A LOCAL SENSITIVITY ANALYSIS FOR THE KINETIC CUCKER-SMALE EQUATION WITH RANDOM INPUTS

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Abstract. We present a local sensitivity analysis for the kinetic Cucker-Smale (C-S) equation with random inputs. This is a companion work to our previous local sensitivity analysis for the particle C-S model. Random imputs in the coefficients of the kinetic C-S equation can be caused by diverse sources such as the incomplete measurement and interactions with unknown environments, and will enter the problem through the communication function or initial data. For the proposed random kinetic C-S equation, we present sufficient conditions for the pathwise well-posedness and flocking estimates. For the local sensitivity analysis, we study the propagation of regularity of the kinetic density function in random space.

1. Introduction

The purpose of this paper is to extend the local sensitivity analysis [21] for the particle C-S model to the corresponding mesoscopic model. The jargon "*flocking*" denotes a collective behavior in which particles in a many-body system organize into an ordered motion using the environmental information based on simple rules [4, 5, 7, 37, 45, 46], e.g., flocking of birds, swarming of fish and herding of sheep, etc. Recently, due to emerging applications [34, 39, 40] in sensor networks, robot systems and unmanned aerial vehicles, research on the collective dynamics has received lots of attention from diverse scientific disciplines. After Vicsek's seminal work [48] on the collective dynamics, several physical and mathematical models were proposed in literature [11, 12, 37, 38, 47]. Among them, our main interest lies on the kinetic C-S equation which can be derived from the mean-field limit from the particle C-S model with random inputs. In literature, the particle and kinetic C-S models have been extensively studied from diverse perspectives, to name a few, collision avoidance [1, 9], effects of white noises [2, 10, 17, 18], kinetic limit [16, 19], dynamics of kinetic model [13, 19, 20], uncertainty quantification(UQ) problems [3, 6], general networks [8, 42], variants [30, 31, 32, 33] of the C-S model, etc. In this paper we will focus on the kinetic C-S equation with a random comunication weight and random initial input.

Let $f = f(t, x, v, z)$ be a one-particle distribution function of the C-S ensemble at position x with velocity $v \in \mathbb{R}^d$, random vector z, here $z = (z_1, \cdots, z_m)$ is a random vector defined on the sample space Ω. The random vector z registers the random effect on the communication

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weight. We also assume that each component random variables z_i are i.i.d., and let $\pi = \pi(z)$ be a probability density function for the random vector z. Note that the communication between particles, denoted by $\psi = \psi(x, z)$, is usually determined a priori by empirical data, thus inevitably contains uncertainty, modeled by z. In this situation, the dynamics of the kinetic density f is governed by the mean-field kinetic equation with random inputs:

$$
\begin{aligned}
&\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F_a[f]f) = 0, \quad x, v \in \mathbb{R}^d, \ z \in \Omega, \ t > 0, \\
&F_a[f](t, x, v, z) := -\int_{\mathbb{R}^{2d}} \psi(x - x_*, z)(v - v_*) f(t, x_*, v_*, z) dv_* dx_*,
\end{aligned} \tag{1.1}
$$

where $\psi(x, z) =: \tilde{\psi}(|x|, z)$ satisfies several structural properties such as the positivity, boundedness, monotonicity and Lipschitz continuity in the first argument: there exists a positive random variable $\psi_M(z) > 0$ such that

$$
\begin{aligned}
&0 < \psi(x, z) \le \psi_M(z) < \infty, \quad \psi(-x, z) = \psi(x, z), \quad (x, z) \in \mathbb{R}^d \times \Omega, \\
&(\tilde{\psi}(|x_2|, z) - \tilde{\psi}(|x_1|, z))(|x_2| - |x_1|) \le 0, \quad \psi(\cdot, z) \in \mathrm{Lip}(\mathbb{R}; \mathbb{R}_+).
\end{aligned} \tag{1.2}
$$

For each sample of z, i.e., the randomness is quenched, then (1.1) becomes a deterministic kinetic C-S equation which has been extensively studied in literature [4, 7, 16, 19, 20, 31, 32, 33].

In this paper, we are mainly interested in the effect of randomness of (1.1) on flocking dynamics and regularity of the solution in the random kinetic equation (1.1), with also random initial data $f^0(x, v, z)$, via the local sensitivity analysis [41, 44]. Note that the kinetic density function $f(t, x, v, z + dz)$ can be expanded in z-variable via Taylor's expnasion:

$$
f(t, x, v, z + dz) = f(t, x, v, z) + \sum_{i=1}^m \frac{\partial f}{\partial z_i} dz_i + \sum_{i,j=1}^m \frac{\partial^2 f}{\partial z_i \partial z_j} dz_i dz_j + \cdots .
$$

Then, we can define the sensitivity matrices consisting of coefficients:

$$
S^1 := \Big(\frac{\partial f}{\partial z_k} \Big), \quad S^2 := \Big(\frac{\partial^2 f}{\partial z_i \partial z_j} \Big), \cdots .
$$

Thus, the local sensitivity analysis deals with the regularity and stability estimates for the sensitivity matrices $S^i, \ i = 1, 2, \cdots$. Such analysis for a wide class of random kinetic equations have been extensively investigated by the group of the second author in [21, 22, 24, 25, 28, 29, 35, 36, 43], where the regularity and sensitivity were studied using weighted Sobolve energy estimates and coercivity or hypocoercivity (for perturnative solution near the global equilibrium) of the kinetic operators. For our kinetic flocking model, we use a mixed norm $H^k_\pi(L^\infty_{x,v})$ (to be defined in (1.3)), based on method of characteristics. This is the standard tool in Vlasov theory for perturbative solutions near vacuum, where energy methods do not work well [14].

There are two main results in this paper. First, we derive pathwise $W^{m,\infty}_{x,v}$-estimate and asymptotic flocking estimates for (1.1) along the sample paths. More precisely, for a fixed random vector z, we provide $W^{m,\infty}_{x,v}$-estimate and flocking estimate using the Lyapunov functional approach as in the deterministic case. Under the suitable regularity and compact support assumptions on the initial data, we show that there exists a unique regular solution process $f = f(t, x, v, z)$ to (1.1) - (1.2) such that for some nonnegative process $\Phi(t, z)$,

$$\int_{\mathbb{R}^{2d}} |v - v_c(z)|^2 f(t, x, v, z) dv dx \le \Phi(t, z) \int_{\mathbb{R}^{2d}} |v - v_c(z)|^2 f^0(x, v, z) dv dx, \quad z \in \Omega,\ t \ge 0,$$

(see Theorem 3.1 and Theorem 3.2). Second, we provide the pathwise local stability estimates. For the regular solutions processes f and $\tilde{f}$ with compact supports, there exists a nonnegative process $C(t, z)$ such that

$$\sum_{l=0}^{m} \left\| \partial_z^l f(t, z) - \partial_z^l \tilde{f}(t, z) \right\|_{L^\infty_{x,v}} \le C(t, z) \sum_{l=0}^{m} \| \partial_z^l f^0(z) - \partial_z^l \tilde{f}^0(z) \|_{L^\infty_{x,v}}, \quad z \in \Omega,\ t \in (0, T),$$

(see Theorem 4.3 and Theorem 4.4 for detailed description). In Proposition 4.1, we also show that the solution process f is bounded in any finite time interval in terms of mixed norm $H^m_\pi(L^\infty_{x,v})$: for $T \in (0, \infty)$,

$$\|f(t)\|_{H^m_\pi(L^\infty_{x,v})} \le C(T) \|f^0\|_{H^m_\pi(L^\infty_{x,v})}, \quad t \in [0, T),$$

where the mixed norm $H^m_\pi(L^\infty_{x,v})$ is defined at the end of this section.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the particle and kinetic C-S models with random inputs, and present several elementary estimates along the sample paths. In Section 3, we study a pathwise well-posedness and asymptotic flocking estimate using the Lyapunov functional approach. In Section 4, we study the propagation of H^k_z-regularity and local sensitivity on the random input parameter for (1.1) - (1.2) in a mixed norm. Finally, Section 5 is devoted to the brief summary of our main results and future direction.

Gallery of notation: Let $\pi : \Omega \to \mathbb{R}_+ \cup \{0\}$ be a nonnegative p.d.f. function, and let $y = y(z)$ be a scalar-valued random function defined on Ω. Then, we define the expected value as

$$\mathbb{E}[\varphi] := \int_\Omega \varphi(z) \pi(z) dz,$$

a weighted L^2-space:

$$L^2_\pi(\Omega) := \{ y : \ \Omega \to \mathbb{R} \mid \int_\Omega |y(z)|^2 \pi(z) dz < \infty \},$$

with an inner product and norm:

$$\langle y_1, y_2 \rangle_{L^2_\pi(\Omega)} := \int_\Omega y_1(z) y_2(z) \pi(z) dz, \qquad \|y\|_{L^2_\pi(\Omega)} := \Big(\int_\Omega |y(z)|^2 \pi(z) dz \Big)^{\frac{1}{2}} = \sqrt{\mathbb{E}[|y|^2]}.$$

For $k \in \mathbb{Z}_+ \cup \{0\}$, set

$$\|y\|_{H^k_\pi(\Omega)} := \Big(\sum_{\ell=0}^{k} \|\partial_z^\ell y\|^2_{L^2_\pi(\Omega)} \Big)^{\frac{1}{2}}, \quad k \ge 1, \qquad \|y\|_{H^0_\pi(\Omega)} := \|y\|_{L^2_\pi(\Omega)}.$$

Let $h = h(x, v, z)$ be scalar-valued random function defined on the extended phase space $\mathbb{R}^{2d} \times \Omega$. For such h, we define a mixed norm $H^k_\pi(L^\infty_{x,v})$ as follows.

$$\|h\|^2_{H^k_\pi(L^\infty_{x,v})} := \sum_{|\alpha| \le k} \|\partial_z^\alpha h(x, v)\|^2_{L^2_\pi(\Omega; L^\infty(\mathbb{R}^{2d}))}. \tag{1.3}$$

Moreover, as long as there is no confusion, we suppress π and Ω dependence in $L^2_\pi(\Omega)$-norm and $H^k_\pi(\Omega)$-norm:

$$\|y\|_{L^2_z} := \|y\|_{L^2_\pi(\Omega)}, \quad \|y\|_{H^k_z} := \|y\|_{H^k_\pi(\Omega)}.$$

For a vector-valued function $y(z) = (y^1(z), \cdots, y^d(z)) \in \mathbb{R}^d$, we set

$$\|y(z)\| := \Big(\sum_{i=1}^d |y^i(z)|^2 \Big)^{\frac{1}{2}}, \qquad \|y\|_{L^2_z} := \Big(\sum_{i=1}^d \|y^i\|^2_{L^2_z} \Big)^{\frac{1}{2}}.$$

2. Preliminaries

In this section, we discuss particle and kinetic C-S models with random inputs, and study their basic properties on the propagation of velocity moments.

2.1. The C-S model with random inputs.

Consider an ensemble consisting of N identical C-S particles in a random communication registered by $\psi = \psi(x, z)$. Let $(x_i(t, z), v_i(t, z)) \in \mathbb{R}^{2d}$ be the position-velocity processes of the i-th C-S particle. Then, their dynamics is governed by Cauchy problem for the random C-S model:

$$\begin{cases} \partial_t x_i(t, z) = v_i(t, z), \ t > 0, \quad z \in \Omega, \ i = 1, \cdots, N, \\ \partial_t v_i(t, z) = \dfrac{1}{N} \displaystyle\sum_{j=1}^N \psi(x_j(t, z) - x_i(t, z), z)(v_j(t, z) - v_i(t, z)), \\ (x_i(0, z), v_i(0, z)) = (x_i^0(z), v_i^0(z)). \end{cases} \tag{2.1}$$

Note that for each fixed $z \in \Omega$, system (2.1) becomes a deterministic C-S model which has been extensively studied in literature (see [7] for detailed survey and references therein).

