on $n,m,r,\alpha,A,\varepsilon_1,\varepsilon_2,C^*$, independent of z, and might be different in different equations.

5.1 The first order z-derivatives of U, V and E

In this section, we will consider the first order z-derivatives, and prove their bounds that will be used for the next induction step.

Consider equation (3.5) in the case of $n = 0, m = 1$,

$$\begin{aligned} \frac{d\partial_z U}{ds} &= -s[\partial_u E(\partial_z U + s\partial_z V) + \partial_z E], \qquad \partial_z U(t; u, v, t, z) = 0, \\ \frac{d\partial_z V}{ds} &= \partial_u E(\partial_z U + s\partial_z V) + \partial_z E, \qquad \partial_z V(t; u, v, t, z) = 0. \end{aligned} \tag{5.1}$$

Using the Gronwall type argument (see (6.15)-(6.19) in [6]), one can obtain:

$$\begin{aligned} |\partial_z U(s; u, v, t, z)| &\leqslant C(||\partial_z E||_r + 1)(t+1)e^{-rt}, \\ |\partial_z V(s; u, v, t, z)| &\leqslant C(||\partial_z E||_r + 1)e^{-rt}; \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} |\partial_z U_\infty(u, v, t, z)| &\leqslant C(||\partial_z E||_r + 1)(t+1)e^{-rt}, \\ |\partial_z V_\infty(u, v, t, z)| &\leqslant C(||\partial_z E||_r + 1)e^{-rt}. \end{aligned} \tag{5.3}$$

Then if we consider equation (3.6) in the case of $n = 0, m = 1$,

$$\begin{aligned} \partial_z \partial_u E = &\int_{-\infty}^{\infty} \partial_v f_e(V_\infty, z)\partial_z V_\infty + \partial_z f_e(V_\infty, z)dw \\ &+ \int_{-\infty}^{\infty} \partial_u g_\infty(U_\infty, V_\infty, z)\partial_z U_\infty + \partial_v g_\infty(U_\infty, V_\infty, z)\partial_z V_\infty + \partial_z g_\infty(U_\infty, V_\infty, z)dw. \end{aligned} \tag{5.4}$$

Define

$$I = \int_{-\infty}^{\infty} \partial_v f_e \partial_z V_\infty + \partial_z f_e dw, \;\; J = \int_{-\infty}^{\infty} \partial_u g_\infty \partial_z U_\infty + \partial_v g_\infty \partial_z V_\infty + \partial_z g_\infty dw.$$

We first deal with I. We transform I by adding and subtracting some terms.

$$\begin{aligned} I = &\int_t^\infty \int_{-\infty}^{\infty} \partial_v f_e(w, z)\partial_z E(u - wt + ws, s)dwds (= I_1) \\ &+ \int_{-\infty}^{\infty} (\partial_v f_e(V_\infty, z) - \partial_v f_e(w))\partial_z V_\infty dw (= I_2) \\ &+ \int_{-\infty}^{\infty} \partial_z f_e(V_\infty, z) - \partial_z f_e(w)dw (= I_3) \\ &+ \int_{-\infty}^{\infty} \partial_v f_e(w, z)(\partial_z V_\infty - \partial_z w)dw - \int_t^\infty \int_{-\infty}^{\infty} \partial_v f_e(w, z)\partial_z E(u - wt + ws, s)dwds (= I_4). \end{aligned} \tag{5.5}$$

By the Taylor series expansion of f_e and (4.12), one can see obviously that the second and third terms have the estimate

$$|I_2| + |I_3| \leqslant C\varepsilon_1(||E||_r + 1)e^{-rt} + O(||\partial_z E||_r ||E||_r e^{-2rt}). \tag{5.6}$$

