ON A UNIFORMLY SECOND ORDER NUMERICAL METHOD FOR THE ONE-DIMENSIONAL DISCRETE-ORDINATE TRANSPORT EQUATION AND ITS DIFFUSION LIMIT WITH INTERFACE*

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Abstract. In this paper, we study a uniformly second order numerical method for the discrete-ordinate transport equation in the slab geometry in the diffusive regimes with interfaces. At the interfaces, the scattering coefficients have discontinuities, so suitable interface conditions are needed to define the unique solution. We first approximate the scattering coefficients by piecewise constants determined by their cell averages, and then, following the work of De Barros and Larsen [12], obtain the analytic solution at each cell, using which to piece together the numerical solution with the neighboring cells using the interface conditions. We show that this method is asymptotic-preserving, which preserves the discrete diffusion limit with the correct interface condition. Moreover, we show that this method is quadratically convergent *uniformly* in the diffusive regime, even with the boundary layers. This is 1) the first *sharp* uniform convergence result for linear transport equations in the diffusive regime, a problem that involves both transport and diffusive scales; and 2) the first uniform convergence *valid up to the boundary* even if the boundary layers exist, so the boundary layer does not need to be resolved numerically. Numerical examples are presented to justify the uniform convergence.

Key words. Linear transport equation, discrete-ordinate method, diffusion limit, interface, asymptotic preserving, uniform numerical convergence, boundary layer

AMS subject classifications. 65D25, 70B05, 65L12

1. Introduction.

The transport equation with discontinuous coefficients is important since it arises in many important applications, ranging from neutron transport, radiative transfer, high frequency waves in heterogeneous and random media, semiconductor device simulation, to quantum mechanics. We consider particles in a bounded domain that interact with a background through absorption and scattering processes. In highly diffusive media, the mean free path (the average distance a particle travels between interaction with the background media) is small compared to the typical length scales. This small ratio is embodied by the introduction of a dimensionless parameter ϵ into the transport equation. In one space dimension, with z_L, z_R being the left and right boundaries respectively, the phase space density $\Psi(z,\mu)$ over $[z_L, z_R] \times [-1,1]$ is governed by a scaled linear transport equation:

$$\mu\partial_z\Psi + \frac{\sigma_T}{\epsilon}\Psi = \Big(\frac{\sigma_T}{\epsilon} - \epsilon\sigma_a\Big)\frac{1}{2}\int_{-1}^{1}\Psi(z,\mu')d\mu' + \epsilon q, \tag{1.1a}$$

with boundary conditions

$$\Psi(z_L,\mu) = \Psi_L(\mu), \quad \mu > 0\,; \qquad \Psi(z_R,\mu) = \Psi_R(\mu), \quad \mu < 0. \tag{1.1b}$$

Here σ_T, σ_a, q are the total scattering and absorption coefficients and the source respectively. They are $O(1)$ and have $O(1)$ derivatives with respect to z, *except at*

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finite number of points. Namely, they are piecewise smooth. Their discontinuities correspond to *interfaces* between different materials or media.

Clearly (1.1) does not admit a unique solution when the coefficients σ_T, σ_a, q are discontinuous. Interface conditions are needed to select the unique solution. Assume the interface is located at $z = 0$. In most physical applications, Ψ is continuous crossing the interface:

$$[\Psi(\cdot, \mu)]\Big|_{z=0} = 0 \tag{1.2}$$

where we use $[\Psi]$ to present the jump of Ψ at a point.

The starting point of numerical methods for solving the transport equation is the discrete-ordinate method [8, 28, 32], which uses a quadrature formula to approximate the scattering term. Let

$$V = \{-M, \cdots, -1, 1, \cdots, M\}.$$

The discrete-ordinates approximation to (1.1) is

$$\mu_m \frac{d\psi_m}{dz}(z) + \frac{\sigma_T(z)}{\epsilon}\psi_m(z) = \left(\frac{\sigma_T(z)}{\epsilon} - \epsilon\sigma_a(z)\right)\sum_{n\in V}\psi_n(z)w_n + \epsilon q(z)\,, \quad m \in V \tag{1.3a}$$

with boundary conditions

$$\psi_m(z_L) = \psi_{Lm} = \Psi_L(\mu_m); \qquad \psi_{-m}(z_R) = \psi_{Rm} = \Psi_R(-\mu_m)\,, \quad \mu_m > 0, \tag{1.3b}$$

and the interface condition

$$[\psi_m]\Big|_{z=0} = 0\,. \tag{1.3c}$$

Here $\{\mu_m, w_m | m \in V\}$ is an order-2M quadrature set with weight w_m normalized as

$$\sum_{n\in V} w_n = 1\,, \quad \sum_{n\in V} w_n\mu_n^2 = 1/3\,. \tag{1.4}$$

μ_m are ordered as

$$1 > \mu_M > \mu_{M-1} > \cdots > \mu_1 > 0 > \mu_{-1} > \mu_{-2} > \cdots > \mu_{-M+1} > \mu_{-M} > -1 \tag{1.5}$$

and are symmetric:

$$\mu_m = -\mu_{-m}, \qquad w_m = w_{-m}, \qquad m \in V. \tag{1.6}$$

The quadrature set is studied in [19, 20] and the commonly used $\{\mu_m, w_m\}$ satisfies conditions (1.4)-(1.6). In this paper we will concentrate on the discrete ordinate equations (1.3) and always use ψ to denote $(\psi_{-M}, \cdots, \psi_M)^T$.

Even when the scattering coefficients are smooth, numerical solutions to the transport equation or the discrete-ordinate equations are challenging when $\epsilon << 1$ since it requires the numerical resolution of the small scale, as in a typical multiscale problem. Asymptotic-preserving (AP) schemes have been proved rather successful for such problems. A scheme for such a problem is AP if it preserves the diffusion limit of the transport equation or its discrete-ordinate approximation, when $\epsilon \to 0$, at the discrete level. This idea was first advocated in the work of Larsen, Morel and Miller

[30, 31] for steady transport equations, and then generalized by Jin and Levermore for boundary-value problems [19, 20]. It was systematically generalized to time dependent transport equations in the diffusive regimes by Jin, Klar, Pareschi, Toscani, etc., see [18, 22, 23, 27], and for hyperbolic systems with stiff relaxations, see [7, 17, 21]. If a method is AP, the correct (macroscopic) physical behavior, valid when the small parameter ϵ is small, can be captured even if the numerical computation is *underresolved* (the mesh size and time step much larger than the small space/time parameters). It was proved in [14] for the linear transport equation with boundary conditions that an AP scheme converges *uniformly* with respect to ϵ. The convergence rate obtained in [14] is not sharp, due to technical reasons. For more recent development of AP schemes for plasmas and fluids, see [9, 11, 10].

Compared to a domain decomposition approach, which couples the transport and diffusion equations through interface coupling conditions, the AP approach avoids the difficulty of finding the coupling condition, which is the bottleneck for most of the multiscale methods.

When there is an interface, corresponding to the discontinuity of the scattering coefficients, the AP method should also capture the correct interface condition for all ϵ. Depending on the applications, there are two kinds of interfaces for this problem. One often arises in neutron transport equation, where the density Ψ is continuous at the interface. The other one arises in radiative transfer equations as an approximation of high frequency waves in random and heterogeneous media, where the energy flux (corresponding to the μ-integral of $\mu\Psi$ here) is continuous at the interface. In this paper we will focus on the first case. For recent works for the second case, we refer to [1, 24, 25].

In this paper, we propose a uniformly second order numerical method for the discrete-ordinate equations with boundary and interface conditions (1.3). The idea of this method is elicited from an exponential tridiagonal difference scheme for the singular perturbation parabolic problems [4, 5, 26], and the "spectral Green's function" method by De Barros and Larsen [12]. The idea is to first approximate the coefficients –here they are the scattering coefficients and source–by their cell averages in each grid cell. Such a constant coefficient problem in each cell can then be solved *analytically* by piecing them together with neighboring cells using the interface conditions, following the idea of [12], thus resulting a finite difference scheme for the entire domain. Philosophically this is similar to the Godunov method for hyperbolic system of conservation laws [15], which uses the exact solution of the Riemann problem in each cell.

The major contribution of this paper is to prove that this method is AP and second order, *uniformly in* ϵ, convergent to the discrete-ordinate equations with boundary and interfaces (1.3). Even for smooth coefficients, this result is stronger, in two aspects, than the only available uniform convergence result in this direction by Golse, Jin and Levermore in [14]. First, using the technique of [14] which uses the diffusion limit directly in the proof, the uniform convergence rate is only *half* of the rate for fixed ϵ. This is purely a technical restriction rather than the actual numerical performance of an AP method. Our result is *sharp*, namely, we prove the quadratic convergence in ϵ for this method, which is the same order as the order with a fixed ϵ. Second, most of the numerical methods for the same problem needs to resolve the boundary layer, while we here prove that our method is uniformly second order *all the way to the boundary*, namely, *the boundary layer does not need to be resolved numerically.*

Our scheme and its analysis rely heavily on the eigenfunction expansion of the constant coefficient, one-dimensional discrete-ordinate equations.

An outline of the remainder of this article follows. An interface analysis is given in section 2 to derive the interface condition for the diffusion limit for the discrete-ordinate equation with discontinuous coefficients. In section 3 the explicit form of this method is derived and proved to be AP. The uniform quadratic convergence with respect to ϵ is proved in section 4. Numerical examples are presented in section 5 to test and confirm our theoretical results. Finally, we conclude with a discussion in section 6.

2. The diffusion limit with interface.

When $\epsilon \to 0$, the classical asymptotic and boundary layer analysis for (1.3) give [19, 30]

$$\psi_m(z) = \phi(z) - \epsilon\frac{\mu_m}{\sigma_T}\partial_z\phi + O(\epsilon^2) \tag{2.1}$$

where ϕ satisfies the diffusion equation:

$$-\frac{d}{dz}\frac{1}{3\sigma_T(z)}\frac{d}{dz}\phi(z) + \sigma_a\phi(z) = q(z) \tag{2.2a}$$

with the boundary conditions

$$\begin{aligned} \phi - \epsilon\tfrac{\lambda}{\sigma_T}\partial_z\phi\Big|_{z=z_L} &= \textstyle\sum_{m=1}^{M}\psi_{Lm}\alpha_m, \\ \phi + \epsilon\tfrac{\lambda}{\sigma_T}\partial_z\phi\Big|_{z=z_R} &= \textstyle\sum_{m=-1}^{-M}\psi_{Rm}\alpha_m. \end{aligned} \tag{2.2b}$$

Here the extrapolation length λ is given by

$$\lambda = \sum_{m=1}^{M}\mu_m - \sum_{m=1}^{M-1}\frac{1}{\xi_m} \tag{2.3}$$

and the discrete H-function α_m is

$$\alpha_m = \prod_{n=1}^{M-1}\left(\mu_m - \frac{1}{\xi_n}\right)\prod_{k=1,k\neq m}^{M}\frac{1}{\mu_m - \mu_k} \tag{2.4}$$

with $\xi_k \in (1/\mu_{k+1}, 1/\mu_k)$ $(1 \le k \le M-1)$ the unique (positive, simple) root of

$$1 = \sum_{m\in V} w_m l_m^{(k)}, \tag{2.5}$$

where

$$l_m^{(k)} = \frac{1}{1-\mu_m\xi_k}. \tag{2.6}$$

For $-M+1 \le k \le -1$, let $\xi_k = -\xi_{-k}$ and $l_m^{(k)}$ the same as in (2.6). $l_m^{(k)}$ have the following properties:

$$\sum_{m\in V}\mu_m w_m l_m^{(k)} = 0, \qquad \sum_{m\in V}\mu_m^2 w_m l_m^{(k)} = 0\,, \tag{2.7}$$

and

$$\sum_{m\in V} w_m\mu_m l_m^{(k)} l_m^{(n)} = \left\{\begin{array}{cc} 0 & n\neq k \\ c^{(k)}\neq 0 & n=k \end{array}\right. . \tag{2.8}$$

Let $c^{(-M)} = c^{(M)} = 1/3$, then $c^{(k)}$ in (2.8) satisfy

$$\sum_{1\le|k|\le M-1} \frac{1}{c^{(k)}} l_n^{(k)} l_m^{(k)} + \frac{\mu_n}{c^{(-M)}} + \frac{\mu_m}{c^{(M)}} = \begin{cases} 0 & m \ne n \\ \frac{1}{w_n \mu_n} & m = n \end{cases} . \tag{2.9}$$

These properties of $l_m^{(k)}$ play an important role in the derivation of this numerical method and the consequent uniform convergence proof. Their proof is given in the Appendix.