Next, we present definition of *pathwise mono-cluster flocking* for the C-S ensemble. Set the first and second velocity moments:

$$m_1(t, z) := \sum_{i=1}^N v_i(t, z), \quad m_2(t, z) := \sum_{i=1}^N |v_i(t, z)|^2, \quad t \geq 0, \ z \in \Omega.$$

Proposition 2.1. [21] *Let $\{(x_i(t, z), v_i(t, z))\}_{i=1}^N$ be a solution process to the C-S model* (2.1) *with zero total momentum:*

$$m_1(0, z) = 0.$$

Then, we have

$$(i) \ m_1(t, z) = 0, \quad \partial_z^k m_1(t, z) = 0, \quad t > 0, \ z \in \Omega, \ k \geq 1.$$
$$(ii) \ \partial_t m_2(t, z) = -\frac{1}{N} \sum_{i,j=1}^N \psi(x_j(t, z) - x_i(t, z), z)|v_j(t, z) - v_i(t, z)|^2.$$

Remark 2.1. *It follows from Proposition 2.1 that the total momentum is conserved and total energy is non-increasing along the C-S flow* (2.1).

2.2. **The kinetic C-S model with random inputs.** Consider an system of N C-S particles with random inputs on the phase space $\mathbb{R}^{2d}$, with N very large. In this case, it becomes computationally expensive to integrate the infinite number of ODE system (2.1). Thus, we introduce a one-particle distribution function $f = f(t,x,v,z)$ for the infinite ensemble. Via the mean-field limit $N \to \infty$ in (2.1), the kinetic density f satisfies the Vlasov equation (see [16, 19] for rigorous justification):

$$
\begin{aligned}
&\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(F_a[f]f) = 0, \quad x, v \in \mathbb{R}^d,\ z\in\Omega,\ t>0,\\
&F_a[f](t,x,v,z) = -\int_{\mathbb{R}^{2d}} \psi(x-x_*,z)(v-v_*)f(t,x_*,v_*,z,t)dv_*dx_*,\\
&f(0,x,v,z) = f^0(x,v,z).
\end{aligned}
\tag{2.2}
$$

For notational simplicity, we introduce a simplified notation:

$$
\sigma := (x,v), \quad d\sigma := dxdv,
$$

and suppress t-dependance in f as well:

$$
f(\sigma,z) := f(t,\sigma,z), \quad \sigma\in\mathbb{R}^{2d},\ z\in\Omega,\ t\ge 0.
$$

Next, we define the k-th velocity momentum of f as follows.

$$
\mathcal{M}_k(t,z) := \int_{\mathbb{R}^{2d}} |v|^k f(\sigma,z)d\sigma.
$$

We now list a few basic propertieis of the moments of f.

Lemma 2.1. *Let $f = f(\sigma,z)$ be a regular solution process to (2.2) decaying fast enough at infinity in the phase space $\mathbb{R}^{2d}$. Then, for $z\in\Omega$ and $t\ge 0$,*

(i) $\displaystyle\int_{\mathbb{R}^{2d}} f(\sigma,z)d\sigma = \int_{\mathbb{R}^{2d}} f^0(\sigma,z)d\sigma, \quad \int_{\mathbb{R}^{2d}} vf(\sigma,z)d\sigma = \int_{\mathbb{R}^{2d}} vf^0(\sigma,z)d\sigma.$

(ii) $\displaystyle\frac{\partial}{\partial t}\int_{\mathbb{R}^{2d}} |v|^2 f(\sigma,z)d\sigma = -\int_{\mathbb{R}^{4d}} \psi(x-x_*,z)|v-v_*|^2 f(\sigma,z)f(\sigma_*,z)d\sigma d\sigma_*.$

(iii) $\displaystyle\frac{\partial}{\partial t}\int_{\mathbb{R}^{2d}} |f(\sigma,z)|^p d\sigma = d(p-1)\int_{\mathbb{R}^{4d}} \psi(x-x_*,z)f(\sigma_*,z)f^p(\sigma,z)d\sigma_* d\sigma, \quad p\in[1,\infty).$

Proof. The proof is elementary, as in its deterministic counterpart [20]. Thus, we omit the proof. □

3. Local sensity analysis: Zeroth-order estimates

In this section, we briefly discuss the pathwise well-posedness and asymptotic flocking estimates of smooth solution to the kinetic C-S equation (1.1) and (1.2).

3.1. **Pathwise well-posedness.** As mentioned in Introduction, the deterministic kinetic C-S equation admits a global C^1-solution in any finite-time interval, as long as the initial datum is $\mathcal{C}^1$-regular and compactly supported in x and v (see [20] for details). More precisely, we can derive a $W^{k,\infty}$-estimate using the method of characteristics in any finite time interval (see [20] for detailed estimates). In the standard Vlasov theory [14] in L^1-framework, $W^{k,\infty}$-estimate is standard. For this reason, we use the mixed norm $H^m_\pi(L^\infty_{x,v})$ throughout the paper. In contrast, when initial datum is a Radon measure, global measure-valued solutions to (1.1) was also studied in [19].

Now, we present the pathwise well-posedness of the kinetic C-S equation (2.2). Since the local existence can be obtained in a standard manner, we obtain a priori $W^{1,\infty}$-estimates and use the continuation principle to yield the unique global solution of (1.1). First, we define the projections of the support in phase space: for $(t,z) \in \mathbb{R}_+ \times \Omega$,

$$\mathscr{R}(t,z) := \overline{\{x \in \mathbb{R}^d \mid f(t,x,v,z) \neq 0 \quad \text{for some } v \in \mathbb{R}^d\}},$$
$$\mathscr{P}(t,z) := \overline{\{v \in \mathbb{R}^d \mid f(t,x,v,z) \neq 0 \quad \text{for some } x \in \mathbb{R}^d\}}.$$

Throughout the paper, for the simplicity of presentation, we assume that Ω is one-dimensional, i.e., $z \in \mathbb{R}$. First, we study the uniform boundedness of the velocity support $\mathscr{P}(t,z)$ in any finite time interval. For this, we consider forward characteristics curves: For $x, v \in \mathbb{R}^d$ and $z \in \Omega$, set

$$(x(t,z), v(t,z)) = (x(t;0,x,v,z), v(t;0,x,v,z)),$$

as a solution process to the following ODE system:

$$\begin{cases} \frac{\partial}{\partial t} x(t,z) = v(t,z), \quad t > 0, \\ \frac{\partial}{\partial t} v(t,z) = F_a[f](t,x,v,z), \\ (x(0,z), v(0,z)) = (x,v). \end{cases} \tag{3.1}$$

Now, we introduce two functionals $R(t,z)$ and $P(t,z)$ measuring the sizes of projected supports of f as follows:

$$R(t,z) := \sup_{x \in \mathscr{R}(t,z)} |x| \quad \text{and} \quad P(t,z) := \sup_{v \in \mathscr{P}(t,z)} |v|. \tag{3.2}$$

Lemma 3.1. *Let $f = f(t,x,v,z)$ be a regular solution process to* (1.1) - (1.2) *whose initial process f^0 has compact support in x and v for each $z \in \Omega$. Then, the following estimates hold.*

(1) There exists a positive random variable $D^0(z)$ such that

$$P(t,z) \le D^0(z)(1+t), \qquad R(t,z) \le R(0,z) + \frac{1}{2} D^0(z)(2t + t^2), \quad t \ge 0, \ z \in \Omega.$$

(2) If ψ satisfies (1.1) *and the extra positive lower bound condition* (3.3)*: there exists a positive random variable $\psi_m(z)$ such that*

$$\inf_{(x,z) \in \mathbb{R}^d \times \Omega} \psi(x,z) \ge \psi_m(z) > 0. \tag{3.3}$$

Then, we have

$$P(t,z) \le D^1(z), \qquad R(t,z) \le R(0,z) + D^1(z)t, \quad t \ge 0, \ z \in \Omega,$$

where $D^1(z)$ is a nonnegative random variable such that

$$D^1(z) := 2 \max \left\{ P(0,z), \ \frac{\psi_M(z)\sqrt{\mathcal{M}_2(z)}}{\psi_m(z)\sqrt{\mathcal{M}_0(z)}} \right\}.$$

Proof. The proof is essentially the same as the determinstic counterpart. See [20]. □

Remark 3.1. *The results of Lemma 3.1 imply*

$$\mathbb{E}[P(t)] \le \mathbb{E}[P(0)] + t\mathbb{E}[D^0], \quad \mathbb{E}[R(t)] \le \mathbb{E}[R(0)] + \left(t + \frac{1}{2}t^2\right) \mathbb{E}[D^0].$$

If ψ has the extra lower bound condition (3.3)*, then*

$$\mathbb{E}[P(t)] \le \mathbb{E}[D^1], \quad \mathbb{E}[R(t)] \le \mathbb{E}[R(0)] + t\mathbb{E}[D^1].$$

Now, we are ready to present the global existence of the unique $\mathcal{C}^k$ solution process to (1.1) for each $z \in \Omega$. Although the global existence of $\mathcal{C}^1$ solution is given in [20], we extend its analysis to the higher regular solution for later analysis.

Theorem 3.1. *Suppose that the following assumptions hold.*

(1) The communication weight is sufficiently regular in the sense that

$$\psi \in L^\infty(\Omega; W^{k,\infty}(\mathbb{R}^d)).$$

(2) Initial process f^0 is compactly supported and $\mathcal{C}^k$-regular in the sense that there exists $C_i(z),\ i = 1, 2$ such that

$$R(0, z) + P(0, z) \le C_1(z), \qquad \sum_{0\le|\alpha|+|\beta|\le k} \|\partial_v^\alpha \partial_x^\beta f^0(z)\|_{L^\infty_{x,v}} \le C_2(z), \quad \textit{for each} \quad z \in \Omega.$$

Then for any $T \in (0,\infty)$, there exists a unique $\mathcal{C}^k$-regular solution process $f = f(t, z) \in \mathcal{C}^k(\mathbb{R}^{2d} \times [0, T))$ to (1.1) for each $z \in \Omega$.