For I_4, similarly as before,

$$\begin{aligned}|I_4| \leqslant &\int_t^\infty \int_{-\infty}^\infty |\partial_v f_e(w)\partial_u E(U+Vs,s,z)[\partial_z U + s\partial_z V]|dwds\\ &+\int_t^\infty \int_{-\infty}^\infty |\partial_v f_e(w)[\partial_z E(U+Vs,s,z) - \partial_z E(u-wt+ws,s)]|dwds\\ \leqslant & C\varepsilon_1(t+1)(||\partial_z E||_r||\partial_q E||_r + ||\partial_u\partial_z E||_r||E||_r+1)e^{-2rt}.\end{aligned}\tag{5.7}$$

Therefore, combine (5.5)-(5.7), we obtain

$$|I_2|+|I_3|+|I_4| \leqslant C\varepsilon_1(||E||_r+1)e^{-rt} + O((t+1)e^{-2rt}).\tag{5.8}$$

Now we deal with J term by term. Let

$$\begin{aligned}J=&\int_{-\infty}^\infty \partial_u g_\infty(U_\infty,V_\infty,z)\partial_z U_\infty dw (= J_1)\\ &+\int_{-\infty}^\infty \partial_v g_\infty(U_\infty,V_\infty,z)\partial_z V_\infty dw (= J_2)\\ &+\int_{-\infty}^\infty \partial_z g_\infty(U_\infty,V_\infty,z) dw (= J_3).\end{aligned}\tag{5.9}$$

Using the Taylor expansion of g_∞, gives

$$\begin{aligned}J_2 =& \int_{-\infty}^\infty [\partial_v g_\infty(U_\infty,V_\infty,z) - \partial_v g_\infty(u-wt,w,z)]\partial_z(V_\infty)dw\\ &+\int_{-\infty}^\infty \partial_v g_\infty(u-wt,w,z)\partial_z(V_\infty)dw\\ =&\int_{-\infty}^\infty [\partial_u\partial_z g_\infty(u-wt,w,z)(U_\infty - u+wt) + \partial_v^2 g_\infty(u-wt,w,z)(V_\infty - w)]\partial_z(V_\infty)dw\\ &+\int_{-\infty}^\infty \partial_v g_\infty(u-wt,w,z)\partial_z(V_\infty)dw + O((t+1)||\partial_z E||_r||E||_r e^{-3rt}),\end{aligned}\tag{5.10}$$

where the first term is obvious of $O((t+1)||\partial_z E||_r||E||_r e^{-2rt})$, therefore one gets

$$|J_2| \leqslant C\varepsilon_2(||\partial_z E||_r + 1)e^{-rt}.\tag{5.11}$$

Similarly using the Taylor series expansion

$$\begin{aligned}J_3 =& \int_{-\infty}^\infty [\partial_z g_\infty(U_\infty,V_\infty,z) - \partial_z g_\infty(u-wt,w,z)]dwds\\ &+\int_{-\infty}^\infty \partial_z g_\infty(u-wt,w,z)dw\\ =&\int_{-\infty}^\infty \partial_u\partial_z g_\infty(u-wt,w,z)(U_\infty - u+wt)dw + \int_{-\infty}^\infty \partial_v\partial_z g_\infty(u-wt,w,z)(V_\infty - w)dw\\ &+\int_{-\infty}^\infty \partial_z g_\infty(u-wt,w,z)dw + O((t+1)||E||_r e^{-2rt}).\end{aligned}\tag{5.12}$$

The second and third terms can be bounded by $C\varepsilon_2||E||_r e^{-rt}$ and $C\varepsilon_2 e^{-rt}$ respectively, but for the first term one needs to deal with it more carefully,

$$\begin{aligned}&\int_{-\infty}^{\infty} \partial_u\partial_z g_\infty(u-wt,w,z)(U_\infty - u + wt)dw\\&= \int_t^{\infty} -sds\int_{-\infty}^{\infty}[\partial_u\partial_z g_\infty(u-wt,w,z)E(U+Vs,s,z)]dw.\end{aligned}\tag{5.13}$$