One of the main numerical difficulties to solve the transport equation (1.3), when $\epsilon \ll 1$, is the demand that the mesh size $\Delta z \ll \epsilon$. This is prohibitively expensive especially in higher dimensions. However, when $\epsilon \ll 1$, the transport equation (1.3) is well approximated by the diffusion equation (2.2) which is much more efficient for numerical approximation.

At the interface, an interface condition should be provided for the diffusion equation. This condition can be found using the analysis similar to the boundary layer analysis that gives (2.2b). When there is an interface between two different media where σ_T, σ_a and q have jump discontinuities, one can expect that two boundary layers exist on both sides of the interface. The interface conditions should be derived by matching the interface condition (1.3c) with the diffusion approximation valid away from the interface. Consider the interface condition

$$[\psi_m]\big|_{z=0} = 0\,, \qquad m \in V. \tag{2.10}$$

Applying the stretching transformation on both sizes of $x = 0$,

$$x^{(1)} = \frac{1}{\epsilon}\int_0^z \sigma_T(s)\,ds, \quad z > 0; \qquad x^{(2)} = \frac{1}{\epsilon}\int_0^z \sigma_T(s)\,ds, \quad z < 0,$$

to (1.3a) respectively gives

$$\psi_m(z) = \psi_m^{(1)}(x^{(1)}) + O(\epsilon^2), \quad z > 0; \qquad \psi_m(z) = \psi_m^{(2)}(x^{(2)}) + O(\epsilon^2), \quad z < 0,$$

where ψ^i satisfies

$$\mu_m \partial_{x^{(i)}} \psi_m^{(i)} + \psi_m^{(i)} - \sum_{n\in V} w_n \psi_n^{(i)} = 0, \qquad \text{in} \quad X^{(i)} \times V. \tag{2.11}$$

Here $i = 1, 2$, $X^{(1)} = \mathbb{R}^+$, $X^{(2)} = \mathbb{R}^-$.

The interface condition for (2.11) corresponding to (2.10) is

$$\psi_m^{(1)}(0^-) = \psi_m^{(2)}(0^+), \qquad m \in V. \tag{2.12}$$

The differential equation (2.11) on $X^{(i)}$, being a transport equation with constant coefficients, can be solved exactly [19]:

$$\psi_m^{(i)}(x^{(i)}) = c_1^{(i)} + c_2^{(i)}(x^{(i)} - \mu_m) + \sum_{1\le|k|\le M-1} A^{(i,k)} l_m^{(k)} \exp(-\xi_k x^{(i)}), \tag{2.13}$$

where the eigenvalue ξ_k and corresponding eigenfunction $l_m^{(k)}$ are defined in (2.5) and (2.6).

The coefficients $c_1^{(i)}$, $c_2^{(i)}$, $A^{(i,k)}$ can be determined by $\psi_m^{(i)}(0)$ through the following boundary conditions:

$$\psi_m^{(i)}(0^{\pm}) = c_1^{(i)} - \mu_m c_2^{(i)} + \sum_{1\le|k|\le M-1} A^{(i,k)} l_m^{(k)}.$$

The interface condition (2.12) implies

$$(c_1^{(1)} - c_1^{(2)}) - \mu_m(c_2^{(1)} - c_2^{(2)}) + \sum_{1\le|k|\le M-1} \big(A^{(1,k)} - A^{(2,k)}\big) l_m^{(k)} = 0.$$

Since the eigenfunctions 1, μ_m, $l_m^{(k)}$ $(1 \le |k| \le M-1)$ are linearly independent (this will be proved in the Appendix), one gets

$$c_1^{(1)} = c_1^{(2)}, \quad c_2^{(1)} = c_2^{(2)}, \quad A^{(1,k)} = A^{(2,k)}, \quad \text{for } 1 \le |k| \le M-1. \tag{2.14}$$

Furthermore, since the diffusion approximation contains no growing exponentials on both sides, one must have

$$A^{(1,n)} = 0, \quad -M+1 \le n \le -1; \qquad A^{(2,n)} = 0, \quad 1 \le n \le M-1. \tag{2.15}$$

Thus

$$\psi_m^{(i)}(x^{(i)}) = c_1^{(i)} + c_2^{(i)}(x^{(i)} - \mu_m). \tag{2.16}$$

We now match these solution with the interior (diffusion) solution. Away from the interface, the solution of (1.3a) can be expressed as

$$\psi_m^{(i)}(z) = \phi^{(i)}(z) - \epsilon\frac{\mu_m}{\sigma_T}\partial_z\phi^{(i)}(z) + O(\epsilon^2).$$

Comparing this with (2.16) at, say, $x^{(i)} = O(\epsilon^\beta)$ with $\beta > 1$ on both side of the interface, and note that $\partial_x = \frac{\epsilon}{\sigma_T}\partial_z$, gives ([19])

$$\begin{aligned} &c_1^{(1)} = \phi^{(1)}(0^+), \qquad c_1^{(2)} = \phi^{(2)}(0^-);\\ &c_2^{(1)} = \frac{\epsilon}{\sigma_T(0^+)}\partial_z\phi^{(1)}(0^+), \qquad c_1^{(2)} = \frac{\epsilon}{\sigma_T(0^-)}\partial_z\phi^{(2)}(0^-). \end{aligned}$$

Now (2.14) yields the interface conditions of the diffusion limit

$$\phi^{(1)}(0^+) = \phi^{(2)}(0^-), \quad \frac{1}{\sigma_T(0^+)}\partial_z\phi^{(1)}(0^+) = \frac{1}{\sigma_T(0^-)}\partial_z\phi^{(2)}(0^-). \tag{2.17}$$

In summary the diffusion limit with interface at $x = 0$ is

$$-\frac{d}{dz}\Big(\frac{1}{3\sigma_T}\phi_z\Big) + \sigma_a\phi = q \tag{2.18a}$$

with the boundary condition

$$\begin{aligned} \phi - \epsilon\tfrac{\lambda}{\sigma_T}\partial_z\phi\Big|_{z=z_L} &= \textstyle\sum_{m=1}^{M} \psi_{Lm}\alpha_m,\\ \phi + \epsilon\tfrac{\lambda}{\sigma_T}\partial_z\phi\Big|_{z=z_R} &= \textstyle\sum_{m=-1}^{-M} \psi_{Rm}\alpha_m\,, \end{aligned} \tag{2.18b}$$

and the interface condition

$$[\phi]\big|_{x=0} = \Big[\frac{1}{\sigma_T}\partial_z\phi\Big]\Big|_{x=0} = 0. \tag{2.18c}$$

3. The numerical method and its AP property.

3.1. Derivation of the method. Assume σ_a, σ_T, q are discontinuous at some discrete set of points. Generate a set of grid points $z_L = z_0 < z_1 < \cdots < z_N = z_R$, so all discontinuities of σ_a, σ_T, q are grid points. Let $\triangle z = \max_{i=0,\cdots,N-1} |z_{i+1} - z_i|$. Define the *cell averages*

$$\sigma_{ai} = \frac{1}{z_{i+1}-z_i}\int_{z_i}^{z_{i+1}} \sigma_a(z)dz, \quad \sigma_{Ti} = \frac{1}{z_{i+1}-z_i}\int_{z_i}^{z_{i+1}} \sigma_T(z)dz,$$

$$q_i = \frac{1}{z_{i+1}-z_i}\int_{z_i}^{z_{i+1}} q(z)dz, \quad i = 0, \cdots, N-1,$$

and $\tilde{\sigma}_a$, $\tilde{\sigma}_T$, $\tilde{q}$ to be *piecewise constant* functions:

$$\tilde{\sigma}_a(z) = \sigma_{ai}, \quad \tilde{\sigma}_T(z) = \sigma_{Ti}, \quad \tilde{q}(z) = q_i, \qquad z \in (z_i, z_{i+1}], \qquad i = 0, \ldots, N-1. \tag{3.1}$$

We get an approximation of the discrete ordinate equation (1.3a):

$$\mu_m \frac{d\tilde{\psi}_m}{dz}(z) + \frac{\tilde{\sigma}_T(z)}{\epsilon}\tilde{\psi}_m(z) = \left(\frac{\tilde{\sigma}_T(z)}{\epsilon} - \epsilon\tilde{\sigma}_a(z)\right)\sum_{n\in V}\tilde{\psi}_n(z)w_n + \epsilon\tilde{q}(z), \quad m \in V, \tag{3.2}$$

with the same boundary condition (1.3b). The rest of the method pretty much follows the framework of "spectral Green's function" method by De Barros and Larsen [12] (see also later works [3, 13], except that we are using the explicit eigenvalues and eigenfunctions and the corresponding discrete W-functions derived in [19].