Proof. The proof is basically the same as that of Theorem 3.1 in [20] for $k = 1$. Consider the nonlinear transport operator $\mathcal{T}$:

$$\mathcal{T} := \partial_t + v \cdot \nabla_x + F_a[f] \cdot \nabla_v.$$

Then, one has

$$\mathcal{T}(f) = d \int_{\mathbb{R}^{2d}} \psi(x - x_*, z) f(\sigma_*, z) f(\sigma, z) d\sigma_* \le d\|\psi\|_{L^\infty_{x,z}} \mathcal{M}_0(z) \|f(z)\|_{L^\infty_{x,v}}.$$

For higher order estimate, we calculate $\mathcal{T}(\partial_v^\alpha \partial_x^\beta f(z))$ for $|\alpha|$, $|\beta| \ge 1$ as follows:

$$\begin{aligned}
&\mathcal{T}(\partial_v^\alpha \partial_x^\beta f(z)) \\
&\quad = -\sum_{|\mu|=1} \binom{\alpha}{\mu} \partial_v^\mu(v) \cdot \nabla_x \partial_v^{\alpha-\mu} \partial_x^\beta f - \sum_{|\gamma|=1} \binom{\beta}{\gamma} \nabla_v \cdot \partial_x^\gamma F_a[f] \partial_v^\alpha \partial_x^{\beta-\gamma} f \\
&\qquad - \sum_{\substack{|\mu'|=1 \\ 0\le|\gamma'|\le|\beta|}} \binom{\alpha}{\mu'} \binom{\beta}{\gamma'} \partial_v^{\mu'} \partial_x^{\gamma'} F_a[f] \cdot \nabla_v \partial_v^{\alpha-\mu'} \partial_x^{\beta-\gamma'} f.
\end{aligned}$$

Note that the following estimates hold:

$$\nabla_v \cdot \partial_x^\gamma F_a[f] \le d\|\psi\|_\infty \mathcal{M}_0(z), \quad |\partial_v^\mu \partial_x^\gamma F_a[f]| \le \|\psi\|_\infty \mathcal{M}_0(z),$$

where $\|\psi\|_\infty := \|\psi\|_{L^\infty(\Omega; W^{k,\infty}(\mathbb{R}^d))}$. From these estimates, one can easily get

$$\mathcal{T}(\partial_v^\alpha \partial_x^\beta f(z)) \le Bd\|\psi\|_\infty \mathcal{M}_0(z) \mathcal{F}(t, z), \quad \text{for } 0 \le |\alpha| + |\beta| \le k \text{ with } |\alpha|,\ |\beta| \ge 1,$$

where B and $\mathcal{F}(t, z)$ are defined by the following relations:

$$B := \sum_{\substack{|\mu|=1 \\ 0\le|\gamma|\le|\beta|}} \binom{\alpha}{\mu} \binom{\beta}{\gamma}, \qquad \mathcal{F}(t, z) := \sum_{0\le|\alpha|+|\beta|\le k} \|\partial_v^\alpha \partial_x^\beta f(z)\|_{L^\infty_{x,v}}.$$

Similarly, we calculate $\mathcal{T}(\partial_v^\alpha f(z))$ and $\mathcal{T}(\partial_x^\beta f(z))$ as follows:

$$\mathcal{T}(\partial_v^\alpha f(z)) = -\nabla_v \cdot F_a[f]\partial_v^\alpha f - \sum_{|\mu|=1} \binom{\alpha}{\mu} \left(\partial_v^\mu(v) \cdot \nabla_x \partial_v^{\alpha-\mu} f + \partial_v^\mu F_a[f] \cdot \nabla_v \partial_v^{\alpha-\mu} f \right),$$

$$\mathcal{T}(\partial_x^\beta f(z)) = -\nabla_v \cdot F_a[f]\partial_x^\beta f - \sum_{1\le|\gamma|\le|\beta|} \binom{\beta}{\gamma} \Big(\nabla_v \cdot \partial_x^\gamma F_a[f] \partial_x^{\beta-\gamma} f + \partial_x^\gamma F_a[f] \cdot \nabla_v \partial_x^{\beta-\gamma} f \Big).$$

Since

$$|\partial_x^\gamma F_a[f]| \le 2\|\psi\|_\infty P(t,z)\mathcal{M}_0(z),$$

one can obtain

$$\begin{aligned} \mathcal{T}(\partial_v^\alpha f(z)) &\le Bd\|\psi\|_\infty \mathcal{M}_0(z)\mathcal{F}(t,z), \\ \mathcal{T}(\partial_x^\beta f(z)) &\le 2Bd\|\psi\|_\infty P(t,z)\mathcal{M}_0(z)\mathcal{F}(t,z). \end{aligned}$$

We combine all above results to get a Gronwall's inequality:

$$\frac{\partial}{\partial t}\mathcal{F}(t,z) \le C(z)P(t,z)\mathcal{F}(t,z).$$

Therefore, by using Lemma 3.1 and Grönwall's lemma, one can complete the proof. □

3.2. **Pathwise flocking estimate.** For the pathwise flocking estimate, we use the Lyapunov functional approach in [20]. First we define the mean bulk velocity v_c as follows:

$$v_c(t,z) := \frac{1}{\mathcal{M}_0(z)} \int_{\mathbb{R}^{2d}} v f(t,\sigma,z) d\sigma, \quad z \in \Omega,\ t \ge 0.$$

Then, it follows from Lemma 2.1 that

$$v_c(t,z) = v_c(0,z) =: v_c(z), \quad t \ge 0.$$

Next, we define the Lyapunov functional given in [20] which measures the velocity variance around the mean value $v_c(z)$: for $(t,z) \in \mathbb{R}_+ \times \Omega$,

$$\mathcal{L}[f](t,z) := \int_{\mathbb{R}^{2d}} |v - v_c(z)|^2 f(t,\sigma,z) d\sigma. \tag{3.4}$$

As discussed in [17], the zero convergence of $\mathcal{L}$ as $t \to \infty$ implies the formation of velocity alignment in probability sense. This can be seen easily from the Chebyshev inequality. More precisely, let f be a probability density function over $\mathbb{R}^{2d}$. Then, for any $\varepsilon > 0$ and $z \in \Omega$,

$$\begin{aligned} \mathcal{L}[f](t,z) &= \int_{\mathbb{R}^{2d}} |v - v_c(z)|^2 f dvdx \ge \int_{|v - v_c(z)| > \varepsilon} |v - v_c(z)|^2 f dvdx \\ &\ge \varepsilon^2 \int_{|v - v_c(z)| > \varepsilon} f dvdx = \varepsilon^2 \mathbb{P}[|v - v^c(0)| > \varepsilon]. \end{aligned}$$

This implies

$$\lim_{t\to\infty} \mathbb{P}[|v - v_c(z)| > \varepsilon] \le \frac{1}{\varepsilon^2} \lim_{t\to\infty} \mathcal{L}[f](t,z) = 0.$$

Lemma 3.2. *Let f be a global solution process to* (1.1) *with initial process f^0 which is compactly supported in x and v for each $z \in \Omega$. Then for each $z \in \Omega$, we have*

$$\frac{\partial}{\partial t}\mathcal{L}[f](t,z) = -\int_{\mathbb{R}^{4d}} \psi(x - x_*, z)|v_* - v|^2 f(\sigma_*, z) f(\sigma, z) d\sigma_* d\sigma.$$

Proof. The proof is almost the same as that of Lemma 4.2 in [20], so we skip the details. □

Now, Lemma 3.2 yields the exponential decay of the functional $\mathcal{L}[f]$.

Theorem 3.2. *Let f be a global solution process to* (1.1) *with initial process f^0 whose support is compactly supported in x and v for each $z \in \Omega$. Then,*

$$\mathcal{L}[f](t,z) \le \mathcal{L}[f^0](z)\exp\left(-2\mathcal{M}_0(z)\int_0^t \varphi(s,z)ds\right), \quad z \in \Omega,\ t \ge 0, \tag{3.5}$$

where $\varphi(t,z)$ is defiend by

$$\varphi(t,z) := \tilde{\psi}\Big(2R(0,z) + D^0(z)(2t+t^2), z\Big).$$

Proof. It follows from Lemma 3.1 that

$$\psi(x-x_*,z) \ge \varphi(t,z) \quad \text{for any } x, x_* \in \mathcal{R}(t,z). \tag{3.6}$$

On the other hand, we use (3.6) and Lemma 3.2 to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{L}[f](t,z) &= -\int_{\mathbb{R}^{4d}} \psi(x-x_*,z)|v_*-v|^2 f(\sigma_*,z)f(\sigma,z)d\sigma_* d\sigma \\ &\le -\varphi(t,z)\int_{\mathbb{R}^{4d}} |v_*-v|^2 f(\sigma_*,z)f(\sigma,z)d\sigma_* d\sigma \\ &= -2\varphi(t,z)\mathcal{M}_0(z)\mathcal{L}[f](t,z),\end{aligned}$$

where we used the identity:

$$|v-v_*|^2 = |v-v_c^0(z)|^2 + |v_*-v_c^0(z)|^2 - 2(v-v_c^0(z))\cdot(v_*-v_c^0(z)).$$

Then, Grönwall's lemma completes the proof. □

Remark 3.2. *Suppose that initial datum f^0 is deterministic:*

$$f^0(\sigma,z) = f^0(\sigma), \quad \textit{for all } z \in \Omega.$$

Hence, $\mathcal{L}[f^0]$ is also deterministic:

$$\mathbb{E}[\mathcal{L}[f^0](\cdot)] = \mathcal{L}[f^0].$$

The relation (3.5) *yields*

$$\mathbb{E}[\mathcal{L}[f](t,\cdot)] \le \mathcal{L}[f^0]\mathbb{E}\Big[\exp\left(-2\mathcal{M}_0(z)\int_0^t \varphi(s,z)ds\right)\Big].$$

4. Local sensitivity analysis: Higher-order estimates

In this section, we present two local sensitivity analysis for regular solution processes such as the propagation of H_z^k-regularity and $L_{x,v}^\infty$-stability. Recall that for the simplicity of notation, we assume that the random space Ω is one-dimensional, i.e. $z \in \mathbb{R}$.

For a given $m \in \mathbb{Z}_+$, applying the operator ∂_z^m to (1.1) gives

$$\partial_t(\partial_z^m f) + v\cdot\nabla_x(\partial_z^m f) + \sum_{0\le l\le m}\binom{m}{l}\nabla_v\cdot\Big(\partial_z^l(F_a[f])\partial_z^{m-l}f\Big) = 0, \tag{4.1}$$

As in previous section, we introduce the following notation: for $m \in \mathbb{Z}_+$,

$$\mathscr{R}_m(t,z) := \overline{\{x \in \mathbb{R}^d \mid \partial_z^m f(x,v,z,t) \neq 0 \quad \text{for some } v \in \mathbb{R}^d\}},$$
$$\mathscr{P}_m(t,z) := \overline{\{v \in \mathbb{R}^d \mid \partial_z^m f(x,v,z,t) \neq 0 \quad \text{for some } x \in \mathbb{R}^d\}}.$$

and $\mathscr{P}_0(t,z) := \mathscr{P}(t,z)$, $\mathscr{R}_0(t,z) := \mathscr{R}(t,z)$.

For a fixed $(x,v,z) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega$, we introduce the forward characteristics $(x^m(t,z), v^m(t,z))$ associated with (4.1):

$$\begin{aligned} &\frac{\partial}{\partial t} x^m(t,z) = v^m(t,z), \\ &\frac{\partial}{\partial t} v^m(t,z) = F_a[f](t, x^m(t,z), v^m(t,z), z). \end{aligned} \tag{4.2}$$

Recall that

$$\sigma := (x,v), \quad d\sigma := dvdx \quad \text{and} \quad d\sigma_* := dv_* dx_*.$$

4.1. **Propagation of pathwise-regularity.** In this subsection, we study propagation of regularity in random space for f in terms of $L^\infty_{x,v}$-norm.

Note that for a fixed $z \in \Omega$, $\partial_z^m f$ satisfies the deterministic equation:

$$\partial_t(\partial_z^m f) + v \cdot \nabla_x(\partial_z^m f) + \sum_{0 \le l \le m} \binom{m}{l} \nabla_v \cdot \Big(\partial_z^l (F_a[f]) \partial_z^{m-l} f \Big) = 0 \quad x, v \in \mathbb{R}^d, \ z \in \Omega, \ t > 0,$$

and we can use all estimates available for the deterministic kinetic C-S equation in [19, 20].

Lemma 4.1. *Let $f = f(t,x,v,z)$ be a regular solution process to* (1.1) *decaying fast enough at infinity in phase space for each $z \in \Omega$ and $t > 0$. Then, we have*

$$(i) \ \frac{\partial}{\partial t} \int_{\mathbb{R}^{2d}} \partial_z^m f(\sigma, z) d\sigma = 0, \quad \frac{\partial}{\partial t} \int_{\mathbb{R}^{2d}} v \partial_z^m f(\sigma, z) d\sigma = 0.$$

$$\begin{aligned} (ii) \ &\frac{\partial}{\partial t} \int_{\mathbb{R}^{2d}} |v|^2 \partial_z^m f(\sigma, z) d\sigma \\ &= - \sum_{l=0}^m \sum_{\nu=0}^l \binom{m}{l} \binom{l}{\nu} \int_{\mathbb{R}^{4d}} |v - v_*|^2 (\partial_z^\nu \psi) \partial_z^{l-\nu} f(\sigma_*, z) \partial_z^{m-l} f(\sigma, z) d\sigma_* d\sigma. \end{aligned}$$

Proof. (i) The first estimate follows from the divergence structure of (4.1), and the second estimate can be treated as follows.