Using the same method as in (4.18) to (4.25)

$$\left|\int_{-\infty}^{\infty} \partial_u\partial_z g_\infty(u-wt,w,z)(U_\infty - u + wt)dw\right| \leqslant C(||\partial_u E||_r + ||E||_r + 1)e^{-rt},\tag{5.14}$$

Now, combining (5.12)-(5.14), one has

$$|J_3| \leqslant C(||\partial_u E||_r + ||E||_r + 1)e^{-rt}.\tag{5.15}$$

Considering J_1, we transform I_1 by adding and subtracting some terms,

$$\begin{aligned}J_1 =& \int_{-\infty}^{\infty}[\partial_u g_\infty(U_\infty, V_\infty, z) - \partial_u g_\infty(u-wt,w)]\partial_z U_\infty dw\\&+\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt,w,z)\partial_z U_\infty dw\\=&\int_{-\infty}^{\infty}\partial_u^2 g_\infty(u-wt,w,z)(U_\infty-u+wt)\partial_z U_\infty dw\\&+\int_{-\infty}^{\infty}\partial_u\partial_v g_\infty(u-wt,w,z)(V_\infty - w)\partial_z U_\infty dw\\&+\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt,w,z)\partial_z U_\infty dw + O((t+1)^2e^{-3rt}).\end{aligned}\tag{5.16}$$

Then the first and second terms can be estimated directly by assumptions of Theorem 2.5. The third term can also be easily estimated by the same method in (4.18)-(4.25)

$$\begin{aligned}&\int_t^{\infty}\int_{-\infty}^{\infty}\left|\partial_u^2 g_\infty(u-wt,w,z)(U_\infty-u+wt)\partial_z U_\infty\right| dw\\&\leqslant C(t+1)||\partial_z E||_r||E||_r e^{-2rt},\\&\int_t^{\infty}\int_{-\infty}^{\infty}\left|\partial_u\partial_v g_\infty(u-wt,w,z)(V_\infty-w)\partial_z U_\infty\right| dw\\&\leqslant C||\partial_z E||_r||E||_r e^{-2rt},\\&\left|\int_{-\infty}^{\infty}\partial_u g_\infty(u-wt,w,z)\partial_z U_\infty dw\right|\\&\leqslant C\varepsilon_2(||\partial_u\partial_z E||_r + ||\partial_z E||_r + 1)e^{-rt}.\end{aligned}\tag{5.17}$$

Therefore, combining (5.10)-(5.17) yields

$$|J_1| + |J_2| + |J_3| \leqslant C\varepsilon_2(||\partial_u\partial_z E||_r + ||\partial_z E||_r + 1)e^{-rt} + C(||\partial_u E||_r + ||E||_r + 1)e^{-rt}.\tag{5.18}$$

Finally, combining (5.8) and (5.18), we obtain

$$|I_2| + |I_3| + |I_4| + |J| \leqslant C(\varepsilon_1 + \varepsilon_2 + 1)e^{-rt} + C\varepsilon_2(||\partial_u \partial_z E||_r + ||\partial_z E||_r)e^{-rt}. \tag{5.19}$$

Then by Proposition 2.4, if $||\partial_u \partial_z E||_r + ||\partial_z E||_r < \infty$, then from section 4.2, $||\partial_u E||_r + ||E||_r < C$, thus

$$|\partial_z E(u,t,z)| + |\partial_u \partial_z E(u,t,z)| \leqslant C(\varepsilon_1 + \varepsilon_2 + 1)e^{-rt} + C\varepsilon_2(||\partial_u \partial_z E||_r + ||\partial_u E||_r)e^{-rt}. \tag{5.20}$$