Note that on $[z_i, z_{i+1}]$, (3.2) is the constant coefficient discrete-ordinate equation, thus can be solved exactly like in [19]. In order to find the solution of

$$\mu_m \frac{d\psi_m}{dz}(z) + \frac{\sigma_{Ti}}{\epsilon}\psi_m(z) = \left(\frac{\sigma_{Ti}}{\epsilon} - \epsilon\sigma_{ai}\right)\sum_{n\in V}\psi_n(z)w_n + \epsilon q_i, \tag{3.3}$$

firstly we seek the general solution of the homogeneous equation

$$\mu_m \frac{d\psi_m}{dz}(z) + \frac{\sigma_{Ti}}{\epsilon}\psi_m(z) = \left(\frac{\sigma_{Ti}}{\epsilon} - \epsilon\sigma_{ai}\right)\sum_{n\in V}\psi_n(z)w_n, \quad z \in [z_i, z_{i+1}]. \tag{3.4}$$

Inserting

$$\psi_m(z) = l_m e^{-\frac{\sigma_{Ti}}{\epsilon}\xi z} \tag{3.5}$$

into (3.4) gives the eigenfunction equation

$$(1-\mu_m\xi)l_m = \left(1 - \epsilon^2\frac{\sigma_{ai}}{\sigma_{Ti}}\right)\sum_{n\in V} l_n w_n. \tag{3.6}$$

Hence l_m must have the form

$$l_m = \frac{\text{constant}}{1-\mu_m\xi},$$

where the 'constant' is independent of m but depends on i. Substituting this l_m into (3.6), one gets the characteristic equation

$$\frac{1}{1-\epsilon^2\frac{\sigma_{ai}}{\sigma_{Ti}}} = \sum_{n\in V}\frac{w_n}{1-\mu_n\xi}. \tag{3.7}$$

It has been shown by Chandrasekha [8] that when $\sigma_{ai}\neq 0$, all the eigenvalues are different from each other and appear in positive/negative pairs. When $\sigma_{ai}=0$, 0 is a double root and the other eigenvalues occur in positive/negative pairs. Let $\xi_n=-\xi_{-n}\geq 0$, for $n=1,\cdots,M$. When $\sigma_{ai}=0$, $\xi_{-M}=\xi_M=0$. Now, because the general solutions have different forms, we consider the two different cases:

(i) When $\sigma_{ai}\neq 0$, there are $2M$ eigenfunctions of the form (3.5). Hence the general solution of (3.4) can be expressed as

$$\psi_m(z)=\sum_{n\in V}A^{(n)}l_m^{(n)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_n z\big),\quad z\in[z_i,z_{i+1}], \tag{3.8}$$

where $A^{(n)}$ are undetermined and $l_m^{(n)}$ are the eigenfunctions corresponding to the nonzero eigenvalues ξ_n which is given by

$$l_m^{(n)}=\frac{1}{1-\mu_m\xi_n}.$$

In the Appendix we will prove that, if $\sigma_{ai}\neq 0$, for $1\leq|n|,|k|\leq M$,

$$\sum_{m\in V}w_m\mu_m l_m^{(k)}l_m^{(n)}=\left\{\begin{array}{cc}0 & n\neq k\\ c^{(k)} & n=k\end{array}\right., \tag{3.9}$$

where $c^{(k)}$ is some nonzero constant.
In addition, it is easy to check that

$$\psi_m=\frac{q_i}{\sigma_{ai}} \tag{3.10}$$

is a special solution of the inhomogeneous equation (3.3). Then the general solution of (3.3) has the following form

$$\psi_m(z)=\sum_{1\leq|n|\leq M}A^{(n)}l_m^{(n)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_n z\big)+\frac{q_i}{\sigma_{ai}},\quad z\in[z_i,z_{i+1}]. \tag{3.11}$$

By the aid of these expressions, we can find the relation between $\tilde{\psi}(z_i)$ and $\tilde{\psi}(z_{i+1})$ as follows:
$A^{(n)}$ can be determined by $\tilde{\psi}_m(z_i)$ from

$$\tilde{\psi}_m(z_i)=\sum_{n\in V}A^{(n)}l_m^{(n)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_n z_i\big)+\frac{q_i}{\sigma_{ai}}.$$

Multiplying both sides of the above equation by $w_m\mu_m l_m^{(k)}$ and summing over V give

$$\begin{aligned}&\sum_{m\in V}w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_i)\\ &=\sum_{m\in V}w_m\mu_m l_m^{(k)}\sum_{n\in V}A^{(n)}l_m^{(n)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_n z_i\big)+\frac{q_i}{\sigma_{ai}}\sum_{m\in V}w_m\mu_m l_m^{(k)}\\ &=\sum_{n\in V}A^{(n)}\sum_{m\in V}w_m\mu_m l_m^{(k)}l_m^{(n)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_n z_i\big)+\frac{q_i}{\sigma_{ai}}\sum_{m\in V}w_m\mu_m l_m^{(k)}.\end{aligned}$$

By using (3.9), $A^{(k)}$ can be given by $\tilde{\psi}(z_i)$ as

$$A^{(k)} = \frac{1}{c^{(k)}}\Big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_i) - \frac{q_i}{\sigma_{ai}}\sum_{m\in V} w_m\mu_m l_m^{(k)}\Big)\exp\big(\frac{\sigma_{Ti}}{\epsilon}\xi_k z_i\big). \tag{3.12a}$$

Similarly $A^{(k)}$ can also be expressed by $\tilde{\psi}(z_{i+1})$ as

$$A^{(k)} = \frac{1}{c^{(k)}}\Big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_{i+1}) - \frac{q_i}{\sigma_{ai}}\sum_{m\in V} w_m\mu_m l_m^{(k)}\Big)\exp\big(\frac{\sigma_{Ti}}{\epsilon}\xi_k z_{i+1}\big). \tag{3.12b}$$

Equating the two equations of (3.12), the relations between $\tilde{\psi}_m(z_i)$ and $\tilde{\psi}_m(z_{i+1})$ are:

$$\begin{aligned}&\Big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_i) - \frac{q_i}{\sigma_{ai}}\sum_{m\in V} w_m\mu_m l_m^{(k)}\Big)\exp\big(\frac{\sigma_{Ti}}{\epsilon}\xi_k(z_i - z_{i+1})\big)\\ &= \sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_{i+1}) - \frac{q_i}{\sigma_{ai}}\sum_{m\in V} w_m\mu_m l_m^{(k)}, \qquad k>0 \qquad (3.13a)\\ &\Big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_{i+1}) - \frac{q_i}{\sigma_{ai}}\sum_{m\in V} w_m\mu_m l_m^{(k)}\Big)\exp\big(\frac{\sigma_{Ti}}{\epsilon}\xi_k(z_{i+1} - z_i)\big)\\ &= \sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_i) - \frac{q_i}{\sigma_{ai}}\sum_{m\in V} w_m\mu_m l_m^{(k)}, \qquad k<0. \qquad (3.13b)\end{aligned}$$

The reason why we express the relations by (3.13a) and (3.13b) is to guarantee that the coefficients would not become exponentially large when $\epsilon \ll 1$. *Since the interface condition (2.10) requires* $\tilde{\psi}$ *to be continuous, i.e. two neighboring intervals* $[z_{i-1}, z_i]$, $[z_i, z_{i+1}]$ *use a common* $\tilde{\psi}(z_i)$, (3.13a)(3.13b) are in fact a particular finite difference scheme.

(ii) When $\sigma_{ai} = 0$, the general solution of the homogeneous equation (3.4) is

$$\psi_m(z) = c_1 + c_2\Big(\frac{\sigma_{Ti}}{\epsilon}z - \mu_m\Big) + \sum_{1\le|k|\le M-1} A^{(k)} l_m^{(k)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_k z\big). \tag{3.14}$$

Here c_1, c_2, $A^{(n)}$ are undetermined, the function 1 and $\frac{\sigma_{Ti}}{\epsilon}z - \mu_m$ are solutions of the eigenfunction equation (3.6) corresponding to the double root zero and $l_m^{(k)}$ are the eigenfunctions corresponding to the nonzero eigenvalues ξ_k which is given by (2.6). The special solution of the inhomogeneous equation (3.3), when $\sigma_{ai} = 0$, is

$$\psi_m = -\frac{3}{2}\sigma_{Ti}q_i z^2 + 3\epsilon q_i\mu_m z - \frac{\epsilon^2 q_i}{\sigma_{Ti}}(3\mu_m^2 - 1). \tag{3.15}$$

Thus the general solution of (3.3) becomes

$$\begin{aligned}\psi_m(z) = c_1 + c_2\Big(\frac{\sigma_{Ti}}{\epsilon}z - \mu_m\Big) + \sum_{1\le|n|\le M-1} A^{(n)} l_m^{(n)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_n z\big)\\ -\frac{3}{2}\sigma_{Ti}q_i z^2 + 3\epsilon q_i\mu_m z - \frac{\epsilon^2 q_i}{\sigma_{Ti}}(3\mu_m^2 - 1).\end{aligned} \tag{3.16}$$

The relation between $\tilde{\psi}(z_i)$ and $\tilde{\psi}(z_{i+1})$ can be derived as follows:
At $z = z_i$, (3.16) gives

$$\begin{aligned}\tilde{\psi}_m(z_i) = c_1 + c_2\Big(\frac{\sigma_{Ti}}{\epsilon}z_i - \mu_m\Big) + \sum_{1\le|n|\le M-1} A^{(n)} l_m^{(n)}\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi_n z_i\big)\\ -\frac{3}{2}\sigma_{Ti}q_i z_i^2 + 3\epsilon q_i\mu_m z_i - \frac{\epsilon^2 q_i}{\sigma_{Ti}}(3\mu_m^2 - 1).\end{aligned} \tag{3.17}$$

For $1 \leq |k| \leq M-1$, multiplying both sides of (3.17) by $w_m \mu_m l_m^{(k)}$, summing over V and using the properties of $l_m^{(n)}$ for $\sigma_{ai} = 0$ in (2.8)(2.7), one gets

$$A^{(k)} = \frac{1}{c^{(k)}} \exp\big(\frac{\sigma_{Ti}}{\epsilon}\xi_k z_i\big)\Big(\sum_{m\in V} w_m \mu_m l_m^{(k)} \tilde{\psi}_m(z_i) + 3\epsilon^2 \frac{q_i}{\sigma_{Ti}} \sum_{m\in V} w_m \mu_m^3 l_k^{(m)}\Big). \quad (3.18)$$

Then multiplying (3.17) by $w_m\mu_m$ and $w_m\mu_m^2$ respectively and summing over V, we obtain

$$\begin{aligned} c_1 &= -\frac{3}{2}\sigma_{Ti} q_i z_i^2 + 9\epsilon^2 \frac{q_i}{\sigma_{Ti}} \sum_{m\in V} w_m \mu_m^4 - \epsilon^2 \frac{q_i}{\sigma_{Ti}} \\ &\quad + 3\sum_{m\in V} w_m \mu_m \big(\frac{\sigma_{Ti}}{\epsilon} z_i + \mu_m\big)\tilde{\psi}_m(z_i), \qquad (3.19) \\ c_2 &= 3\epsilon q_i z_i - 3\sum_{m\in V} w_m \mu_m \tilde{\psi}_m(z_i). \qquad (3.20) \end{aligned}$$

Similarly, $A^{(k)}$, c_1, c_2 can also be expressed in terms of $\tilde{\psi}(z_{i+1})$ and we consequently get the relations between $\tilde{\psi}(z_i)$ and $\tilde{\psi}(z_{i+1})$:

$$\begin{aligned} &\exp\big(\frac{\sigma_{Ti}}{\epsilon}\xi_k(z_i - z_{i+1})\big)\Big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_i) + 3\epsilon^2\frac{q_i}{\sigma_{Ti}}\sum_{m\in V} w_m\mu_m^3 l_m^{(k)}\Big) \\ &= \sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_{i+1}) + 3\epsilon^2\frac{q_i}{\sigma_{Ti}}\sum_{m\in V} w_m\mu_m^3 l_m^{(k)}, \qquad k>0 \qquad (3.21a) \\ &\exp\big(\frac{\sigma_{Ti}}{\epsilon}\xi_k(z_{i+1} - z_i)\big)\Big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_{i+1}) + 3\epsilon^2\frac{q_i}{\sigma_{Ti}}\sum_{m\in V} w_m\mu_m^3 l_m^{(k)}\Big) \\ &= \sum_{m\in V} w_m\mu_m l_m^{(k)}\tilde{\psi}_m(z_i) + 3\epsilon^2\frac{q_i}{\sigma_{Ti}}\sum_{m\in V} w_m\mu_m^3 l_m^{(k)}, \qquad k<0 \qquad (3.21b) \\ &-\frac{3}{2}\sigma_{Ti}q_i z_i^2 + 3\sum_{m\in V} w_m\mu_m\big(\frac{\sigma_{Ti}}{\epsilon}z_i + \mu_m\big)\tilde{\psi}_m(z_i) \\ &= -\frac{3}{2}\sigma_{Ti}q_i z_{i+1}^2 + 3\sum_{m\in V} w_m\mu_m\big(\frac{\sigma_{Ti}}{\epsilon}z_{i+1} + \mu_m\big)\tilde{\psi}_m(z_{i+1}), \qquad (3.21c) \\ &3\epsilon q_i z_i - 3\sum_{m\in V} w_m\mu_m\tilde{\psi}_m(z_i) = 3\epsilon q_i z_{i+1} - 3\sum_{m\in V} w_m\mu_m\tilde{\psi}_m(z_{i+1}). \qquad (3.21d) \end{aligned}$$

These equations are also finite difference equations.