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{\mathbb{R}^{2d}} v \partial_z^m f(\sigma, z) d\sigma \\ &\quad = -d \sum_{0 \le l \le m} \binom{m}{l} \int_{\mathbb{R}^{2d}} (\partial_z^l F_a[f]) (\partial_z^{m-l} f) d\sigma \\ &\quad = -d \sum_{l=0}^m \sum_{\nu=0}^l \binom{m}{l} \binom{l}{\nu} \int_{\mathbb{R}^{4d}} (v - v_*) \partial_z^\nu \psi(z) \partial_z^{l-\nu} f(\sigma_*, z) \partial_z^{m-l} f(\sigma, z) d\sigma_* d\sigma. \end{aligned}$$

Note that the following relation holds:

$$
\begin{aligned}
&\sum_{l=0}^{m}\sum_{\nu=0}^{l}\binom{m}{l}\binom{l}{\nu}\partial_z^{\nu}\psi(z)\partial_z^{l-\nu}f(\sigma_*,z)\partial_z^{m-l}f(\sigma,z)\\
&\qquad\qquad=\sum_{l=0}^{m}\sum_{\nu=0}^{l}\binom{m}{l}\binom{l}{\nu}\partial_z^{\nu}\psi(z)\partial_z^{l-\nu}f(\sigma,z)\partial_z^{m-l}f(\sigma_*,z).
\end{aligned}
\tag{4.3}
$$

We use (4.3) and change of variable $(x,v)\leftrightarrow(x_*,v_*)$ to derive the desired result.

(ii) One can easily follow the calculation in (ii). We use the same argument in Lemma 2.1 and change of variable to yield the desired estimate. □

Next, define two functionals that measure the size of projected supports of $\partial_z^m f(t,\cdot,z)$, as we did in Section 3: For $m\in\mathbb{Z}_+,\ t\geq 0,\ z\in\Omega$, set

$$
R_m(t,z):=\sup_{x\in\mathscr{R}_m(t,z)}|x|,\qquad P_m(t,z):=\sup_{v\in\mathscr{P}_m f(t,z)}|v|.
\tag{4.4}
$$

For $m=0$, it follows from (3.1) and (3.2) that

$$
R_0(t,z)=R(t,z),\qquad P_0(t,z)=P(t,z),\quad t\geq 0,\ z\in\Omega.
$$

Below, we estimate the sizes of R_m and P_m in the following lemma.

Lemma 4.2. *Let $f=f(t,x,v,z)$ be a regular solution process to* (1.1) - (1.2) *with initial process f^0 whose z-derivative $\partial_z^m f^0$ has compact support in x and v for each $z\in\Omega$. Then, the functionals R_m and P_m defined in* (4.4) *satisfy the following estimates:*

(1) For $m\in\mathbb{Z}_+$, there exists a positive random variable $D_m^0(z)$ such that

$$
P_m(t,z)\leq D_m^0(z)(1+t),\qquad R_m(t,z)\leq R_m(0,z)+\frac{1}{2}D_m^0(z)(2t+t^2),\quad t\geq 0,\ z\in\Omega.
$$

(2) If ψ satisfies (1.2) *and extra positive lower bound condition below: there exists a positive random variable $\psi_m(z)$ such that*

$$
\inf_{(x,z)\in\mathbb{R}^d\times\Omega}\psi(x,z)\geq\psi_m(z)>0.
$$

Then,

$$
P_m(t,z)\leq D_m^1(z),\qquad R_m(t,z)\leq R_m(0,z)+D_m^1(z)t,\quad t\geq 0,\ z\in\Omega,
$$

where $D_m^1(z)$ is a nonnegative random variable such that

$$
D_m^1(z):=2\max\left\{P_m(0,z),\ \frac{\psi_M(z)\sqrt{\mathcal{M}_2(z)}}{\psi_m(z)\sqrt{\mathcal{M}_0(z)}}\right\},
$$

Proof. (1) For $z\in\Omega$, let $(x^m(t,z),v^m(t,z))$ be the characteristic curve to (4.2) such that

$$
P_m(t,z)=|v^m(t,z)|.
$$

Then,

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}P_m(t,z)^2 &= \int_{\mathbb{R}^{2d}} \psi(x^m(t,z)-x_*,z)(v_*-v^m(t,z))\cdot v^m(t,z) f(\sigma_*,z)d\sigma_* \\
&\le \int_{\mathbb{R}^{2d}} \psi(x^m(t,z)-x_*,z) v_*\cdot v^m(t,z) f(\sigma_*,z)d\sigma_* \\
&\le \psi_M(z) P_m(t,z)\sqrt{\mathcal{M}_2(t,z)}\sqrt{\mathcal{M}_0(t,z)} \\
&\le \psi_M(z) P_m(t,z)\sqrt{\mathcal{M}_2(z)}\sqrt{\mathcal{M}_0(z)},
\end{aligned} \tag{4.5}
$$

where we used Lemma 2.1 and $\psi_M(z) := \max_x \psi(x,z)$.

Next, we divide (4.5) by $P_m(t,z)$ and integrate with respect to t to obtain

$$
P_m(t,z) \le P_m(0,z) + t\psi_M(z)\sqrt{\mathcal{M}_2(z)}\sqrt{\mathcal{M}_0(z)}.
$$

This yields the desired estimate (3.4) with

$$
D_m^0(z) := \max\Big\{P_m(0,z), \psi_M(z)\sqrt{\mathcal{M}_2(z)}\sqrt{\mathcal{M}_0(z)}\Big\}.
$$

For $R_m(t,z)$, we choose the particle trajectory $(x^m(t,z), v^m(t,z))$ that yields $R_m(t,z)$:

$$
R_m(t,z) = |x^m(t,z)|.
$$

Then, by using $(4.2)_1$ and estimate of $P_m(t,z)$, one has

$$
\begin{aligned}
R_m(t,z) &\le R_m(0,z) + \int_0^t |v^m(s,z)|ds \le |x^m(0,z)| + D_m^0(z)\int_0^t (1+s)ds \\
&\le R_m(0,z) + \frac{1}{2}D_m(z)(2t+t^2).
\end{aligned}
$$

(ii) Again it follows from the first equality in (3.6) that

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}P_m(t,z)^2 &= \int_{\mathbb{R}^{2d}} \psi(x^m(t,z)-x_*,z)(v_*-v^m(t,z))\cdot v^m(t,z) f(\sigma_*,z)d\sigma_* \\
&= \int_{\mathbb{R}^{2d}} \psi(x^m(t,z)-x_*,z) v_*\cdot v^m(t,z) f(\sigma_*,z)d\sigma_* \\
&\quad - P_m(t,z)^2 \int_{\mathbb{R}^{2d}} \psi(x^m(t,z)-x_*,z) f(\sigma_*,z)d\sigma_* \\
&\le -\psi_m(z)P_m(t,z)^2\mathcal{M}_0(z) + \psi_M(z)P(t,z)\sqrt{\mathcal{M}_2(z)}\sqrt{\mathcal{M}_0(z)}.
\end{aligned} \tag{4.6}
$$

Dividing (4.6) by $P_m(t,z)$ and use Gronwall's lemma, one finds

$$
P_m(t,z) \le P_m(0,z)e^{-\psi_m(z)\mathcal{M}_0(z)t} + \frac{\psi_M(z)\sqrt{\mathcal{M}_2(z)}}{\psi_m(z)\sqrt{\mathcal{M}_0(z)}} \le D_m^1(z),
$$

where $D_m^1(z)$ is defined by

$$
D_m^1(z) := 2\max\left\{P_m(0,z),\ \frac{\psi_M(z)\sqrt{\mathcal{M}_2(z)}}{\psi_m(z)\sqrt{\mathcal{M}_0(z)}}\right\}.
$$

Thus, we have the desired uniform boundedness. On the other hand, by the same analysis in (**??**), one has

$$
R_m(t,z) \le R_m(0,z) + D_1(z)t.
$$

□

Before we present our main theorem in this subsection, we provide a preliminary lemma.

Lemma 4.3. *For each $z \in \Omega$ and $m \in \mathbb{N}$, let $f = f(t,x,v,z)$ be a regular solution process to* (1.1) - (1.2) *where every z-derivative of initial process $\partial_z^l f^0$ and f^0 itself are compactly supported in x and v. Then, there exist $C_i(t,z),\ i = 1,2$ such that*

$$(i)\ |\partial_z^l F_a[f]| + |\partial_x^\mu \partial_z^l F_a[f]| \le C_1(t,z) \sum_{0\le\nu\le l} \|\partial_z^\nu f\|_{L^\infty_{x,v}},$$

$$(ii)\ |\partial_{v_i}\partial_z^l F_a[f]| + |\partial_{v_i}\partial_x^\mu \partial_z^l F_a[f]| \le C_2(t,z) \sum_{0\le\nu\le l} \|\partial_z^\nu f\|_{L^\infty_{x,v}},$$

where $|\mu| \le k,\ l \le m$ and $\|\psi\|_\infty := \|\psi\|_{W^{m,\infty}(\Omega;W^{k,\infty}(\mathbb{R}^d))}$.

Proof. (i) It follows from $(1.1)_2$ that

$$\begin{aligned}
|\partial_z^l F_a[f](t,x,v,z)| &= \left| \sum_{0\le\nu\le l} \binom{l}{\nu} \int_{\mathscr{P}_\nu(t,z)\times\mathscr{R}_\nu(t,z)} \partial_z^{l-\nu}\psi(x-x_*,z)(v_*-v)\partial_z^\nu f(\sigma_*,z)d\sigma_* \right| \\
&\le 2\|\psi\|_\infty \sum_{0\le\nu\le l} \binom{l}{\nu} P_\nu(t,z) \int_{\mathscr{P}_\nu(t,z)\times\mathscr{R}_\nu(t,z)} |\partial_z^\nu f(\sigma_*,z)|d\sigma_* \\
&\le 2\|\psi\|_\infty \sum_{0\le\nu\le l} \binom{l}{\nu} P_\nu(t,z) \int_{\mathscr{P}_\nu(t,z)\times\mathscr{R}_\nu(t,z)} \|\partial_z^\nu f(z)\|_{L^\infty_{x,v}} d\sigma_* \\
&\le 2\|\psi\|_\infty \sum_{0\le\nu\le l} \binom{l}{\nu} P_\nu(t,z) \left(P_\nu(t,z)R_\nu(t,z)\right)^d \|\partial_z^\nu f\|_{L^\infty_{x,v}} \\
&\le C_1(t,z) \sum_{0\le\nu\le l} \|\partial_z^\nu f\|_{L^\infty_{x,v}},
\end{aligned}$$

where we used

$$|\mathscr{P}_\nu(t,z)\times\mathscr{R}_\nu(t,z)| \le (P_\nu(t,z)R_\nu(t,z))^d,$$

and $C_1(t,z)$ is given by

$$C_1(t,z) := 2\|\psi\|_\infty \sum_{0\le\nu\le l} \binom{l}{\nu} P_\nu(t,z) \left(P_\nu(t,z)R_\nu(t,z)\right)^d.$$

For $\partial_x^\mu \partial_z^l F_a[f]$, one can use similar estimate:

$$\begin{aligned}
|\partial_x^\mu \partial_z^l F_a[f]| &\le \sum_{0\le\nu\le l} \binom{l}{\nu} \int_{\mathscr{P}_\nu(t,z)\times\mathscr{R}_\nu(t,z)} |\partial_x^\mu \partial_z^{l-\nu}\psi(x-x_*,z)||v-v_*||\partial_z^\nu f(\sigma_*,z)|d\sigma_* \\
&\le 2\|\psi\|_\infty \sum_{0\le\nu\le l} \binom{l}{\nu} P_\nu(t,z) \left(P_\nu(t,z)(t,z)R_\nu(t,z)\right)^d \|\partial_z^\nu f\|_{L^\infty_{x,v}} \\
&\le C_1(t,z) \sum_{0\le\nu\le l} \|\partial_z^\nu f\|_{L^\infty_{x,v}}.
\end{aligned}$$