Then by choosing ε_2 small enough ($C\varepsilon_2 < 1$), we can get

$$||\partial_u \partial_z E||_r + ||\partial_z E||_r \leqslant C. \tag{5.21}$$

Remark 5.8 *In the above proof, under the assumption that $||\partial_u \partial_z E||_r + ||\partial_z E||_r < \infty$, we obtain a uniform bound on this inequality. Actually similar to Remark 4.7, if one constructs $\partial_z E$ as in [6], then this assumption is automatically satisfied, because one can use a similar contraction map to prove $\partial_z E$ exists and $||\partial_z \partial_u E||_r + ||\partial_z E||_r < \infty$. More specifically, one can first prove that E can be a $z-$Lipschitz continuous function with Lipschitz constant decaying as e^{-rt}, then one can get that its partial derivative of z is not large and then use the contraction map to prove it exists. Since the detailed analysis is too tedious and not our main focus, we omit this proof.*

5.2 The higher-order mixed u,z-derivatives of U, V

In this section, we consider the higher order mixed u,z-derivatives of U, V and find their relations to the higher order u,z-derivatives of E by using the induction method.

First consider (3.5) for $m > 1$.

$$\begin{aligned}
&\frac{d}{ds}(\partial_u^n \partial_z^m U(s;u,v,t,z)) = -s\partial_u^n \partial_z^m (E(U+Vs,s,z)),\\
&\partial_u^n \partial_z^m Z(t;u,v,t,z) = 0;\\
&\frac{d}{ds}(\partial_u^n \partial_z^m V(s;u,v,t,z)) = \partial_u^n \partial_z^m (E(U+Vs,s,z)),\\
&\partial_u^n \partial_z^m V(t;u,v,t,z) = 0.
\end{aligned} \tag{5.22}$$

If we assume that

$$||\partial_u^\alpha \partial_z^\beta E||_r < C, \ \forall \alpha \leqslant n, \beta \leqslant m, (\alpha+\beta) < n+m,$$

and

$$|\partial_u^\alpha \partial_z^\beta (U+Vs)| \leqslant C(||\partial_u^\alpha \partial_z^\beta E||_r + 1)e^{-rt} < \infty,$$

then we can simplify (3.6) as

$$\partial_u^n \partial_z^m (E(U+Vs,s,z)) = \partial_u E \partial_u^n \partial_z^m (U+Vs) + \partial_u^n \partial_z^m E(\partial_z(U+Vs))^n + O((t+1)e^{-2rt}). \tag{5.23}$$

Plug this into (5.22), one can get

$$\begin{aligned} &|\partial_u^n \partial_z^m U(s;u,v,z,t)| \leqslant C(||\partial_u^n \partial_z^m E||_r + 1)(t+1)e^{-rt}, \\ &|\partial_u^n \partial_z^m V(s;u,v,z,t)| \leqslant C(||\partial_u^n \partial_z^m E||_r + 1)e^{-rt}. \end{aligned} \tag{5.24}$$

We have proved (5.24) for the case $m = 0$ and $n = 0$ *or* $1, m = 1$. Therefore, by induction one can get for all $n \geqslant 0$ and $m \geqslant 0$, if $||\partial_u^\alpha \partial_z^\beta E||_r + 1)e^{-rt} < \infty$ for $\alpha \leqslant n, \beta \leqslant m$, one has

$$\begin{aligned} &|\partial_u^n \partial_z^m Z| \leqslant C(||\partial_u^n \partial_z^m E||_r + 1)(t+1)e^{-rt}, \\ &|\partial_u^n \partial_z^m V| \leqslant C(||\partial_u^n \partial_z^m E||_r + 1)e^{-rt}. \end{aligned} \tag{5.25}$$

5.3 Higher-order mixed u,z-derivatives of E and proof of Theorem 2.5

Now all tools have been prepared, we then need two inductions in two different directions to finish our proof.