We have now derived the finite difference scheme whose solution is the exact solution of (3.2). This is clearly a second order scheme for fixed ϵ. When σ_T, σ_A and q are all constants, this method has no spatial error [12].

Consider the matrix form of this special finite difference scheme, the unknowns should be arranged as

$$\Big(\tilde{\psi}_{-M}(z_0), \cdots, \tilde{\psi}_M(z_0), \cdots, \tilde{\psi}_{-M}(z_i), \cdots, \tilde{\psi}_M(z_i), \cdots, \tilde{\psi}_{-M}(z_N), \cdots, \tilde{\psi}_M(z_N)\Big)^T,$$

then the coefficient matrix of the finite difference equations becomes

$$\begin{pmatrix} 0_M & I_M & & & & \\ & B_1^0 & B_1^1 & & & \\ & & \ddots & \ddots & & \\ & & & B_N^0 & B_N^1 & \\ & & & & I_M & 0_M \end{pmatrix},$$

where B_j^i $(i = 0, 1; j = 1, \cdots, N)$ are $2M \times 2M$ matrices and $0_M, I_M$ represent the $M \times M$ zero matrix and identity matrix respectively.

3.2. Asymptotic preservation. Note that this numerical method solves (3.2) exactly. Let $\epsilon \ll 1$, $\Delta z/\epsilon \gg 1$. By using the same asymptotic analysis as for (1.3a) [30], the solution of (3.2) inside the interval (z_i, z_{i+1}) can be expressed as

$$\tilde{\psi}_m(z) = \tilde{\phi}(z) - \epsilon \frac{\mu_m}{\tilde{\sigma}_T} \partial_z \tilde{\phi}(z) + O(\epsilon^2),$$

where $\tilde{\psi}(z)$ satisfies the diffusion equation:

$$-\frac{d}{dz}\left(\frac{1}{3\tilde{\sigma}_T}\tilde{\phi}_z\right) + \tilde{\sigma}_a \tilde{\phi} = \tilde{q}, \quad z_i < z < z_{i+1}, \quad \forall i \tag{3.22a}$$

subject to the boundary conditions

$$\begin{aligned} \tilde{\phi} - \epsilon \tfrac{\lambda}{\tilde{\sigma}_T} \partial_z \tilde{\phi}\Big|_{z=z_L} &= \textstyle\sum_{m=1}^{M} \psi_{Lm} \alpha_m, \\ \tilde{\phi} + \epsilon \tfrac{\lambda}{\tilde{\sigma}_T} \partial_z \tilde{\phi}\Big|_{z=z_R} &= \textstyle\sum_{m=-1}^{-M} \psi_{Rm} \alpha_m. \end{aligned} \tag{3.22b}$$

Here the extrapolation length λ and discrete H-function α_m are the same as in (2.3)(2.4). The same interface analysis in section 2 can be applied here as well, leading to the connection conditions at the nodes as

$$\tilde{\phi}(z_i^-) = \tilde{\phi}(z_i^+), \qquad \frac{1}{\tilde{\sigma}_T} \partial_z \tilde{\phi}(z_i^-) = \frac{1}{\tilde{\sigma}_T} \partial_z \tilde{\phi}(z_i^+). \tag{3.22c}$$

In particular, at the interface $z_i = 0$,

$$\tilde{\phi}(0^-) = \tilde{\phi}(0^+), \qquad \frac{1}{\tilde{\sigma}_T} \partial_z \tilde{\phi}(0^-) = \frac{1}{\tilde{\sigma}_T} \partial_z \tilde{\phi}(0^+). \tag{3.22d}$$

Clearly, (3.22) is consistent to the continuous diffusion limit (2.18). In fact it is basically the scheme using the same idea of the numerical method for (2.18) [4]. From the classical estimate of the continuous dependence of coefficients for elliptic equations, we have

$$\|\phi - \tilde{\phi}\|_\infty < C_{dif} \Delta z,$$

where C_{dif} is some constant independent of ϵ. Then the method is AP.

Remark: Since the method is AP, an analysis similar to that of [14] implies that it is uniformly convergent with respect to ϵ. Since the it is of second order for (3.22), the uniform convergence using the method of [14] is *first order*. In the next section, we will prove a uniformly *second* order convergence.

4. Uniform quadratic convergence. In this section an error estimate for this method is given. We can prove that it has the quadratic convergence uniformly with respect to ϵ, valid to the boundary. The main result is the following theorem.

THEOREM 4.1. *Consider the boundary-interface value problem of the discrete ordinate equations (1.3) with solution* $\psi = (\psi_{-M}, \cdots \psi_{-1}, \psi_1, \cdots, \psi_M)^T$. *Assume* $\sigma_T(z) > 0$, $\sigma_a(z) > 0$, $q(z)$ *are piecewise* C^2 *functions. Let* $\tilde{\sigma}_T(z)$, $\tilde{\sigma}_a(z)$, $\tilde{q}(z)$ *be the piecewise cell-average constants defined in (3.1) and* $\tilde{\psi} = (\tilde{\psi}_{-M}, \cdots, \tilde{\psi}_{-1}, \tilde{\psi}_1 \cdots, \tilde{\psi}_M)^T$ *be the solution of (3.2) with the same boundary condition (1.3b). Then there exists an* $\epsilon_0 > 0$, *s.t. if* $\epsilon \le \epsilon_0$, *we have*

$$\|\psi - \tilde{\psi}\|_{L^\infty([z_L,z_R]\times V)} = \max_{z\in[z_L,z_R]} \left\{ \max_{m\in V} \left\{ |\psi_m - \tilde{\psi}_m| \right\} \right\} < C_{tra}\Delta z^2.$$

Here C_{tra} *is independent of* ϵ *and* Δz.

Proof. Let the error $e_m = \psi_m - \tilde{\psi}_m$. (1.3a) minus (3.2) gives

$$\begin{aligned} &\mu_m \frac{de_m}{dz} + \frac{\tilde{\sigma}_T}{\epsilon} e_m - \left(\frac{\tilde{\sigma}_T}{\epsilon} - \epsilon\tilde{\sigma}_a \right) \sum_{n\in V} e_n w_n \\ &= -\frac{\sigma_T - \tilde{\sigma}_T}{\epsilon}\psi_m + \left(\frac{\sigma_T - \tilde{\sigma}_T}{\epsilon} - \epsilon\big(\sigma_a - \tilde{\sigma}_a\big) \right) \sum_{n\in V} \psi_n w_n + \epsilon\big(q - \tilde{q}\big) \\ &\equiv \mathfrak{Q}_m, \end{aligned} \tag{4.1}$$

while the boundary conditions become

$$e_{mL} = e_m(z_L) = 0, \quad \mu_m > 0\,; \quad e_{mR} = e_m(z_R) = 0, \quad \mu_m < 0. \tag{4.2}$$

Step 1: We try to derive some moment equations of (4.1).
Introduce some moments:

$$\begin{aligned} \tilde{e}_k &= \textstyle\sum_{m\in V} \mu_m w_m l_m^{(k)} e_m, \quad \text{for } 1 \le |k| \le M-1, \\ \tilde{e}_{-M} &= \textstyle\sum_{m\in V} w_m \mu_m e_m, \qquad \tilde{e}_M = \sum_{m\in V} w_m \mu_m^2 e_m, \end{aligned} \tag{4.3}$$

where $l_m^{(k)}$ are defined as in (2.6). Let

$$\mathbf{e} = (e_{-M}, \cdots, e_M)^T, \quad \tilde{\mathbf{e}} = (\tilde{e}_{-M}, \cdots, \tilde{e}_M)^T. \tag{4.4}$$

In order to find the equation that $\tilde{\mathbf{e}}$ satisfies, we have to express $\mathbf{e}$ in terms of $\tilde{\mathbf{e}}$. With $c^{(k)}$ $(1 \le |k| \le M-1)$ given in (2.8) and $c^{(-M)} = c^{(M)} = 1/3$, from (2.9), we have for $n \in V$,

$$\begin{aligned} &\sum_{1\le|k|\le M-1} \frac{1}{c^{(k)}} l_n^{(k)} \tilde{e}_k + \frac{\mu_n}{c^{(-M)}} \tilde{e}_{-M} + \frac{1}{c^{(M)}} \tilde{e}_M \\ &= \sum_{1\le|k|\le M-1} \frac{1}{c^{(k)}} l_n^{(k)} \sum_{m\in V} \mu_m w_m l_m^{(k)} e_m + \frac{\mu_n}{c^{(-M)}} \sum_{m\in V} w_m \mu_m e_m \\ &\quad + \frac{1}{c^{(M)}} \sum_{m\in V} w_m \mu_m^2 e_m \end{aligned} \tag{4.5}$$

$$= \sum_{m\in V} w_m \mu_m e_m \Big(\sum_{1\le|k|\le M-1} \frac{1}{c^{(k)}} l_n^{(k)} l_m^{(k)} + \frac{\mu_n}{c^{(-M)}} + \frac{\mu_m}{c^{(M)}} \Big) = e_n. \tag{4.6}$$

For $m \in V$, let

$$l_m^{(-M)} = \mu_m, \qquad l_m^{(M)} = 1. \tag{4.7}$$

(4.6) then is

$$e_n = \sum_{k \in V} \frac{1}{c^{(k)}} l_n^{(k)} \tilde{e}_k. \tag{4.8}$$

Write (4.1) in the following form:

$$\mu_m \frac{de_m}{dz} + \frac{\tilde{\sigma}_T}{\epsilon} e_m - \frac{\tilde{\sigma}_T}{\epsilon} \sum_{n \in V} e_n w_n = -\epsilon \tilde{\sigma}_a \sum_{n \in V} e_n w_n + \mathfrak{Q}_m. \tag{4.9}$$