(ii) For $\partial_{v_i}\partial_z^l F_a[f]$, by direct estimate, one has

$$\begin{aligned}
|\partial_{v_i}\partial_z^l F_a[f]| &\le \sum_{0\le\nu\le l}\binom{l}{\nu}\int_{\mathscr{P}_\nu(t,z)\times\mathscr{R}_\nu(t,z)}|\partial_z^{l-\nu}\psi(x-x_*,z)||\partial_z^\nu f(\sigma_*,z)|d\sigma_*\\
&\le \|\psi\|_\infty \sum_{0\le\nu\le l}\binom{l}{\nu}\left(P_\nu(t,z)R_\nu(t,z)\right)^d\|\partial_z^\nu f\|_{L^\infty_{x,v}}\\
&\le C_2(t,z)\sum_{0\le\nu\le l}\|\partial_z^\nu f\|_{L^\infty_{x,v}},
\end{aligned}$$

where $C_2(t,z)$ is defined by

$$C_2(t,z) := \|\psi\|_\infty \sum_{0\le\nu\le l}\binom{l}{\nu}\left(P_\nu(t,z)R_\nu(t,z)\right)^d.$$

For $\partial_{v_i}\partial_x^\mu\partial_z^l F_a[f]$, similarly one has

$$\begin{aligned}
|\partial_{v_i}\partial_x^\mu\partial_z^l F_a[f]| &\le \sum_{0\le\nu\le l}\binom{l}{\nu}\int_{\mathscr{P}_\nu(t,z)\times\mathscr{R}_\nu(t,z)}|\partial_x^\mu\partial_z^{l-\nu}\psi(x-x_*,z)||\partial_z^\nu f(\sigma_*,z)|d\sigma_*\\
&\le \|\psi\|_\infty \sum_{0\le\nu\le l}\binom{l}{\nu}\left(P_\nu(t,z)R_\nu(t,z)\right)^d\|\partial_z^\nu f\|_{L^\infty_{x,v}}\\
&\le C_2(t,z)\sum_{0\le\nu\le l}\|\partial_z^\nu f\|_{L^\infty_{x,v}}.
\end{aligned}$$

□

Remark 4.1. *From Lemma 4.2, one can deduce that*

$$P_\nu(\cdot,z) := \mathcal{O}(t), \quad R_\nu(\cdot,z) := \mathcal{O}(t^2), \quad \text{for each } z\in\Omega.$$

Thus, this yields the following estimates: for each $z\in\Omega$, we have

$$C_1(\cdot,z) = \mathcal{O}(t^{3d+1}), \quad C_2(\cdot,z) = \mathcal{O}(t^{3d}),$$

where C_1 and C_2 are constants given in Lemma 4.3.

Now we present the global existence of a unique regular solution process to (4.1) for each $z\in\Omega$. First we would like to consider the case $m=1$.

Theorem 4.1. *For each $z\in\Omega$ and $k\ge 2$, suppose that the initial process f^0 satisfies the following conditions:*

(1) Initial processes $f^0(z)$ and $\partial_z f^0(z)$ are compactly supported in phase space for each $z\in\Omega$:

$$R_l(t,z),\quad P_l(t,z)<\infty, \quad \text{for each } l=0,\ 1, \quad \text{and each} \quad z\in\Omega.$$

(2) Initial processes $f^0(z)$ and $\partial_z^l f^0(z)$ are $\mathcal{C}^k$-regular and $\mathcal{C}^{k-1}$-regular, respectively and bounded for each $z\in\Omega$:

$$\sum_{\substack{0\le l\le 1\\ 0\le|\alpha|+|\beta|\le k-l}}\|\partial_v^\alpha\partial_x^\beta\partial_z^l f^0(\cdot,\cdot,z)\|_{L^\infty_{x,v}}<\infty.$$

(3) The communication weight ψ belongs to $W^{1,\infty}(\Omega; W^{k,\infty}(\mathbb{R}^d))$.

Then, for any $T \in (0,\infty)$, there exists a unique $\mathcal{C}^{k-1}$-regular solution process $\partial_z f(z) \in \mathcal{C}^{k-1}(\mathbb{R}^{2d} \times [0,T))$ to (4.1) for each $z \in \Omega$.

Proof. The proof is almost the same as Theorem 3.1. So we provide a priori $W^{k-1,\infty}$-estimate for $\partial_z f$. Again consider the nonlinear transport operator

$$\mathcal{T} := \partial_t + v \cdot \nabla_x + F_a[f] \cdot \nabla_v.$$

• Case A (zeroth-order estimate): It follows from (4.1) that

$$\mathcal{T}(\partial_z f) = -\nabla_v \cdot F_a[f]\partial_z f - \nabla_v \cdot \partial_z F_a[f] f - \partial_z F_a[f] \cdot \nabla_v f.$$

By Remark 3.1, f is $\mathcal{C}^k$-regular and bounded. Set

$$\mathcal{F}^0(t,z) := \sum_{0 \le |\alpha|+|\beta| \le k} \|\partial_v^\alpha \partial_x^\beta f(t,\cdot,\cdot,z)\|_{L^\infty_{x,v}},$$

Then, $\mathcal{F}^0(t,z)$ is bounded for each $t \le T$ and $z \in \Omega$, and

$$\begin{aligned}
\mathcal{T}(\partial_z f) &\le d\|\psi\|_\infty \mathcal{M}_0(z)|\partial_z f| + d\|\psi\|_\infty \Big(\mathcal{M}_0(z) + (P_1(t,z)R_1(t,z))^d \|\partial_z f\|_{L^\infty_{x,v}}\Big)|f| \\
&\quad + 2\|\psi\|_\infty \Big(P_0(t,z)\mathcal{M}_0(z) + P_1(t,z)\left(P_1(t,z)R_1(t,z)\right)^d \|\partial_z f\|_{L^\infty_{x,v}}\Big)|\nabla_v f| \\
&\le C(t,z)(1+\mathcal{F}^0(t,z))\Big(\|\partial_z f\|_{L^\infty_{x,v}} + 1\Big) \\
&\le C(t,z)\Big(\|\partial_z f\|_{L^\infty_{x,v}} + 1\Big),
\end{aligned} \tag{4.7}$$

where $\|\psi\|_\infty := \|\psi\|_{W^{k,\infty}(\mathbb{R}^d)\times W^{1,\infty}(\Omega)}$ and $C(t,z)$ is a constant that depends on $d, f^0, P_l(t,z), R_l(t,z)$, but independent of $\partial_z f$, since

$$\begin{aligned}
\nabla_v \cdot \partial_z F_a[f] &= d\int_{\mathbb{R}^{2d}} \partial_z \psi(z) f_*(z) + \psi(z)\partial_z f(\sigma_*,z) d\sigma_* \\
&\le d\|\psi\|_\infty \Big(\mathcal{M}_0(z) + (P_1(t,z)R_1(t,z))^d \|\partial_z f\|_{L^\infty_{x,v}}\Big), \\
|\partial_z F_a[f]| &\le \int_{\mathbb{R}^{2d}} \|v_* - v\| |\partial_z \psi(z) f(\sigma_*,z) + \phi(z)\partial_z f(\sigma_*,z)| d\sigma_* \\
&\le 2\|\psi\|_\infty \Big(P_0(t,z)\mathcal{M}_0(z) + P_1(t,z)\left(P_1(t,z)R_1(t,z)\right)^d \|\partial_z f\|_{L^\infty_{x,v}}\Big), \\
\nabla_v \cdot F_a[f] &\le d\|\psi\|_\infty \mathcal{M}_0(z).
\end{aligned}$$

• Case B (Higher-order estimate): Now we estimate higher-order terms. First, consider the term:

$$\partial_v^\alpha \partial_x^\beta \partial_z f, \qquad \text{where } \ 1 \le |\alpha|, |\beta| \le k-1 \text{ and } |\alpha|+|\beta| \le k-1.$$

Then, it follows from (1.1) that we have

$$\begin{aligned}
\mathcal{T}(\partial_v^\alpha \partial_x^\beta \partial_z f) &= -\nabla_v \cdot \partial_z F_a[f] \partial_v^\alpha \partial_x^\beta f - \partial_z F_a[f] \cdot \partial_v^\alpha \partial_x^\beta (\nabla_v f) - \nabla_v \cdot F_a[f] \partial_v^\alpha \partial_x^\beta (\partial_z f) \\
&\quad - \sum_{|\mu_1|=1} G_1(\alpha,\beta,\mu)(f) - \sum_{\substack{0 \le |\mu_2| \le 1 \\ 0 \le |\mu_3| \le |\beta| \\ |\mu_2|+|\mu_3| \ne 0}} H_1(\alpha,\beta,\mu_2,\mu_3)(f),
\end{aligned}$$

where G_1 and H_1 are given by the following relations:

$$
\begin{aligned}
&G_1(\alpha,\beta,\mu_1)(f) = \binom{\alpha}{\mu_1}\partial_v^{\mu_1}(v)\cdot\partial_v^{\alpha-\mu_1}\partial_x^{\beta}(\nabla_x\partial_z f),\\
&H_1(\alpha,\beta,\mu_2,\mu_3)(f)\\
&\quad = \binom{\alpha}{\mu_2}\binom{\beta}{\mu_3}\Big(\partial_v^{\mu_2}\partial_x^{\mu_3}(\nabla_v\cdot\partial_z F_a[f])\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}f + \partial_v^{\mu_2}\partial_x^{\mu_3}(\partial_z F_a[f])\cdot\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}(\nabla_v f)\\
&\qquad + \partial_v^{\mu_2}\partial_x^{\mu_3}(\nabla_v\cdot F_a[f])\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}\partial_z f + \partial_v^{\mu_2}\partial_x^{\mu_3}(F_a[f])\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}(\nabla_v\partial_z f)\Big).
\end{aligned}
$$

One can easily see that

$$
|G_1(\alpha,\beta,\mu_1)(f)| \le \binom{\alpha}{\mu_1}\|\partial_v^{\alpha-\mu_1}\partial_x^{\beta}(\nabla_x\partial_z f)\|_{L^\infty_{x,v}}.
$$

Here, by using Theorem 3.1, one finds

$$
|\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}f|,\quad |\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}\nabla_v f| \le \mathcal{F}^0(t,z)\le C(t,z),
$$

Combining this with Lemma 4.3 gives

$$
\begin{aligned}
&|H_1(\alpha,\beta,\mu_2,\mu_3)(f)|\\
&\quad\le C(t,z)\left(\sum_{l=0}^{1}\|\partial_z^l f\|_{L^\infty_{x,v}} + \|\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}\partial_z f\|_{L^\infty_{x,v}} + \|\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}(\nabla_v\partial_z f)\|_{L^\infty_{x,v}}\right).
\end{aligned}
$$