Firstly, we need to induct on ∂_u^n with fixed ∂_z^m. Because of (2.14), (4.30), (5.21), we can assume

$$||\partial_u^\alpha \partial_z^\beta E||_r < C, \ \forall 1 \leqslant \alpha \leqslant n, \beta \leqslant m.$$

One can see (4.30),(5.21) imply that this assumption is true for $n = 1, m = 1$.

By equation (3.7),

$$\begin{aligned} \partial_u^{n+1} \partial_z^m (E(u,t,q)) = \int_{-\infty}^{\infty} & \partial_u^n \partial_z^m (f_e(V_\infty(u-wt,w,t,z),z)) \\ & + \partial_u^n \partial_z^m (g_\infty(U_\infty(u-wt,w,t,z), V_\infty(u-wt,w,t,z),z))dw. \end{aligned} \tag{5.26}$$

By equation (3.8),

$$I = \int_{-\infty}^{\infty} \partial_u^n \partial_z^m (f_e(V_\infty(u-wt,w,t,z),z))dw = I_1 + I_2, \tag{5.27}$$

where

$$I_1 = \int_{-\infty}^{\infty} \partial_v f_e \partial_u^n \partial_z^m (V_\infty) dw,$$

$$I_2 = \int_{-\infty}^{\infty} C_{\alpha,\beta,\delta,\zeta} \sum_{\substack{\alpha+\beta \leqslant n+m, \\ \alpha \geqslant 1, 0 \leqslant \beta \leqslant m}} \partial_v^\alpha \partial_z^\beta f_e \prod_{\substack{\sum_i^\alpha \delta_i = n, \sum_i^\alpha \zeta_i = m-\beta, \\ \delta_i \geqslant 0, \zeta_i \geqslant 0, \delta_i + \zeta_i \geqslant 1}} \partial_u^{\delta_i} \partial_z^{\zeta_i} (V_\infty) dw.$$

Then by using the similar method as in (4.11),

$$|I_1| \leqslant \int_{-\infty}^{\infty} |\partial_v f_e(w) \partial_u^n \partial_z^m (V_\infty)| dw \leqslant C\varepsilon_1(||\partial_u^n \partial_z^m E||_r + 1)e^{-rt}. \tag{5.28}$$

Since I_2 contains one or more components of $\partial_u^l\partial_z^m V_\infty$ for $1\leqslant l\leqslant n, q\leqslant m, l+q<m$, therefore, one can get

$$\begin{aligned}|I_2| &\leqslant \int_{-\infty}^{\infty} C_{\alpha,\beta,\delta,\zeta} \sum_{\substack{\alpha+\beta\leqslant n+m,\\ \alpha\geqslant 1, 0\leqslant\beta\leqslant m}} |\partial_v^\alpha\partial_z^\beta f_e| \prod_{\substack{\sum_i^\alpha \delta_i=n, \sum_i^\alpha \zeta_i=m-\beta,\\ \delta_i\geqslant 0,\zeta_i\geqslant 0,\delta_i+\zeta_i\geqslant 1}} |\partial_u^{\delta_i}\partial_z^{\zeta_i}(V_\infty)|dw \\ &\leqslant C\varepsilon_1 e^{-rt}.\end{aligned} \tag{5.29}$$

From (5.28) and (5.29), one obtains

$$|I|\leqslant |I_1|+|I_2|\leqslant C\varepsilon_1 e^{-rt}+C\varepsilon_1||\partial_u^n\partial_z^m E||_r e^{-rt}. \tag{5.30}$$

Now we estimate the second term in (5.26),

$$\begin{aligned}&\int_{-\infty}^{\infty}\partial_u^n\partial_z^m(g_\infty(U_\infty(u-wt,w,t,z),V_\infty(u-wt,w,t,z),z))dw \\ &=\int_{-\infty}^{\infty}\partial_u g_\infty\partial_u^n\partial_z^m(U_\infty)dw(=J_1)+\int_{-\infty}^{\infty}\partial_v g_\infty\partial_u^n\partial_z^m(V_\infty)dw(=J_2)+\int_{-\infty}^{\infty}R(w,t)dw(=J_3).\end{aligned} \tag{5.31}$$