For$1 \le |k| \le M-1$, multiplying both sides of (4.9) by $w_m l_m^{(k)}$ and summing over V give

$$\begin{aligned} &\sum_{m \in V} w_m \mu_m l_m^{(k)} \frac{de_m}{dz} + \frac{\tilde{\sigma}_T}{\epsilon} \sum_{m \in V} w_m l_m^{(k)} e_m - \frac{\tilde{\sigma}_T}{\epsilon} \sum_{m \in V} w_m l_m^{(k)} \sum_{n \in V} e_n w_n \\ &= -\epsilon \tilde{\sigma}_a \sum_{m \in V} w_m l_m^{(k)} \sum_{n \in V} e_n w_n + \sum_{m \in V} w_m l_m^{(k)} \mathfrak{Q}_m. \end{aligned} \tag{4.10}$$

From (2.5)(2.6),

$$\begin{aligned} &\frac{\tilde{\sigma}_T}{\epsilon} \sum_{m \in V} w_m l_m^{(k)} e_m - \frac{\tilde{\sigma}_T}{\epsilon} \sum_{m \in V} w_m l_m^{(k)} \sum_{n \in V} e_n w_n \\ &= \frac{\tilde{\sigma}_T}{\epsilon} \Big(\sum_{m \in V} w_m l_m^{(k)} e_m - \sum_{m \in V} e_m w_m \Big) = \xi_k \frac{\tilde{\sigma}_T}{\epsilon} \sum_{m \in V} \mu_m w_m l_m^{(k)} e_m. \end{aligned}$$

Substituting this into (4.10) gives

$$\begin{aligned} &\sum_{m \in V} w_m \mu_m l_m^{(k)} \frac{de_m}{dz} + \xi_k \frac{\tilde{\sigma}_T}{\epsilon} \sum_{m \in V} \mu_m w_m l_m^{(k)} e_m \\ &= -\epsilon \tilde{\sigma}_a \sum_{m \in V} e_m w_m + \sum_{m \in V} w_m l_m^{(k)} \mathfrak{Q}_m. \end{aligned} \tag{4.11}$$

Similarly, multiplying both sides of (4.9) by w_m and $w_m \mu_m$, summing over V, one gets

$$\sum_{m \in V} w_m \mu_m \frac{de_m}{dz} = -\epsilon \tilde{\sigma}_a \sum_{m \in V} e_m w_m + \sum_{m \in V} w_m \mathfrak{Q}_m, \tag{4.12}$$

$$\sum_{m \in V} w_m \mu_m^2 \frac{de_m}{dz} + \frac{\tilde{\sigma}_T}{\epsilon} \sum_{m \in V} w_m \mu_m e_m = \sum_{m \in V} w_m \mu_m \mathfrak{Q}_m, \tag{4.13}$$

Substituting (4.3)(4.8) into (4.11)-(4.13) gives a system of equations that $\tilde{e}_k$ satisfy:

$$\begin{aligned} \frac{d\tilde{e}_k}{dz} + \xi_k \frac{\tilde{\sigma}_T}{\epsilon} \tilde{e}_k &= -\epsilon \tilde{\sigma}_a \sum_{m \in V} w_m \sum_{n \in V} \frac{1}{c^{(n)}} l_m^{(n)} \tilde{e}_n + \sum_{m \in V} w_m l_m^{(k)} \mathfrak{Q}_m \\ &= -\epsilon \tilde{\sigma}_a \sum_{m \in V \setminus \{M\}} \frac{1}{c^{(m)}} \tilde{e}_m + \sum_{m \in V} w_m l_m^{(k)} \mathfrak{Q}_m, \end{aligned} \tag{4.14a}$$

$$\frac{d\tilde{e}_{-M}}{dz} = -\epsilon\tilde{\sigma}_a \sum_{m\in V\setminus\{M\}} \frac{1}{c^{(m)}}\tilde{e}_m + \sum_{m\in V} w_m \mathfrak{Q}_m, \tag{4.14b}$$

$$\frac{d\tilde{e}_M}{dz} + \frac{\tilde{\sigma}_T}{\epsilon}\tilde{e}_{-M} = \sum_{m\in V} w_m\mu_m\mathfrak{Q}_m. \tag{4.14c}$$

Step 2: Write the integral formulations of $\tilde{\mathbf{e}}$ defined by (4.14), including its values at the boundaries.

Let

$$\tilde{\alpha}(z) = \int_{z_L}^{z} \tilde{\sigma}_T(x)dx. \tag{4.15}$$

Using the method of integrating factor, for $k = 1, \cdots, M-1$, multiplying both sides of (4.14a) by $\exp\big(\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k\big)$ and integrating from z_L to z gives

$$\begin{aligned}&\tilde{e}_k(z) \qquad (4.16)\\ &= \exp\Big(-\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k\Big)\tilde{e}_k(z_L) - \epsilon\int_{z_L}^{z}\exp\Big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\Big)\tilde{\sigma}_a \sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m\,dx\\ &\quad + \int_{z_L}^{z}\exp\Big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\Big)\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m\,dx. \qquad (4.17)\end{aligned}$$

Similarly, for $k = -M+1, \cdots, -1$, integrate from z to z_R:

$$\begin{aligned}\tilde{e}_k(z) &= \exp\Big(\frac{\tilde{\alpha}(z_R)-\tilde{\alpha}(z)}{\epsilon}\xi_k\Big)\tilde{e}_k(z_R)\\ &\quad+\epsilon\int_{z}^{z_R}\exp\Big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\Big)\tilde{\sigma}_a\sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m\,dx\\ &\quad-\int_{z}^{z_R}\exp\big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m\,dx. \end{aligned}\tag{4.18}$$

Integrating (4.14b)(4.14c) from z_L to z gives

$$\tilde{e}_{-M}(z) = \tilde{e}_{-M}(z_L) - \epsilon\int_{z_L}^{z}\tilde{\sigma}_a\sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m\,dx + \int_{z_L}^{z}\sum_{m\in V} w_m\mathfrak{Q}_m\,dx, \tag{4.19}$$

$$\tilde{e}_M(z) = -\int_{z_L}^{z}\frac{\tilde{\sigma}_T(x)}{\epsilon}\tilde{e}_{-M}(x)\,dx + \tilde{e}_M(z_L) + \int_{z_L}^{z}\sum_{m\in V} w_m\mu_m\mathfrak{Q}_m\,dx. \tag{4.20}$$

Like in (4.17)–(4.19), for (4.20), we would like to express $\tilde{e}_M(z)$ by its boundary data. Substituting (4.19) into (4.20) gives

$$\begin{aligned}&\tilde{e}_M(z) + \frac{\tilde{\alpha}(z)}{\epsilon}\tilde{e}_{-M}(z_L) - \int_{z_L}^{z}\tilde{\sigma}_T\int_{z_L}^{x}\tilde{\sigma}_a\sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m\,dsdx\\ &= -\int_{z_L}^{z}\frac{\tilde{\sigma}_T}{\epsilon}\int_{z_L}^{x}\sum_{m\in V} w_m\mathfrak{Q}_m\,dsdx + \tilde{e}_M(z_L) + \int_{z_L}^{z}\sum_{m\in V} w_m\mu_m\mathfrak{Q}_m\,dx. \end{aligned}\tag{4.21}$$

In particular, when $z = z_R$, (4.21) is

$$
\begin{aligned}
&\tilde{e}_M(z_R) - \tilde{e}_M(z_L) + \frac{\tilde{\alpha}(z_R)}{\epsilon}\tilde{e}_{-M}(z_L) \\
&= \int_{z_L}^{z_R} \tilde{\sigma}_T \int_{z_L}^{x} \tilde{\sigma}_a \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} \tilde{e}_m \, dsdx \\
&\quad - \int_{z_L}^{z_R} \frac{\tilde{\sigma}_T}{\epsilon} \int_{z_L}^{x} \sum_{m \in V} w_m \mathfrak{Q}_m \, dsdx + \int_{z_L}^{z} \sum_{m \in V} w_m \mu_m \mathfrak{Q}_m \, dx.
\end{aligned} \tag{4.22}
$$

From (4.22) we can get an expression for $\tilde{e}_{-M}(z_L)/\epsilon$. Substituting it into (4.21), one gets

$$
\begin{aligned}
&\tilde{e}_M(z) + \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)} \int_{z_L}^{z_R} \tilde{\sigma}_T \int_{z_L}^{x} \tilde{\sigma}_a \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} \tilde{e}_m \, dsdx \\
&\quad - \int_{z_L}^{z} \tilde{\sigma}_T \int_{z_L}^{x} \tilde{\sigma}_a \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} \tilde{e}_m \, dsdx \\
&= \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)} \tilde{e}_M(z_R) + \Big(1 - \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)}\Big) \tilde{e}_M(z_L) \\
&\quad + \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)} \int_{z_L}^{z_R} \frac{\tilde{\sigma}_T}{\epsilon} \int_{z_L}^{x} \sum_{m \in V} w_m \mathfrak{Q}_m \, dsdx - \int_{z_L}^{z} \frac{\tilde{\sigma}_T}{\epsilon} \int_{z_L}^{x} \sum_{m \in V} w_m \mathfrak{Q}_m \, dsdx \\
&\quad - \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)} \int_{z_L}^{z_R} \sum_{m \in V} w_m \mu_m \mathfrak{Q}_m \, dx + \int_{z_L}^{z} \sum_{m \in V} w_m \mu_m \mathfrak{Q}_m \, dx.
\end{aligned} \tag{4.23}
$$

Formally, (4.17)(4.18)(4.19)(4.23) is the integral formulation of (4.14). Note that the left side of the combination of these equations can be viewed as an ϵ-independent operator acting on $\tilde{\mathbf{e}}$ and all the perturbations of σ_T, σ_a, q occur in $\mathfrak{Q}_m$. In order to prove the convergence we have to find the explicit expressions of $\tilde{\mathbf{e}}(z_L), \tilde{\mathbf{e}}(z_R)$ by $\mathfrak{Q}_m$ and some operator acting on $\mathbf{e}$.