Then, we again use Lemma 4.3 and combine all the estimates above to get

$$
\begin{aligned}
&\mathcal{T}(\partial_v^{\alpha}\partial_x^{\beta}\partial_z f)\\
&\qquad\le C(t,z)\Bigg(\sum_{l=0}^{1}\|\partial_z^l f\|_{L^\infty_{x,v}} + \|\partial_v^{\alpha}\partial_x^{\beta}(\partial_z f)\|_{L^\infty_{x,v}} + \sum_{|\mu_1|=1}\|\partial_v^{\alpha-\mu_1}\partial_x^{\beta}(\nabla_x\partial_z f)\|_{L^\infty_{x,v}}\\
&\qquad\quad + \sum_{\substack{0\le|\mu_2|\le 1\\ 0\le|\mu_3|\le|\beta|\\ |\mu_2|+|\mu_3|\neq 0}}\Big(\|\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}\partial_z f\|_{L^\infty_{x,v}} + \|\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}(\nabla_v\partial_z f)\|_{L^\infty_{x,v}}\Big)\Bigg)\\
&\qquad\le C(t,z)\left(\mathcal{F}^1(t,z)+1\right),
\end{aligned}
\tag{4.8}
$$

where $C(t,z)$ depends on d, $\mathcal{F}^0(t,z)$, $\|\psi\|_\infty$, $P_l(t,z)$, $R_l(t,z)$ for $l=0$, 1, and $\mathcal{F}^1(t,z)$ is given by

$$
\mathcal{F}^1(t,z) := \sum_{0\le|\alpha|+|\beta|\le k-1}\|\partial_v^{\alpha}\partial_x^{\beta}\partial_z f(t,z)\|_{L^\infty_{x,v}}.
$$

Similarly, for $|\beta|\ge 1$ and $|\alpha|\ge 1$, we compute $\mathcal{T}(\partial_x^{\beta}\partial_z f)$ and $\mathcal{T}(\partial_v^{\alpha}\partial_z f)$ as follows:

(4.9)

$$
\begin{aligned}
(i)\ \mathcal{T}(\partial_x^\beta \partial_z f) &\le C(t,z)\Bigg(\sum_{l=0}^{1}\|\partial_z^l f\|_{L^\infty_{x,v}} + \|\partial_x^\beta(\partial_z f)\|_{L^\infty_{x,v}} \\
&\quad + \sum_{1\le|\mu_3|\le|\beta|}\|\partial_x^{\beta-\mu_3}\partial_z f\|_{L^\infty_{x,v}} + \|\partial_x^{\beta-\mu_3}(\nabla_v\partial_z f)\|_{L^\infty_{x,v}}\Bigg) \\
&\le C(t,z)\left(\mathcal{F}^1(t,z)+1\right), \\
(ii)\ \mathcal{T}(\partial_v^\alpha \partial_z f) &\le C(t,z)\Bigg(\sum_{l=0}^{1}\|\partial_z^l f\|_{L^\infty_{x,v}} + \|\partial_v^\alpha(\partial_z f)\|_{L^\infty_{x,v}} + \sum_{|\mu_1|=1}\|\partial_v^{\alpha-\mu_1}(\nabla_x\partial_z f)\|_{L^\infty_{x,v}} \\
&\quad + \sum_{0\le|\mu_2|\le 1}\|\partial_v^{\alpha-\mu_2}\partial_z f\|_{L^\infty_{x,v}} + \sum_{|\mu_2|=1}\|\partial_v^{\alpha-\mu_2}(\nabla_v\partial_z f)\|_{L^\infty_{x,v}}\Bigg) \\
&\le C(t,z)\left(\mathcal{F}^1(t,z)+1\right).
\end{aligned}
$$

Finally, we combine all estimates (4.7), (4.8) and (4.9) to obtain

$$\frac{\partial}{\partial t}\mathcal{F}^1(t,z) \le C(t,z)\left(\mathcal{F}^1(t,z)+1\right).$$

Then, Grönwall's lemma yields the desired estimate. □

Now we generalize the idea in Theorem 4.1 to the case $m \ge 1$.

Theorem 4.2. *For each $z\in\Omega$ and $m\ge 2$, assume that the initial process $f^0(z) := f(0,z)$ satisfies the followings:*

(1) Every z-derivative of initial process $\partial_z^l f^0(z)$ is compactly supported in the phase space for each $l = 0,\ 1,\ \cdots,\ m$ and each $z\in\Omega$, i.e.

$$R_l(0,z) < \infty, \quad P_l(0,z) < \infty, \quad \text{for each } \ l = 0,\ 1,\cdots,m \quad \text{and each } z\in\Omega.$$

(2) Every z-derivative of initial process $\partial_z^l f^0(z)$is $\mathcal{C}^{m-l+1}$-regular and bounded for each $l = 0,\ 1,\ \cdots,\ m$ and each $z\in\Omega$, i.e.

$$\sum_{\substack{0\le l\le m\\ 0\le|\alpha|+|\beta|\le m-l+1}} \|\partial_v^\alpha\partial_x^\beta\partial_z^l f^0(z)\|_{L^\infty_{x,v}} < \infty.$$

(3) The communication weight ψ belongs to $W^{m+1,\infty}(\mathbb{R}^d\times\Omega)$.

Then, for any $T\in(0,\infty)$, there exists a unique $\mathcal{C}^{m-l+1}$-regular solution process $\partial_z^l f(z) \in \mathcal{C}^{m-l+1}(\mathbb{R}^{2d}\times[0,T))$ to (4.1) for each $l = 1,\ \cdots,\ m$ and each $z\in\Omega$.

Proof. We proceed the proof by induction on m. Note that we have already proved the initial step ($m = 2$ case) in Theorem 4.1. So one just needs to verify the induction step for m. However, to show this, one needs to use induction on l, i.e. to show the following step:

> *For fixed $m\ge 2$ and $l\le m$, under the assumptions in this theorem, if there exists a unique $C^{m-\nu+1}$-regular solution process $\partial_z^\nu f(z)$ for all $\nu < l$, then there also exists a unique C^{m-l+1}-regular solution process $\partial_z^l f(z)$.*

Note that we have also proved the initial step ($l = 1$) in Theorem 4.1. Since it suffices to show the above statement, again we consider the nonlinear transport operator $\mathcal{T} := \partial_t + v \cdot \nabla_x + F_a[f] \cdot \nabla_v$ and define $\mathcal{F}^\nu(t,z)$ as follows:

$$\mathcal{F}^\nu(t,z) := \sum_{0 \le |\alpha|+|\beta| \le m-\nu+1} \|\partial_v^\alpha \partial_x^\beta \partial_z^\nu f(z)\|_{L^\infty_{x,v}}, \quad \text{for } \nu = 0,\ 1,\ \cdots, l. \tag{4.10}$$

• Case A (Zeroth-order estimate): From (4.1),

$$\begin{aligned}\mathcal{T}(\partial_z^l f) &= -\nabla_v \cdot F_a[f] \partial_z^l f - \nabla_v \cdot \partial_z^l F_a[f] f - \partial_z^l F_a[f] \cdot \nabla_v f \\ &\quad - \sum_{\nu=1}^{l-1} \binom{l}{\nu} \Big(\nabla_v \cdot \partial_z^{l-\nu} F_a[f] \partial_z^\nu f + \partial_z^{l-\nu} F_a[f] \cdot \nabla_v \partial_z^\nu f \Big).\end{aligned}$$

Then by using Lemma 4.3, one can estimate $\mathcal{T}(\partial_z^l f)$ as follows:

(4.11)

$$\begin{aligned}\mathcal{T}(\partial_z^l f) &\le C(t,z) \left(|\partial_z^l f| + \sum_{0 \le \nu \le l} \|\partial_z^\nu f\|_{L^\infty_{x,v}} + \sum_{\nu=1}^{l-1} \left(\left(\sum_{r=0}^{l-\nu} \|\partial_z^r f\|_{L^\infty_{x,v}} \right) (|\partial_z^\nu f| + |\nabla_v \cdot \partial_z^\nu f) \right) \right) \\ &\le C(t,z) \left(\mathcal{F}^l(t,z) + \sum_{0 \le \nu \le l} \mathcal{F}^\nu(t,z) + \sum_{\nu=1}^{l-1} \left(\sum_{r=0}^{l-1} \mathcal{F}^r(t,z) \right) \mathcal{F}^\nu(t,z) \right) \\ &\le C(t,z) \Big(\mathcal{F}^l(t,z) + 1 \Big),\end{aligned}$$

where $C(t,z)$ depends on $\mathcal{F}^\nu(t,z)$, d, $\|\psi\|_\infty$, $P_\nu(t,z)$ and $R_\nu(t,z)$ with $0 \le \nu \le l-1$. Since they can be bounded by induction hypothesis, we get the last inequality.

• Case B (Higher-order estimate): It follows from tedious, direct computation that one can obtain from (4.1) that for $1 \le |\alpha|,\ |\beta|$ with $|\alpha| + |\beta| \le m - l + 1$,

$$\partial_t(\partial_v^\alpha \partial_x^\beta \partial_z^l f) + \sum_{0 \le |\mu_1| \le 1} \binom{\alpha}{\mu_1} \partial_v^{\mu_1}(v) \cdot (\nabla_x \partial_v^{\alpha-\mu_1} \partial_x^\beta \partial_z^l f) + \sum_{\substack{0 \le \nu \le l \\ 0 \le |\mu_2| \le 1 \\ 0 \le |\mu_3| \le |\beta|}} H(\alpha, \beta, l, \mu_2, \mu_3, \nu)(f) = 0,$$

where H is given by

$$\begin{aligned}H(\alpha, \beta, l, \mu_2, \mu_3, \nu)(f) = \binom{l}{\nu} \binom{\alpha}{\mu_2} \binom{\beta}{\mu_3} \Big(&\partial_v^{\mu_2} \partial_x^{\mu_3} (\nabla_v \cdot \partial_z^\nu F_a[f]) \partial_v^{\alpha-\mu_2} \partial_x^{\beta-\mu_3} \partial_z^{l-\nu} f \\ &+ \partial_v^{\mu_2} \partial_x^{\mu_3} \partial_z^\nu F_a[f] \cdot \nabla_v (\partial_v^{\alpha-\mu_2} \partial_x^{\beta-\mu_3} \partial_z^{l-\nu} f) \Big).\end{aligned}$$

It follows from Lemma 4.3 that

$$|\partial_v^{\mu_2} \partial_x^{\mu_3} (\nabla_v \cdot \partial_z^\nu F_a[f]) \partial_v^{\alpha-\mu_2} \partial_x^{\beta-\mu_3} \partial_z^{l-\nu} f| \le C(t,z) \left(\sum_{r=0}^{\nu} \|\partial_z^r f\|_{L^\infty_{x,v}} \right) \|\partial_v^{\alpha-\mu_2} \partial_x^{\beta-\mu_3} \partial_z^{l-\nu} f\|_{L^\infty_{x,v}},$$

$$|\partial_v^{\mu_2} \partial_x^{\mu_3} \partial_z^\nu F_a[f] \cdot \nabla_v (\partial_v^{\alpha-\mu_2} \partial_x^{\beta-\mu_3} \partial_z^{l-\nu} f)| \le C(t,z) \left(\sum_{r=0}^{\nu} \|\partial_z^r f\|_{L^\infty_{x,v}} \right) \|\partial_v^{\alpha-\mu_2} \partial_x^{\beta-\mu_3} (\nabla_v \partial_z^{l-\nu} f)\|_{L^\infty_{x,v}}.$$

Now let us define a set $I(l,\beta)$ given by

$I(l,\beta) := \{(\nu,\mu_2,\mu_3) \in (\mathbb{N}\cup\{0\})^{2d+1} \mid \nu \le l, \quad |\mu_2| \le 1, \quad |\mu_3| \le |\beta|, \quad \nu + |\mu_2| + |\mu_3| > 0\}.$

Then using the estimates above, one can get for $(\nu,\mu_2,\mu_3) \in I(l,\beta)$,

$$
\begin{aligned}
&H(\alpha,\beta,l,\mu_2,\mu_3,\nu)(f)\\
&\quad \le C(t,z)\left(\sum_{r=0}^{\nu}\|\partial_z^r f\|_{L^\infty_{x,v}}\right)\left(\|\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}\partial_z^{l-\nu} f\|_{L^\infty_{x,v}} + \|\partial_v^{\alpha-\mu_2}\partial_x^{\beta-\mu_3}(\nabla_v\partial_z^{l-\nu} f)\|_{L^\infty_{x,v}}\right)\\
&\quad \le C(t,z)\left(\sum_{r=0}^{\nu}\mathcal{F}^r(t,z)\right)\mathcal{F}^{l-\nu}(t,z).
\end{aligned}
$$