To deal with J_1 and J_2, we use the same method from (4.16) to (4.25) to get

$$|J_1|+|J_2|\leqslant C\varepsilon_2(||\partial_u^{n+1}\partial_z^m E||_r+||\partial_u^n\partial_z^m E||_r+1)e^{-rt}. \tag{5.32}$$

Note J_3 has the following form

$$J_3=\int_{-\infty}^{\infty}\sum_{l<m}C(l)\partial_u^n\partial_z^{m-l}(\partial_z^l g_\infty)dw+\int_{-\infty}^{\infty}R_1(w,t)dw, \tag{5.33}$$

where

$$R_1(w,t)=R(w,t)-\sum_{l<m}C(l)\partial_u^n\partial_z^{m-l}(\partial_z^l g_\infty).$$

For the first term, one can treat $\partial_z^l g_\infty$ as new g_∞ and go through the same process like in (5.31)-(5.32). For $R_1(w,t)$, it won't make any difference to the case without q, so one can use exactly the same method as in (4.26)-(4.28) to get

$$|J_3|\leqslant Ce^{-rt}. \tag{5.34}$$

Therefore, by assuming $||\partial_u^{n+1}\partial_z^m E||_r<\infty$ and letting ε_2 be small enough ($C\varepsilon_2<1$), we can get

$$||\partial_u^{n+1}\partial_z^m E||_r\leqslant C. \tag{5.35}$$

The first induction suggests that if one has proved $||\partial_z^m E||_r<C$, then one gets a uniform bound for $||\partial_u^n\partial_z^m E||_r<C_N$ for all $n\leqslant N$, where C_N depends on C, N and N is an arbitrary fixed positive integer.

Nextly, we do induction in another direction to prove Theorem 2.5. We can assume

$$||\partial_u^\alpha \partial_z^\beta E||_r < C, \ \forall 0 \leqslant \alpha \leqslant m, 0 \leqslant \beta < m, (\alpha+\beta) \leqslant m.$$

One can see that this assumption is true for $m = 2$, and if one proves $||\partial_z^M E||_r < C$, then this assumption is obviously true for $m = M+1$ by the first induction.

Again using (3.7) for $n = 0$,

$$\begin{aligned}\partial_u \partial_z^m E(u,t,z) = & \int_{-\infty}^{\infty} \partial_z^m f_e(V_\infty(u-wt,w,t,z),z)dw (= I) \\ & + \int_{-\infty}^{\infty} \partial_z^m g_\infty(U_\infty(u-wt,w,t,z), V_\infty(u-wt,w,t,z),z)dw (= J).\end{aligned} \tag{5.36}$$

First we deal with I,

$$I = \int_{-\infty}^{\infty} \partial_v f_e \partial_z^m (V_\infty) dw + \int_{-\infty}^{\infty} R_2(w,t)dw. \tag{5.37}$$

Using the same method in (5.27)-(5.29), we can get

$$\int_{-\infty}^{\infty} |R_2(w,t)|dw \leqslant C\varepsilon_1 e^{-rt}. \tag{5.38}$$

The first term can be similarly reformulated using (5.4)-(5.8),

$$\int_{-\infty}^{\infty} \partial_v f_e \partial_z^m (V_\infty) dw = \int_t^{\infty} \int_{-\infty}^{\infty} \partial_v f_e(w,z) \partial_z^m E(u-wt+ws,s,z)dw + R_3 (\leqslant C\varepsilon_1 e^{-rt}). \tag{5.39}$$

As for J, in (5.9), we have dealt with $\partial_z(g_\infty(U_\infty, V_\infty))$, then if we split it according to the number of z-derivatives on g_∞, it will have the same form as (5.9), therefore by the induction condition and (5.9)-(5.18),

$$|J| \leqslant \int_{-\infty}^{\infty} \partial_u g_\infty \partial_z^m (U_\infty) + \partial_v g_\infty \partial_z^m (V_\infty) + R_4(w,t)dw. \tag{5.40}$$