Particularly, let $z = z_R$, (4.17)(4.18)(4.19) become

$$\begin{aligned} &\tilde{e}_k(z_R) - \exp\Big(-\frac{\tilde{\alpha}(z_R)}{\epsilon}\xi_k\Big)\tilde{e}_k(z_L) \\ =& -\epsilon\int_{z_L}^{z_R}\tilde{\sigma}_a \exp\Big(\frac{\tilde{\alpha}(z)-\tilde{\alpha}(z_R)}{\epsilon}\xi_k\Big)\sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m dz \\ &+\int_{z_L}^{z_R}\exp\big(\frac{\tilde{\alpha}(z)-\tilde{\alpha}(z_R)}{\epsilon}\xi_k\big)\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m dz, \quad k=1,\cdots,M-1 \end{aligned} \tag{4.24a}$$

$$\begin{aligned} &\exp\big(\frac{\tilde{\alpha}(z_R)}{\epsilon}\xi_k\big)\tilde{e}_k(z_R) - \tilde{e}_k(z_L) \\ =& -\epsilon\int_{z_L}^{z_R}\tilde{\sigma}_a \exp\big(\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m dz \\ &+\int_{z_L}^{z_R}\exp\Big(\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k\Big)\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m dz, \quad k=-M+1,\cdots,-1. \end{aligned} \tag{4.24b}$$

$$\begin{aligned} &\tilde{e}_{-M}(z_R) - \tilde{e}_{-M}(z_L) \\ =& -\epsilon\int_{z_L}^{z_R}\tilde{\sigma}_a \sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m dz + \int_{z_L}^{z_R}\sum_{m\in V} w_m\mathfrak{Q}_m dz. \end{aligned} \tag{4.24c}$$

Furthermore, write (4.22) as follows

$$\begin{aligned} &\epsilon\tilde{e}_M(z_R) - \epsilon\tilde{e}_M(z_L) + \tilde{\alpha}(z_R)\tilde{e}_{-M}(z_L) \\ =& \epsilon\int_{z_L}^{z_R}\tilde{\sigma}_T\int_{z_L}^{x}\tilde{\sigma}_a\sum_{m\in V\setminus\{-M\}}\frac{1}{c^{(m)}}\tilde{e}_m\, dsdx \\ &-\int_{z_L}^{z_R}\tilde{\sigma}_T\int_{z_L}^{x}\sum_{m\in V} w_m\mathfrak{Q}_m\, dsdx + \epsilon\int_{z_L}^{z}\sum_{m\in V} w_m\mu_m\mathfrak{Q}_m\, dx. \end{aligned} \tag{4.24d}$$

and the boundary conditions of $\mathbf{e}$ using (4.8) imply

$$\begin{array}{ll} \sum_{k\in V}\frac{1}{c^{(k)}}l_n^{(k)}\tilde{e}_k(z_L)=0, & \text{for } k=1,\cdots,M \\ \sum_{k\in V}\frac{1}{c^{(k)}}l_n^{(k)}\tilde{e}_k(z_R)=0, & \text{for } k=-M,\cdots,-1 \end{array}. \tag{4.24e}$$

Now (4.24) is a system of $4M$ linear equations with $4M$ unkonwns. Denote the right hand side vector by Λ. It is obvious that $\tilde{e}_k(z_L), \tilde{e}_k(z_R), k \in V$ can be determined by Λ through some linear operators $\mathfrak{L}_{Lk}$ and $\mathfrak{L}_{Rk}$ (which are part of the inverse of the coefficient matrix of (4.24)):

$$\tilde{\mathbf{e}}_k(z_L) = \mathfrak{L}_{Lk}\Lambda, \qquad \tilde{\mathbf{e}}_k(z_R) = \mathfrak{L}_{Rk}\Lambda.$$

Λ can be decomposed into two parts, one only contains the fluxes of $\mathfrak{Q}_m$ and the other is some linear operator acting on $\tilde{\mathbf{e}}$. That is

$$\Lambda = \epsilon\Lambda^{(1)}(\tilde{\mathbf{e}}) + \Lambda^{(2)},$$

where

$$\Lambda^{(1)} = (\Lambda^{(1)}_{-M}, \cdots, \Lambda^{(1)}_{M}), \qquad \Lambda^{(2)} = (\Lambda^{(2)}_{-M}, \cdots, \Lambda^{(2)}_{M}),$$

and $\Lambda_k^{(1)}$, $\Lambda_k^{(2)}$ are given as follows:

When $k = 1, \cdots, M-1$, let

$$\Lambda_k^{(1)}(\mathbf{f}) = -\int_{z_L}^{z_R} \tilde{\sigma}_a \exp\big(\frac{\tilde{\alpha}(z) - \tilde{\alpha}(z_R)}{\epsilon}\xi_k\big) \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} f_m dz,$$

$$\Lambda_k^{(2)} = \int_{z_L}^{z_R} \exp\big(\frac{\tilde{\alpha}(z) - \tilde{\alpha}(z_R)}{\epsilon}\xi_k\big) \sum_{m \in V} w_m l_m^{(k)} \mathfrak{Q}_m dz,$$

where $\mathbf{f} = (f_{-M}, \cdots, f_{-1}, f_1, \cdots, f_M) \in \mathbb{R}^{2M}$. When $k = -M+1, \cdots, -1$,

$$\Lambda_k^{(1)}(\mathbf{f}) = -\int_{z_L}^{z_R} \tilde{\sigma}_a \exp\big(\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k\big) \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} f_m dz$$

$$\Lambda_k^{(2)} = \int_{z_L}^{z_R} e^{\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k} \sum_{m \in V} w_m l_m^{(k)} \mathfrak{Q}_m dz.$$

Moreover

$$\Lambda_{-M}^{(1)}(\mathbf{f}) = -\int_{z_L}^{z_R} \tilde{\sigma}_a \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} f_m dz, \qquad \Lambda_{-M}^{(2)} = \int_{z_L}^{z_R} \sum_{m \in V} w_m \mathfrak{Q}_m dz,$$

$$\Lambda_M^{(1)}(\mathbf{f}) = \int_{z_L}^{z_R} \tilde{\sigma}_T \int_{z_L}^{z} \tilde{\sigma}_a \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} f_m dx dz,$$

$$\Lambda_M^{(2)} = -\int_{z_L}^{z_R} \tilde{\sigma}_T \int_{z_L}^{z} \sum_{m \in V} w_m \mathfrak{Q}_m dx dz + \epsilon \int_{z_L}^{z_R} \sum_{m \in V} w_m \mu_m \mathfrak{Q}_m dz.$$

Then

$$\tilde{e}_k(z_L) = \mathfrak{L}_{Lk}\Lambda = \epsilon \mathfrak{L}_{Lk}\Lambda^{(1)}(\tilde{e}) + \mathfrak{L}_{Lk}\Lambda^{(2)}, \quad \tilde{e}_k(z_R) = \mathfrak{L}_{Rk}\Lambda = \epsilon \mathfrak{L}_{Rk}\Lambda^{(1)}(\tilde{e}) + \mathfrak{L}_{Rk}\Lambda^{(2)}. \tag{4.25}$$

Now we have found the expressions of $\tilde{e}_k(z_L)$, $\tilde{e}_k(z_R)$ through the linear operators $\mathfrak{L}_{Lk}$, $\mathfrak{L}_{Rk}$. The combination of (4.17)(4.18)(4.19)(4.23) can be written in the vector form

$$\Pi \tilde{\mathbf{e}}(z) = \Theta(z) + \epsilon \Omega \tilde{\mathbf{e}}(z),$$

where

$$(\Pi \mathbf{f})_k = \begin{cases} f_k, & k = -M, \cdots, M-1 \\ f_M - \int_{z_L}^{z} \tilde{\sigma}_T \int_{z_L}^{x} \tilde{\sigma}_a \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} f_m \, ds dx & \\ \quad + \frac{\tilde{\alpha}_T(z)}{\tilde{\alpha}(z_R)} \int_{z_L}^{z_R} \tilde{\sigma}_T \int_{z_L}^{x} \tilde{\sigma}_a \sum_{m \in V \setminus \{-M\}} \frac{1}{c^{(m)}} f_m \, ds dx, & k = M \end{cases}, \tag{4.26}$$

$$\Theta_k(z) = \begin{cases} \exp\big(-\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\mathfrak{L}_{Lk}\Lambda^{(2)} + \int_{z_L}^{z}\exp\big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m\,dx, \\ \qquad\qquad k=1,\cdots,M-1 \\ \exp\big(\frac{\tilde{\alpha}(z_R)-\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\mathfrak{L}_{Rk}\Lambda^{(2)} - \int_{z}^{z_R}\exp\big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m\,dx, \\ \qquad\qquad k=-M+1,\cdots,-1 \\ \mathfrak{L}_{L(-M)}\Lambda^{(2)} + \int_{z_L}^{z}\sum_{m\in V} w_m\mathfrak{Q}_m\,dx, \qquad k=-M \\ \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)}\mathfrak{L}_{RM}\Lambda^{(2)} + \Big(1-\frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)}\Big)\mathfrak{L}_{LM}\Lambda^{(2)} \\ \quad +\frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)}\int_{z_L}^{z_R}\frac{\tilde{\sigma}_T}{\epsilon}\int_{z_L}^{x}\sum_{m\in V} w_m\mathfrak{Q}_m\,dsdx - \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)}\int_{z_L}^{z_R}\sum_{m\in V} w_m\mu_m\mathfrak{Q}_m\,dx \\ \quad -\int_{z_L}^{z}\frac{\tilde{\sigma}_T}{\epsilon}\int_{z_L}^{x}\sum_{m\in V} w_m\mathfrak{Q}_m\,dsdx + \int_{z_L}^{z}\sum_{m\in V} w_m\mu_m\mathfrak{Q}_m\,dx, \qquad k=M \end{cases}, \tag{4.27}$$

and

$$(\Omega\mathbf{f})_k(z) = \begin{cases} \exp\big(-\frac{\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\mathfrak{L}_{Lk}\Lambda^{(1)}(\mathbf{f}) \\ \quad -\int_{z_L}^{z}\exp\big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\tilde{\sigma}_a\sum_{m\in V\backslash\{-M\}}\frac{1}{c^{(m)}}f_m\,dx, \\ \qquad\qquad k=1,\cdots,M-1 \\ \exp\big(\frac{\tilde{\alpha}(z_R)-\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\mathfrak{L}_{Rk}\Lambda^{(1)}(\mathbf{f}) \\ \quad +\int_{z}^{z_R}\exp\big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z)}{\epsilon}\xi_k\big)\tilde{\sigma}_a\sum_{m\in V\backslash\{-M\}}\frac{1}{c^{(m)}}f_m\,dx, \\ \qquad\qquad k=-M+1,\cdots,-1 \\ \mathfrak{L}_{L(-M)}\Lambda^{(1)}(\mathbf{f}) - \int_{z_L}^{z}\tilde{\sigma}_a\sum_{m\in V\backslash\{-M\}}\frac{1}{c^{(m)}}f_m\,dx, \qquad k=-M \\ \frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)}\mathfrak{L}_{RM}\Lambda^{(1)}(\mathbf{f}) + \Big(1-\frac{\tilde{\alpha}(z)}{\tilde{\alpha}(z_R)}\Big)\mathfrak{L}_{LM}\Lambda^{(1)}(\mathbf{f}), \qquad k=M \end{cases}. \tag{4.28}$$

Π, Ω are operators. It is obvious that Π is invertible (since it is diagonal except the last row) and independent of ϵ, thus

$$\tilde{\mathbf{e}}(z) = \Pi^{-1}\Theta(z) + \epsilon\Pi^{-1}\Omega\tilde{\mathbf{e}}(z). \tag{4.29}$$

Step 3: A Neumann series can be constructed corresponding to (4.29):

$$\tilde{\mathbf{e}}(z) = \sum_{i=0}^{\infty}\tilde{\mathbf{e}}^{(i)}(z) \tag{4.30}$$

with

$$\tilde{\mathbf{e}}^{(0)}(z) = \Pi^{-1}\Theta(z)$$

and

$$\tilde{\mathbf{e}}^{(i)}(z) = (\epsilon\Pi^{-1}\Omega\tilde{\mathbf{e}}^{(i-1)})(z) = \epsilon^i\big((\Pi^{-1}\Omega)^i\Theta(z)\big). \tag{4.31}$$

If the sum in (4.30) is convergent, (4.30) is just the solution of (4.29) [6].