This yields the following estimates:

$$
\begin{aligned}
&\mathcal{T}(\partial_v^\alpha\partial_x^\beta\partial_z^l f)\\
&\qquad = -\sum_{|\mu_1|=1}\binom{\alpha}{\mu_1}\partial_v^{\mu_1}(v)\cdot(\nabla_x\partial_v^{\alpha-\mu_1}\partial_x^\beta\partial_z^l f) - (\nabla_v\cdot F_a[f])\partial_v^\alpha\partial_x^\beta\partial_z^l f\\
&\qquad\quad - \sum_{(\nu,\mu_2,\mu_3)\in I(l,\beta)} H(\alpha,\beta,l,\mu_2,\mu_3,\nu)(f)\\
&\qquad \le C(t,z)\Bigg(\sum_{|\mu_1|=1}\|\nabla_x\partial_v^{\alpha-\mu_1}\partial_x^\beta\partial_z^l f)\|_{L^\infty_{x,v}} + \|\partial_v^\alpha\partial_x^\beta\partial_z^l f\|_{L^\infty_{x,v}}\\
&\qquad\quad + \sum_{(\nu,\mu_2,\mu_3)\in I(l,\beta)}\left(\sum_{r=0}^{\nu}\mathcal{F}^r(t,z)\right)\mathcal{F}^{l-\nu}(t,z)\Bigg)\\
&\qquad \le C(t,z)\Bigg((1+\mathcal{F}^0(t,z))\mathcal{F}^l(t,z) + \left(\sum_{r=0}^{l-1}\mathcal{F}^r(t,z)\right)\mathcal{F}^0(t,z)\\
&\qquad\quad + \sum_{\nu=1}^{l-1}\left(\sum_{r=0}^{\nu}\mathcal{F}^r(t,z)\right)\mathcal{F}^{l-\nu}(t,z)\Bigg)\\
&\qquad \le C(t,z)\left(\mathcal{F}^l(t,z)+1\right),
\end{aligned}
\tag{4.12}
$$

where $C(t,z)$ depends on d, $\|\psi\|_\infty$, $\mathcal{F}^\nu(t,z)$, $P_\nu(t,z)$ and $R_\nu(t,z)$ with $0 \le \nu \le l-1$, but independent of $\partial_z^l f$. From the assumptions and induction hypothesis, for each $z \in \Omega$, one has that $C(t,z)$ can be bounded by a constant.

We use (4.12) to get estimates for $\mathcal{T}(\partial_v^\alpha\partial_z^l f)$ and $\mathcal{T}(\partial_x^\beta\partial_z^l f)$:

$$\mathcal{T}(\partial_v^\alpha\partial_z^l f),\ \mathcal{T}(\partial_x^\beta\partial_z^l f) \le C(t,z)\left(\mathcal{F}^l(t,z)+1\right).$$

Therefore, combination of (4.10), (4.11) and (4.12) yields

$$\frac{\partial}{\partial t}\mathcal{F}^l(t,z) \le C(t,z)\left(\mathcal{F}^l(t,z)+1\right), \quad 0 < t \le T,\ z \in \Omega.$$

The Grönwall's lemma gives

$$\mathcal{F}^l(t,z) \le C(t,z)\Big(\mathcal{F}^l(0,z)+1\Big), \quad 0 \le t \le T.$$

Again, this yields the boundedness of $\mathcal{F}^l(t,z)$ for any $t \le T$ and each $z \in \Omega$. $\square$

4.2. **Pathwise stability analysis.** In this subsection, we present $L^\infty_{x,v}$-stability of (4.1). First we present pathwise L^∞- stability in the following theorem.

Theorem 4.3. *For each $z \in \Omega$, suppose that two initial processes $f^0(z) := f(0,z)$ and $\tilde{f}^0(z) := \tilde{f}(0,z)$ satisfy the following conditions:*

(1) Initial processes are compactly supported in the phase space for each $z \in \Omega$:

$$R(0,z) + \tilde{R}(0,z) < \infty, \quad P(0,z) + \tilde{P}(0,z) < \infty, \quad \text{for each } \ z \in \Omega.$$

(2) Initial processes $f^0(z)$ and $\tilde{f}^0(z)$ are $\mathcal{C}^1$-regular and bounded for each $z \in \Omega$:

$$\sum_{0 \le |\alpha|+|\beta| \le 1} \|\partial_v^\alpha \partial_x^\beta f^0(z)\|_{L^\infty_{x,v}}, \quad \sum_{0 \le |\alpha|+|\beta| \le 1} \|\partial_v^\alpha \partial_x^\beta \tilde{f}^0(z)\|_{L^\infty_{x,v}} < \infty.$$

(3) The communication weight $\psi(\cdot,z)$ belongs to $W^{1,\infty}(\mathbb{R}^d)$.

Then, for any $T \in (0,\infty)$, there exists a nonnegative random variable $C(\cdot,z) \in L^\infty(0,T)$ for each $z \in \Omega$ such that

$$\|f(t,z) - \tilde{f}(t,z)\|_{L^\infty_{x,v}} \le C(t,z)\|f^0(z) - \tilde{f}^0(z)\|_{L^\infty_{x,v}}, \quad t \in (0,T).$$

Proof. From Theorem 3.1 in [20], for each $z \in \Omega$, there uniquely exist $\mathcal{C}^1$-regular processes $f(t,z)$ and $\tilde{f}(t,z)$ in $\mathcal{C}^1(\mathbb{R}^{2d} \times [0,T))$ whose initial processes are $f^0(z)$ and $\tilde{f}^0(z)$ respectively, and

$$\begin{aligned} \mathcal{F}^0(t,z) &:= \sum_{0 \le |\alpha|+|\beta| \le 1} \|\partial_v^\alpha \partial_x^\beta f(t,z)\|_{L^\infty_{x,v}} < \infty, \\ \tilde{\mathcal{F}}^0(t,z) &:= \sum_{0 \le |\alpha|+|\beta| \le 1} \|\partial_v^\alpha \partial_x^\beta \tilde{f}(t,z)\|_{L^\infty_{x,v}} < \infty. \end{aligned}$$

Now, by using (1.1) one gets

$$\partial_t(f - \tilde{f}) + v \cdot \nabla_x(f - \tilde{f}) + \nabla_v \cdot (F_a[f] - F[\tilde{f}])\tilde{f}) + \nabla_v \cdot (F_a[f](f - \tilde{f})) = 0. \tag{4.13}$$

Then

$$\begin{aligned} |\nabla_v \cdot (F_a[f] - F[\tilde{f}])| &\le d \int_{\mathbb{R}^{2d}} \psi(x - x_*, z)|f(\sigma_*, z) - \tilde{f}(\sigma_*, z)| d\sigma_* \\ &\le C_1(t,z)\|f(t,z) - \tilde{f}(t,z)\|_{L^\infty_{x,v}}, \\ |F_a[f] - F[\tilde{f}]| &\le \int_{\mathbb{R}^{2d}} \psi(x - x_*, z)|v - v_*\|f(\sigma_*, z) - \tilde{f}(\sigma_*, z)| d\sigma_* \\ &\le C_2(t,z)\|f(t,z) - \tilde{f}(t,z)\|_{L^\infty_{x,v}}, \end{aligned} \tag{4.14}$$

where $C_i(\cdot,z) = O(t^{3d+1})$ for each $z \in \Omega$ and $i = 1, \ 2$ by Lemma 3.1.

Now by integrating (4.13) using the relations (4.14) one gets

$$\|(f - \tilde{f})(t,z)\|_{L^\infty_{x,v}} \le \|(f^0 - \tilde{f}^0)(z)\|_{L^\infty_{x,v}} + \int_0^t C(s,z)\|(f - \tilde{f})(s,z)\|_{L^\infty_{x,v}} ds.$$

Note that $C(s,z) \in L^\infty(0,T)$ for each $z \in \Omega$ and its estimate can be followed from Lemma 3.1, Remark 3.1 and the estimate of $\mathcal{F}^0$ and $\tilde{\mathcal{F}}^0$, which has exponential order in t for each $z \in \Omega$. Hence by using Grönwall's lemma one obtains the desired result. $\square$

Similarly, we present the local sensitivity analysis.

Theorem 4.4. *For each $z \in \Omega$ and $m \geq 2$, assume that two initial processes $f^0(z) := f(0,z)$ and $\tilde{f}^0(z) := \tilde{f}(0,z)$ satisfy the following conditions:*

(1) z-derivatives of initial processes $\partial_z^l f^0(z)$ and $\partial_z^l \tilde{f}^0(z)$ are compactly supported in the phase space for each $l = 0,\ 1,\ \cdots,\ m$ and each $z \in \Omega$, i.e.

$$\sum_{0\leq l\leq m} P_l(0,z) + \tilde{P}_l(0,z) + R_l(0,z) + \tilde{R}_l(0,z) < \infty, \quad \textit{for each} \ \ z \in \Omega.$$

(2) z-derivatives of initial processes $\partial_z^l f^0(z)$ and $\partial_z^l \tilde{f}^0(z)$ are $\mathcal{C}^{m-l+1}$-regular and bounded for each $l = 0,\ 1,\ \cdots,\ m$ and each $z \in \Omega$, i.e.

$$\sum_{\substack{0\leq l\leq m\\ 0\leq|\alpha|+|\beta|\leq m-l+1}} \|\partial_v^\alpha \partial_x^\beta \partial_z^l f^0(z)\|_{L^\infty_{x,v}} + \|\partial_v^\alpha \partial_x^\beta \partial_z^l \tilde{f}^0(z)\|_{L^\infty_{x,v}} < \infty.$$

(3) The communication weight ψ belongs to $W^{m,\infty}(\mathbb{R}^d \times \Omega)$.

Then, for any $T \in (0,\infty)$, there exists a nonnegative random variable $C(T,z)$ such that

$$\sum_{l=0}^m \left\|\partial_z^l f(t,z) - \partial_z^l \tilde{f}(t,z)\right\|_{L^\infty_{x,v}} \leq C(T,z) \sum_{l=0}^m \|\partial_z^l f^0(z) - \partial_z^l \tilde{f}^0(z)\|_{L^\infty_{x,v}}, \quad t \in (0,T).$$

Proof. It follows from Theorem 4.2 that, for each $l = 0,\ 1,\ \cdots, m$, there exist unique $\mathcal{C}^{m-l+1}$ regular solution processes $\partial_z^l f(t,z)$ and $\partial_z^l \tilde{f}(t,z)$ to (4.1) corresponding to initial processes $\partial_z^l f^0(z)$ and $\partial_z^l \tilde{f}(z)$ respectively.