Consider the first and second terms, using the same method in (5.9)-(5.17), and for R_4, we use the same method in (5.33)-(5.34) by just adding some z-derivatives on g_∞. Since the z-derivative doesn't touch g_∞ one does not need to consider the influence of the z-derivative, so

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} \partial_u g_\infty \partial_z^m (U_\infty) + \partial_v g_\infty \partial_z^m (V_\infty) dw \right| \\ & \leqslant C\varepsilon_2 (||\partial_u \partial_z^m E||_r + ||\partial_z^m E||_r + 1)e^{-rt}, \end{aligned} \tag{5.41}$$

and

$$\left| \int_{-\infty}^{\infty} R_4(w,t)dw \right| \leqslant Ce^{-rt}. \tag{5.42}$$

Then by Proposition 2.4, if $||\partial_u \partial_z^m E||_r + ||\partial_z^m E||_r < \infty$, then

$$|\partial_z^m E(u,t,z)| + |\partial_u \partial_z^m E(u,t,z)| \leqslant C(\varepsilon_1 + \varepsilon_2 + 1)e^{-rt} + C\varepsilon_2(||\partial_u \partial_z^m E||_r + ||\partial_z^m E||_r + 1)e^{-rt}. \tag{5.43}$$

Then by choosing ε_2 small enough ($C\varepsilon_2 < 1$), we can get

$$||\partial_u \partial_z^m E||_r + ||\partial_z^m E||_r \leqslant C. \tag{5.44}$$

Remark 5.9 *The constant C above will only depend on $n, m, r, \alpha, A, \varepsilon_1, \varepsilon_2, C^*$, although when n and m become large C might be very large. This won't be a problem because one only needs to estimate $\partial_z^k E$ for $k \leqslant K$, and one can see in this case one only needs to take finite number of u and z derivatives ($n \leqslant K$, $n+m \leqslant K$). Therefore, one can have a uniform bound C for finite K.*

Remark 5.10 *In the above two inductions, we need the assumption $||\partial_u^{n+1}\partial_z^m E||_r < \infty$ for the first induction and $||\partial_u\partial_z^m E||_r + ||\partial_z^m E||_r < \infty$ for the second induction. Then we can obtain a uniform bound. Actually similar to before, by using similar contraction map as [6], this bound is automatically satisfied.*

5.4 Proof of Corollary 2.6

In this section, we will use Theorem 2.5 to prove Corollary 2.6. In this section, C is a constant only depending on $k, r, \alpha, A, \varepsilon_1, \varepsilon_2$ and will change in different equation.

Up to now, we have proved $||\partial_u^n\partial_z^m E||_r < C$ for all $n+m \leqslant K$. Then by (3.3), (3.8) and (3.9) for case $n = 0, m = k, 1 \leqslant k \leqslant K$, we get

$$\begin{aligned}\partial_z^k f(u,v,t,z) &= \partial_z^k(f_e(V_\infty(u,v,t),z)) + \partial_z^k(g_\infty(U_\infty(u,v,t),V_\infty(u,v,t),z)) \\ &= \partial_z^k f_e(V_\infty(u,v,t),z) + \partial_z^k g_\infty(U_\infty(u,v,t),V_\infty(u,v,t),z) + R(u,v,t,z),\end{aligned} \tag{5.45}$$

where $R(u,v,t,z)$ is the Langrange remainder of the expansion.