In order to prove the convergence of the Neumann series, we study the properties of the terms in the Neumann series. We first establish several lemmas.

Lemma 4.2. *For an interval $[z_i, z_{i+1}]$, assume $\Delta z = |z_{i+1}-z_i|$, $f \in C^2([z_i, z_{i+1}])$, $g \in C^1([z_i, z_{i+1}])$, and $\eta \in [z_i, z_{i+1}]$ satisfies*

$$f(\eta) = \tilde{f} = \frac{1}{\Delta z}\int_{z_i}^{z_{i+1}} f dz.$$

Then, for $x \geq z_{i+1}$

$$\begin{aligned} \textstyle\int_{z_i}^{z_{i+1}} \big(f-\tilde f\big) g\,dz \sim O(\Delta z^3),\\ \textstyle\int_{z_i}^{z_{i+1}} e^{\frac{\tilde\alpha(z)-\tilde\alpha(x)}{\epsilon}\xi}\big(f-\tilde f\big) g\,dz \sim \tfrac{\xi}{\epsilon}O(\Delta z^3), \quad \xi>0\\ \textstyle\int_{z_i}^{z_{i+1}} e^{\frac{\tilde\alpha(z)}{\epsilon}\xi}\big(f-\tilde f\big) g\,dz \sim \tfrac{\xi}{\epsilon}O(\Delta z^3), \qquad \xi<0 \end{aligned} \tag{4.32}$$

Proof. From the definition of η

$$\begin{aligned} \int_{z_i}^{z_{i+1}} f\,dz &= \int_{z_i}^{z_{i+1}} \big(f(\eta)+f'(\eta)(z-\eta)+O(\Delta z^2)\big)\,dz\\ &= f(\eta)\Delta z + \frac{1}{2}f'(\eta)(z_i+z_{i+1}-2\eta)\Delta z + O(\Delta z^3) = f(\eta)\Delta z. \end{aligned}$$

Thus

$$f'(\eta)(z_i+z_{i+1}-2\eta) \sim O(\Delta z^2).$$

Then

$$\int_{z_i}^{z_{i+1}} \big(f-\tilde f\big) g\,dz = \int_{z_i}^{z_{i+1}} \big(f'(\eta)(z-\eta)+O(\Delta z^2)\big)\big(g(\eta)+O(\Delta z)\big)dz \sim O(\Delta z^3).$$

Assume $y_1, y_2 < 0$, it is easy to check that

$$|\exp(y_1)-\exp(y_2)| < \exp(\max\{y_1,y_2\})|y_1-y_2|.$$

By using this fact, when $\xi > 0$ we have

$$\begin{aligned} &\int_{z_i}^{z_{i+1}} \exp\big(\frac{\tilde\alpha(z)-\tilde\alpha(x)}{\epsilon}\xi\big)\big(f-\tilde f\big) g\,dz\\ =& \exp\big(\frac{\tilde\alpha(z_{i+1})-\tilde\alpha(x)}{\epsilon}\xi\big)\int_{z_i}^{z_{i+1}} \exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi(z_{i+1}-z)\big)\\ &\cdot\big(f'(\eta)(z-\eta)+O(\Delta z^2)\big)\big(g(\eta)+O(\Delta z)\big)\,dz\\ =& \exp\big(\frac{\tilde\alpha(z_{i+1})-\tilde\alpha(x)}{\epsilon}\xi\big)\Big(\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi(z_{i+1}-\eta)\big)\\ &\cdot\Big(\frac{1}{2}f'(\eta)g(\eta)\big(z_i+z_{i+1}-2\eta\big)+O(\Delta z^2)\Big)\Delta z\\ &+f'(\eta)g(\eta)\int_{z_i}^{z_{i+1}} \Big(\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi(z_{i+1}-z)\big)-\exp\big(-\frac{\sigma_{Ti}}{\epsilon}\xi(z_{i+1}-\eta)\big)\Big)\\ &\cdot\big((z-\eta)+O(\Delta z^2)\big)\,dz\Big)\\ \sim& \exp\big(\frac{\tilde\alpha(\eta)-\tilde\alpha(x)}{\epsilon}\xi\big)O(\Delta z^3) + \frac{\xi}{\epsilon}O(\Delta z^3) \sim \frac{\xi}{\epsilon}O(\Delta z^3). \end{aligned}$$

The result for $\xi < 0$ can be obtained similarly. □

Lemma 4.3. *For the linear operator* $\mathfrak{L}_{Lm}$, $\mathfrak{L}_{Rm}$ *defined in (4.25), we have*

$$\|\mathfrak{L}_{Lm}\|_\infty \leq C_l, \quad \|\mathfrak{L}_{Rm}\|_\infty \leq C_l, \qquad \forall m \in V$$

where C_l *is a constant independent of* ϵ *and* Δz.

Proof. The linear operators $\mathfrak{L}_{Lk}$, $\mathfrak{L}_{Rk}$ defined in (4.25) arise in inverting the coefficient matrix of the following system of linear equations:

$$\tilde{e}_k(z_R) - \exp\big(-\frac{\tilde{\alpha}(z_R)}{\epsilon}\xi_k\big)\tilde{e}_k(z_L) = b_k, \quad 1 \le k \le M-1, \tag{4.33a}$$

$$\exp\big(\frac{\tilde{\alpha}(z_R)}{\epsilon}\xi_k\big)\tilde{e}_k(z_R) - \tilde{e}_k(z_L) = b_k, \quad -M+1 \le k \le -1 \tag{4.33b}$$

$$\tilde{e}_{-M}(z_R) - \tilde{e}_{-M}(z_L) = b_{-M}, \qquad \epsilon\tilde{e}_M(z_R) - \epsilon\tilde{e}_M(z_L) + \tilde{\alpha}(z_R)e_{-M}(z_L) = b_M, \tag{4.33c}$$

$$\sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_L) = 0, \quad k>0, \qquad \sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_R) = 0, \qquad k<0, \tag{4.33d}$$

with $\mathbf{b} = (b_{-M}, \cdots, b_M)^T$ being any vector.

Assume

$$\sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_L) = a_k, \quad k<0, \qquad \sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_R) = a_k, \qquad k>0.$$

Using (4.33d)(2.8)(2.7)(4.8), we have, for $1 \le |m| \le M-1$,

$$\begin{aligned}\sum_{k>0} w_k\mu_k l_k^{(m)} a_k &= \sum_{k\in V} w_k\mu_k l_k^{(m)} \sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_R)\\ &= \sum_{n\in V}\frac{1}{c^{(n)}}\tilde{e}_n(z_R)\sum_{k\in V} w_k\mu_k l_k^{(m)} l_k^{(n)} = \tilde{e}_m(z_R),\end{aligned}$$

$$\begin{aligned}\sum_{k<0} w_k\mu_k l_k^{(m)} a_k &= \sum_{k\in V} w_k\mu_k l_k^{(m)} \sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_L)\\ &= \sum_{n\in V}\frac{1}{c^{(n)}}\tilde{e}_n(z_L)\sum_{k\in V} w_k\mu_k l_k^{(m)} l_k^{(n)} = \tilde{e}_m(z_L),\end{aligned}$$

and for $|m| = M$,

$$\sum_{k>0} w_k\mu_k a_k = \sum_{k\in V} w_k\mu_k \sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_R) = \tilde{e}_{-M}(z_R), \quad \sum_{k<0} w_k\mu_k a_k = \tilde{e}_{-M}(z_L),$$

$$\sum_{k>0} w_k\mu_k^2 a_k = \sum_{k\in V} w_k\mu_k^2 \sum_{n\in V}\frac{1}{c^{(n)}}l_k^{(n)}\tilde{e}_n(z_R) = \tilde{e}_M(z_R), \quad \sum_{k<0} w_k\mu_k^2 a_k = \tilde{e}_M(z_L).$$

Inserting these into (4.33) gives

$$\begin{aligned}&\exp\big(\frac{\tilde{\alpha}(z_R)}{\epsilon}\xi_m\big)\sum_{k>0} w_k\mu_k l_k^{(m)} a_k - \sum_{k<0} w_k\mu_k l_k^{(m)} a_k = b_m, \quad -M+1 \le m \le -1,\\ &\sum_{k>0} w_k\mu_k l_k^{(m)} a_k - \exp\big(-\frac{\tilde{\alpha}(z_R)}{\epsilon}\xi_m\big)\sum_{k<0} w_k\mu_k l_k^{(m)} a_k = b_m, \quad 1 \le m \le M-1,\\ &\sum_{k>0} w_k\mu_k a_k - \sum_{k<0} w_k\mu_k a_k = b_{-M},\\ &\epsilon\sum_{k>0} w_k\mu_k^2 a_k - \epsilon\sum_{k<0} w_k\mu_k^2 a_k + \tilde{\alpha}(z_R)\sum_{k<0} w_k\mu_k a_k = b_M.\end{aligned}$$

Let $\mathbf{a} = (a_{-M}, \cdots, a_M)$, $E_1 = \operatorname{diag}\big\{ \exp\big(\frac{\tilde\alpha(z_R)}{\epsilon}\xi_{-M+1}\big) \cdots, \exp\big(\frac{\tilde\alpha(z_R)}{\epsilon}\xi_{-1}\big)\big\}$ and $E_2 = \operatorname{diag}\big\{ \exp\big(-\frac{\tilde\alpha(z_R)}{\epsilon}\xi_1\big), \cdots, \exp\big(-\frac{\tilde\alpha(z_R)}{\epsilon}\xi_{M-1}\big)\big\}$. From the symmetry of w_n, μ_n, ξ_n, the above system of equations is equivalent to

$$B\mathbf{a} = \mathbf{b}, \tag{4.34}$$

where

$$B = \begin{pmatrix} -(w_k\mu_k)_{k<0} & (w_k\mu_k)_{k>0} \\ -\big(w_k\mu_k l_k^{(m)}\big)_{-M+1\le m\le -1, k<0} & E_1\big(w_k\mu_k l_k^{(m)}\big)_{-M+1\le m\le -1, k>0} \\ -E_2\big(w_k\mu_k l_k^{(m)}\big)_{1\le m\le M-1, n<0} & \big(w_k\mu_k l_k^{(m)}\big)_{1\le m\le M-1, k>0} \\ -(\epsilon w_k\mu_k^2 + \tilde\alpha(z_R) w_k\mu_k)_{k<0} & (\epsilon w_k\mu_k^2)_{k>0} \end{pmatrix}. \tag{4.35}$$

Then

$$\begin{aligned} |\tilde e_m(z_R)| &\le \max_{k>0}\{|a_k|\} \sum_{k>0} |w_k\mu_k l_k^{(m)}| \le \|\mathbf{a}\|_\infty \sum_{k>0} |w_k\mu_k l_k^{(m)}| \\ &\le \|B^{-1}\|_\infty \|\mathbf{b}\|_\infty \sum_{k>0} |w_k\mu_k l_k^{(m)}|\,, \end{aligned}$$

and

$$\tilde e_{-M}(z_R) \le \|B^{-1}\|_\infty \|\mathbf{b}\|_\infty \sum_{k>0} |w_k\mu_k|, \quad \tilde e_M(z_R) \le \|B^{-1}\|_\infty \|\mathbf{b}\|_\infty \sum_{k>0} |w_k\mu_k^2|.$$

For any vector $\mathbf{b}$, we have

$$\|\mathfrak{L}_{Rm}\|_\infty \le \sum_{k>0} |w_k\mu_k l_k^{(m)}| \|B^{-1}\|_\infty.$$

Now what remains is to prove $\|B^{-1}\|_\infty$ is uniformly bounded with respect to ϵ.