Then, it follows from (4.1) that

$$\begin{aligned} &\partial_t \partial_z^l (f-\tilde{f}) + v\cdot\nabla_x \partial_z^l (f-\tilde{f}) \\ &\qquad + \nabla_v \cdot \left(\sum_{\nu=0}^l \binom{l}{\nu} \partial_z^\nu (F_a[f] - F[\tilde{f}])\partial_z^l \tilde{f} + \partial_z^\nu F_a[f] \partial_z^{l-\nu}(f-\tilde{f})\right) = 0. \end{aligned} \tag{4.15}$$

Note that the following estimates hold for each ν;

$$\begin{aligned} |\partial_z^\nu (F_a[f] - F[\tilde{f}])| &\leq \sum_{r=0}^\nu \binom{\nu}{r} \int_{\mathbb{R}^{2d}} |v - v_*\|\partial_z^r \psi(x - x_*, z) \partial_z^{\nu-r}(f(\sigma_*, z) - \tilde{f}(\sigma_*, z))| d\sigma_* \\ &\leq C(t,z) \sum_{r=0}^\nu \|\partial_z^r (f-\tilde{f})(t,z)\|_{L^\infty_{x,v}}, \\ |\nabla_v \cdot \partial_z^\nu (F_a[f] - F[\tilde{f}])| &\leq d \sum_{r=0}^\nu \binom{\nu}{r} \int_{\mathbb{R}^{2d}} |\partial_z^r \psi(x - x_*, z) \partial_z^{\nu-r}(f(\sigma_*, z) - \tilde{f}(\sigma_*, z))| d\sigma_* \\ &\leq C(t,z) \sum_{r=0}^\nu \|\partial_z^r (f-\tilde{f})(t,z)\|_{L^\infty_{x,v}}, \end{aligned}$$

where $C(\cdot, z) = O(t^{3d+1})$ and non-decreasing in t for each $z \in \Omega$ by Lemma 4.2. Define $\mathcal{F}^\nu(t,z)$ and $\tilde{\mathcal{F}}^\nu(t,z)$ as (4.10). Thus, by integrating (4.15) over the particle trajectory of $\partial_z^l f$ one obtains

$$\|\partial_z^l (f-\tilde{f})(t,z)\|_{L^\infty_{x,v}} \le \|\partial_z^l (f^0-\tilde{f}^0)(z)\|_{L^\infty_{x,v}} + \int_0^t \tilde{C}(s,z) \sum_{\nu=0}^{l} \|\partial_z^\nu (f-\tilde{f})(s,z)\|_{L^\infty_{x,v}} ds,$$

where $t \in (0,T)$ and $\tilde{C}(\cdot, z) = O(s^{3d+1})$ and non-decreasing in s for each $z \in \Omega$ by Lemma 4.1, Remark 4.1 and Lemma 4.2. Since this holds for each $l = 0, 1, \cdots, m$, one can get

$$\sum_{l=0}^{m} \|\partial_z^l (f-\tilde{f})(t,z)\|_{L^\infty_{x,v}} \le \sum_{l=0}^{m} \|\partial_z^l (f^0-\tilde{f}^0)(z)\|_{L^\infty_{x,v}} + m\int_0^t \tilde{C}(s,z) \sum_{l=0}^{m} \|\partial_z^l (f-\tilde{f})(s,z)\|_{L^\infty_{x,v}} ds.$$

Then, by using Grönwall's lemma one obtains

$$\sum_{l=0}^{m} \|\partial_z^l (f-\tilde{f})(t,z)\|_{L^\infty_{x,v}} \le e^{m\int_0^t \tilde{C}(s,z)ds} \sum_{l=0}^{m} \|\partial_z^l (f^0-\tilde{f}^0)(z)\|_{L^\infty_{x,v}}.$$

We set $C(T,z) := e^{m\int_0^T \tilde{C}(s,z)ds}$ to obtain the desired estimate. $\square$

4.3. **Uniform bound in $H^m_\pi(L^\infty_{x,v})$-norm.** In this subsection, we provide L^2-estimates by using L^∞-estimates for $\partial_z^m f$. To be precise, we estimate $\partial_z^m f$ in a space $L^2_\pi(\Omega; L^\infty_{x,v}(\mathbb{R}^{2d}))$.

Proposition 4.1. *For $T \in (0,\infty)$ and $m \ge 2$, suppose the following conditions:*

(1) For every l =0, 1, $\cdots$, m, z-derivatives of initial process $\partial_z^l f^0$ have compact velocity and position support which are uniform in z, i.e.

$$\sum_{0\le l\le m} \sup_{z\in\Omega} \left(P_l(0,z) + R_l(0,z)\right) < \infty.$$

(2) For every l =0, 1, $\cdots$, m, z-derivatives of initial process $\partial_z^l f^0$ are $\mathcal{C}^{m-l+1}$-regular for each $z \in \Omega$ and belongs to $(L^2_\pi \cap L^\infty)(\Omega; W^{m-l+1,\infty}(\mathbb{R}^{2d}))$, i.e.

$$\sum_{0\le l\le m} \|\partial_z^l f\|_{(L^2_\pi\cap L^\infty)(\Omega; W^{m-l+1,\infty}(\mathbb{R}^{2d}))} < \infty.$$

(3) The communication weight ψ belongs to $W^{m,\infty}(\mathbb{R}^d \times \Omega)$.

Then one can obtain

$$\|f(t)\|^2_{H^m_\pi(L^\infty_{x,v})} \le C(T)\|f^0\|^2_{H^m_\pi(L^\infty_{x,v})}, \quad t \in (0,T).$$

Proof. Multiplying (4.1) by $\partial_z^m f$ and $\pi(z)$, and then integrating the resulting relation over Ω, give

$$\begin{aligned} &\frac{1}{2}\left(\frac{\partial}{\partial t}\|\partial_z^m f\|^2_{L^2_\pi(\Omega)} + v\cdot\nabla_x \|\partial_z^m f\|^2_{L^2_\pi(\Omega)} + F_a[f]\cdot\nabla_v \|\partial_z^m f\|^2_{L^2_\pi(\Omega)}\right) \\ &\quad = -\sum_{1\le l\le m} \binom{m}{l} \int_\Omega \nabla_v \cdot \left(\partial_z^l F_a[f] \partial_z^{m-l} f\right) \partial_z^m f(\sigma,z)\pi(z)dz \\ &\qquad - \int_\Omega \nabla_v \cdot F_a[f] |\partial_z^m f|^2 \pi(z) dz. \end{aligned} \tag{4.16}$$

Note that the coefficients in the R.H.S. of (4.16) can be estimated using Lemma 4.3: for $0 \le l \le m$,

$$
\begin{aligned}
&|\partial_z^l F_a[f]| \le 2\|\psi\|_\infty \sum_{0\le\nu\le l}\left(\binom{l}{\nu} P_\nu(t,z)\,(P_\nu(t,z)R_\nu(t,z))^d\,\|\partial_z^p f(z)\|_{L^\infty_{x,v}}\right),\\
&|\nabla_v\cdot\partial_z^l F_a[f]| \le d\|\psi\|_\infty \sum_{0\le\nu\le l}\left(\binom{l}{\nu}(P_\nu(t,z)R_\nu(t,z))^d\,\|\partial_z^p f(z)\|_{L^\infty_{x,v}}\right).
\end{aligned}
\tag{4.17}
$$

Set

$$
\mathcal{F}^l(t,z) := \sum_{0\le|\alpha|+|\beta|\le m-l+1}\|\partial_v^\alpha\partial_x^\beta\partial_z^l f(z)\|_{L^\infty_{x,v}}.
$$

One can also deduce from Theorem 4.2 that z-dependence of $\mathcal{F}^l(t,z)$ is given by $\mathcal{F}^\nu(0,z)$, $P_\nu(0,z)$ and $R_\nu(0,z)$ for all $0\le\nu\le l$. Thus, by using the uniform boundedness of $P_\nu(0,z)$, $R_\nu(0,z)$ and $\partial_z^\nu f^0$ in the assumption, one gets

$$
\sum_{0\le l\le m}|\nabla\cdot\partial_z^l F_a[f]| + |\nabla_v\partial_z^l f| \le C(t),
\tag{4.18}
$$

where $C(t)$ is a constant that is finite for every $0\le t\le T$. On the other hand, using Young's inequality gives

$$
\begin{aligned}
&\left|\int_\Omega\left(\nabla_v\cdot\partial_z^l F_a[f]\right)\partial_z^{m-l}f(\sigma,z)\partial_z^m f(\sigma,z)\pi(z)dz\right|\\
&\qquad\qquad \le C(t)\left(\|\partial_z^{m-l}f\|^2_{L^2_\pi(\Omega)} + \|\partial_z^m f\|^2_{L^2_\pi(\Omega)}\right),\\
&\left|\int_\Omega \partial_z^l F_a[f]\cdot\nabla_v\partial_z^{m-l}f(\sigma,z)\partial_z^m f(\sigma,z)\pi(z)dz\right|\\
&\qquad\qquad \le C(t)\left(\|\partial_z^m f\|^2_{L^2_\pi(\Omega)} + \sum_{1\le l\le m}\|\partial_z^l f\|^2_{L^2_\pi(L^\infty_{x,v})}\right).
\end{aligned}
\tag{4.19}
$$

Now combining estimates (4.17), (4.18) and (4.19) one gets

$$
\frac{1}{2}\left(\frac{\partial}{\partial t}\|\partial_z^m f\|^2_{L^2_\pi(\Omega)} + v\cdot\nabla_x\|\partial_z^m f\|^2_{L^2_\pi(\Omega)} + F_a[f]\cdot\nabla_v\|\partial_z^m f\|^2_{L^2_\pi(\Omega)}\right) \le C(t)\sum_{0\le l\le m}\|\partial_z^l f\|^2_{L^2_\pi(L^\infty_{x,v})}.
$$

Integration this inequality over the characteristic curve $(x(t;0,x,v), v(t;0,x,v))$ of $\|\partial_z^m f\|^2_{L^2_\pi(\Omega)}$ leads to

$$
\begin{aligned}
&\|\partial_z^m f\|^2_{L^2_\pi(\Omega)}(x,v)\\
&\qquad \le \|\partial_z^m f^0\|^2_{L^2_\pi(\Omega)}(x(0),v(0)) + \int_0^t C(s)\sum_{0\le l\le m}\|\partial_z^l f\|^2_{L^2_\pi(L^\infty_{x,v})}ds\\
&\qquad \le \|\partial_z^m f^0\|^2_{L^2_\pi(L^\infty_{x,v})} + \int_0^t C(s)\sum_{0\le l\le m}\|\partial_z^l f\|^2_{L^2_\pi(L^\infty_{x,v})}ds.
\end{aligned}
$$

This implies

$$
\|\partial_z^m f\|^2_{L^2_\pi(L^\infty_{x,v})} \le \|\partial_z^m f^0\|^2_{L^2_\pi(L^\infty_{x,v})} + \int_0^t C(s)\sum_{0\le l\le m}\|\partial_z^l f\|^2_{L^2_\pi(L^\infty_{x,v})}ds.
\tag{4.20}
$$

Finally, we sum the above inequality (4.20) over m to obtain

$$\|f\|^2_{H^m_\pi(L^\infty_{x,v})} \le \|f^0\|^2_{H^m_\pi(L^\infty_{x,v})} + \int_0^t C(s)\|f\|^2_{H^m_\pi(L^\infty_{x,v})} ds.$$

Then, we apply Grönwall's lemma to complete the proof. □

5. Conclusion

In this paper, we provided a local sensitivity analysis for the kinetic C-S equation with random inputs. In earlier works, research on the continuous C-S model with random effects focus on the white noise perturbations as external stochastic forces. In the current work, we did not assume any specific dependence of randomness in the system dynamics. In the first two authors' recent work [21] on the particle C-S model with random inputs, systematic local sensitivity analysis was performed for the C-S flocking model with finite size of ensemble. Thus, a natural extension of the previous work is to figure out whether similar local sensitivity analysis can be done for the infinite system. In this work, we have assumed the structural symmetric condition in the communication weight even for the presence of random inputs. In fact, this strong structural constraint results in the robustness of the flocking estimates independent of the size of the C-S ensemble. However, in general situation, uncertainty effect might screw up the symmetry of the communication. Thus, it results in the non-existence of global flocking as in the short-ranged communications even for the deterministic C-S equation. The direction toward the destabilizing random effects on the collective dynamics will be an interesting issue for a future research, at both the theoretical and numerical levels. One may also consult some recent numerical studies on uncertain flocking models in [3, 6].

There are many other related models for collective dynamics, decision making and self-organization in complex system in biological and social sciences [45], which are also of great interest to study the influence of uncertain random effects in various input parameters in the models.

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