By the former discussion, V_∞, U_∞ are close to v, u, therefore, we can approximate $\partial_z^k f(u,v,t,z)$ by $\partial_z^k f_e(v,z) + \partial_z^k g_\infty(u,v,z)$. Firstly we deal with f_e by the Taylor series expansion,

$$\partial_z^k f_e(V_\infty(u,v,t),z) - \partial_z^k f_e(v,z) = \partial_v\partial_z^k f_e(v,z)(V_\infty - v) + R_1(u,v,t,z), \tag{5.46}$$

where $R_1(u,v,t,z)$ is the remaining part of the expansion.

Then by the assumption of f_e and (5.25),

$$|\partial_v\partial_z^k f_e(v,z)(V_\infty - v)| \leqslant C\frac{\varepsilon_1}{1+|v|^\alpha}||E||_r e^{-rt} \leqslant C\varepsilon_1||E||_r e^{-rt}, \tag{5.47}$$

$$|R_1(u,v,t,z)| \leqslant Ce^{-2rt}. \tag{5.48}$$

Similarly, to deal with $\partial_z^k g_\infty(U_\infty(u,v,t),V_\infty(u,v,t),z) - \partial_z^k g(u,v,z)$, we again use the Taylor series expansion

$$\begin{aligned}&\partial_z^k g_\infty(U_\infty(u,v,t),V_\infty(u,v,t),z) - \partial_z^k g(u,v,z) = \partial_u\partial_z^k g_\infty(u,v,z)(U_\infty - u) \\ &+ \partial_v\partial_z^k g_\infty(u,v,z)(V_\infty - v) + R_2(u,v,t,z)(\leqslant C(1+t)^2 e^{-2rt}),\end{aligned} \tag{5.49}$$

where $R_2(u,v,t,z)$ is the Lagrange remainder part of the expansion.

Similarly, by the assumption of g_∞ and (5.25),

$$\begin{aligned}\partial_u\partial_z^k g_\infty(u,v,z)(U_\infty - u) &\leqslant \frac{C}{1+|v|^\alpha}||E||_r(1+t)e^{-rt} \leqslant C(1+t)e^{-rt}, \\ \partial_v\partial_z^k g_\infty(u,v,z)(V_\infty - v) &\leqslant \frac{C}{1+|v|^\alpha}||E||_r e^{-rt} \leqslant Ce^{-rt}.\end{aligned} \tag{5.50}$$

Finally, to deal with $R(u,v,t,z)$, one can see there must exist at least one z-derivative on U_∞ or V_∞ in each term, which makes every term of order $(1+t)e^{-rt}$ and the coefficient $||\partial_u^n \partial_z^m E||_r < C$ for $n+m \leqslant K$ and u,v,z-derivatives of f_e, g_∞ are uniformly bounded by ϵ_1 and ϵ_2 respectively. Since we have already got $||\partial_u^n \partial_z^m E||_r < C$ for $n+m \leqslant K$ in Theorem 2.5, approximating f_e, g_∞ like before, we can finally get

$$|\partial_z^k f(u,v,t,z) - \partial_z^k f_e(u,v,z) - \partial_z^k g_\infty(u,v,z)| < C(1+t)e^{-rt} \quad \forall u,v \in [0,2\pi],\ t>0, \tag{5.51}$$

for $\forall 0 \leqslant k \leqslant K$ and C only depends on $r, \alpha, A, \varepsilon_1, \varepsilon_2, C^*, K$.

6 Conclusion

In this paper we analyze the regularity in the random space of the Landau dampling solution constucted in [6] by considering the impact of random uncertainty which could enter the system as a free parameter (with given probability density distribution) from the initial data or the global equilibrium. The solution of [6] is more general than that of [3] (the random regularity of its solution was studied in [16]) in that it replaces the flatness condition in [3] by a more general stability condition. Such a study is valuable in understanding the property of the solution with random perturbation, as well as the property of its numerical approximations.

In the future one may consider more general Landau damping solutions, such as those constructed in [2, 13], and the convergence properties of their numerical approximations.

Acknowledgements. The authors thank Dr. Ruiwen Shu for helpful discussions.

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