Firstly, we recall a basic lemma in matrix theory (see for [16]). Assume $D_1, D_2 \in \mathbb{R}^{n\times n}$. If D_1 is invertible and $\|D_1^{-1}\|_\infty \le \beta_1$, $\|D_1 - D_2\|_\infty \le \beta_2$, $\beta_1\beta_2 < 1$, then D_2 is invertible and $\|D_2^{-1}\|_\infty \le \frac{\beta_1}{1-\beta_1\beta_2}$. For the matrix B given in (4.35), we can decompose it into $B = B_1 + B_2$, where

$$B_1 = \begin{pmatrix} -(w_k\mu_k)_{k<0} & (w_k\mu_k)_{k>0} \\ -\big(w_k\mu_k l_k^{(m)}\big)_{-M+1\le m\le -1, k<0} & 0 \\ 0 & \big(w_k\mu_k l_k^{(m)}\big)_{1\le m\le M-1, k>0} \\ -\tilde\alpha(z_R)(w_k\mu_k)_{k<0} & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ 0 & E_1\big(w_k\mu_k l_k^{(m)}\big)_{-M+1\le m\le -1, k>0} \\ -E_2\big(w_k\mu_k l_k^{(m)}\big)_{1\le m\le M-1, k<0} & 0 \\ -(\epsilon w_k\mu_k^2)_{k<0} & (\epsilon w_k\mu_k^2)_{k>0} \end{pmatrix}.$$

Since all columns of B_1 are linearly independent and no ϵ term appears, B_1 is invertible and $\|B_1^{-1}\|_\infty$ is independent of ϵ. When ϵ becomes small $\|B_2\|_\infty$ either decays exponentially in ϵ because of the elements of E_1, E_2 or linearly. There exists $\epsilon_0 < 1$ such that when $\epsilon < \epsilon_0$, $\|B_1^{-1}\|_\infty \|B_2\|_\infty < 1/2$. Thus $\|B^{-1}\|_\infty < 2\|B_1^{-1}\|_\infty$, which is uniformly bounded with respect to ϵ. Thus we complete the proof for $\mathfrak{L}_{Rm}$. The proof for $\mathfrak{L}_{Lm}$ is almost the same. ∎

According to the diffusion theory of discrete-ordinate equations [14], the solution of (1.3) ψ_m can be written as

$$\psi_m = \phi - \epsilon \frac{\mu_m}{\sigma_T}\frac{d\phi}{dz} + \epsilon^2 g_m + \gamma_{Lm} + \gamma_{R-m}. \tag{4.36}$$

Here ϕ is the solution of the diffusion equation (2.18). γ_{Lm},γ_{Rm} are the boundary layer correctors which decay exponentially to zero as $z \to \infty$, and each satisfies a half-space problem of the form

$$\mu_m \partial_x \gamma_m + \gamma_m - \sum_{n\in V} w_n \gamma_n = 0 \tag{4.37a}$$

over $(0,\infty)\times V$ with boundary conditions

$$\gamma_{Lm} = \psi_{Lm} - \phi(z_L) + \epsilon\frac{\mu_m}{\sigma_T(z_L)}\partial_z\phi(z_L), \quad \text{for } m>0 \tag{4.37b}$$

$$\gamma_{Rm} = \psi_{Rm} - \phi(z_R) - \epsilon\frac{\mu_m}{\sigma_T(z_L)}\partial_z\phi(z_L), \quad \text{for } m>0 \tag{4.37c}$$

where

$$x = \frac{\alpha(z)}{\epsilon} = \frac{1}{\epsilon}\int_{z_L}^{z}\sigma_T(s)ds. \tag{4.38}$$

The next lemma establishes the uniform boundedness of g_m.

Lemma 4.4. *For g_m defined in (4.36), we have*

$$\|g_m\|_{L^\infty([z_L,z_R]\times V)} < C_g,$$

where C_g is independent of ϵ.

Proof. It has been shown in Theorem 3.2 in [14] that, when there is no interface, $\|g_m\|_{L^\infty([z_L,z_R]\times V)}$ is uniformly bounded with respect to ϵ. With an interface at $z=0$, since there is no sharp interface layer on both sides of the interface, by the same analysis as in [14], if

$$\Big\|\Big(\frac{1}{\sigma_T}\frac{d}{dz}\Big)^k\phi\Big\|_{L^\infty([z_L,z_R]\backslash 0)}, \qquad k=1,2,3,4 \tag{4.39}$$

are uniformly bounded, we have $\|g_m\|_{L^\infty([z_L,z_R]\times V)}$ uniformly bounded.

The maximum of ϕ in $[z_L,z_R]$ is at the boundary, or the interface or the interior of the domain. If it is at the boundary, we have $\partial_z\phi|_{z_L}<0$ or $\partial_z\phi|_{z_R}>0$. From (2.18b),

$$\|\phi\|_{L^\infty[z_L,z_R]} < \max\{\sum_{m=1}^{M}\psi_{Lm}w_m, \sum_{m=-1}^{-M}\psi_{Rm}w_m\} \le \max(\|\psi_L\|,\|\psi_R\|)\,.$$

If the maximum is at the interface $z=0$, since $\partial_z\phi(0^-)$, $\partial_z\phi(0^+)$ have the same sign by the interface condition (2.18c), ϕ will decrease or increase simultaneously on both sides of the interface. Then $\partial_z\phi(0^+) = \partial_z\phi(0^-) = 0$ and $\partial_z^2\phi(0^+), \partial_z^2\phi(0^-) < 0$. Thus (2.18a) gives $\|\phi\|_{L^\infty[z_L,z_R]} < \|\frac{q}{\sigma_a}\|_{L^\infty[z_L,z_R]}$. This also holds when the maximum occurs at neither boundary nor interface points. In summary, $\|\phi\|_{L^\infty[z_L,z_R]}$ is bounded independent of ϵ. Use the same discussion as in Lemma 2.1 in [14], we have that

$$\Big\|\Big(\frac{1}{\sigma_T}\frac{d}{dz}\Big)^k\phi\Big\|_{L^\infty([z_L,z_R]\backslash 0)}, \qquad k = 1,2,3,4$$

are also uniformly bounded. The lemma is proved. ∎

LEMMA 4.5. *For Θ defined in (4.27), we have*

$$\|\Theta\|_{L^\infty([z_L,z_R]\times V)} < C_\theta \Delta z^2,$$

where C_θ is some constant independent of ϵ, Δz.

Proof. It is obvious that the integrals in the definition of $\Lambda_k^{(2)}$ can be divided into the sum of $\int_{z_i}^{z_{i+1}}$, $i = 0,1,\cdots,N-1$. So do the integrals in the definition of Θ_k (4.27), by assuming $z \in (z_j, z_{j+1}]$, can be divided into the sum of $\int_{z_i}^{z_{i+1}}$, $i = 0,\cdots,j-1$ and $\int_{z_j}^{z}$. When $1 \le k \le M-1$, all integrals in $\Lambda_k^{(2)}$ and Θ_k defined in the interval $[z_i, z_{i+1}]$ can be controlled by some constant multiplying

$$\int_{z_i}^{z_{i+1}} e^{\frac{\tilde\alpha(z)-\tilde\alpha(z_{i+1})}{\epsilon}\xi_k}\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m dz. \tag{4.40}$$

Let $\Delta z = |z_{i+1} - z_i|$. Inserting (4.36) into the definition of $\mathfrak{Q}_m$ in (4.1) gives

$$\begin{aligned}
&\mathfrak{Q}_m \\
=&-\frac{\sigma_T-\tilde\sigma_T}{\epsilon}\Big(\phi - \epsilon\frac{\mu_m}{\sigma_T}\frac{d\phi}{dz} + \epsilon^2 g_m + \gamma_{Lm} + \gamma_{Rm}\Big) + \epsilon(q-\tilde q)\\
&+\Big(\frac{\sigma_T-\tilde\sigma_T}{\epsilon} - \epsilon(\sigma_a - \tilde\sigma_a)\Big)\Big(\phi + \epsilon^2\sum_{n\in V} w_n g_n + \sum_{n\in V} w_m\gamma_{Lm} + \sum_{n\in V} w_m\gamma_{Rm}\Big)\\
=&\frac{\sigma_T-\tilde\sigma_T}{\sigma_T}\mu_m\frac{d\phi}{dz} - \epsilon(\sigma_a-\tilde\sigma_a)\Big(\phi + \epsilon^2\sum_{n\in V} w_n g_n\Big)\\
&+\epsilon(\sigma_T-\tilde\sigma_T)\Big(\sum_{n\in V} w_n g_n - g_m\Big) + \epsilon(q-\tilde q)\\
&-\frac{\sigma_T-\tilde\sigma_T}{\epsilon}\Big(\gamma_{Lm} - \sum_{n\in V} w_n\gamma_{Ln} + \gamma_{Rm} - \sum_{n\in V} w_n\gamma_{Rn}\Big)
\end{aligned} \tag{4.41}$$

Using the eigenfunction (2.5), the definition of (2.6) and (2.7), one gets

$$\begin{aligned}
\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m =& -\frac{\sigma_T-\tilde\sigma_T}{\epsilon}\xi_k\Big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\gamma_{Lm} + \sum_{m\in V} w_m\mu_m l_m^{(k)}\gamma_{Rm}\Big)\\
&+\epsilon\Big((\sigma_T-\tilde\sigma_T)\sum_{m\in V} w_m g_m - (\sigma_a-\tilde\sigma_a)\phi + (q-\tilde q)\Big)\\
&-\epsilon^3(\sigma_a-\tilde\sigma_a)\sum_{m\in V} w_m g_m - \epsilon(\sigma_T-\tilde\sigma_T)\sum_{m\in V} w_m l_m^{(k)} g_m.
\end{aligned}$$

In order to estimate (4.40), firstly consider

$$-\int_{z_i}^{z_{i+1}} \exp\Big(\frac{\tilde\alpha(x)-\tilde\alpha(z_{i+1})}{\epsilon}\xi_k\Big)\frac{\sigma_T-\tilde\sigma_T}{\epsilon}\xi_k\sum_{m\in V} w_m\mu_m l_m^{(k)}\gamma_{Lm}.$$

Similar to the derivation from (4.9) to (4.11), multiplying both sides of (4.37a) by $w_m l_m^{(k)}$ and summing over V gives

$$\partial_x\big(\sum_{m\in V} w_m\mu_m l_m^{(k)}\gamma_{Lm}\big)+\xi_k\sum_{m\in V} w_m\mu_m l_m^{(k)}\gamma_{Lm}=0.$$

Then

$$\sum_{m\in V} w_m\mu_m l_m^{(k)}\gamma_{Lm}=C\exp(-\xi x)=C\exp\big(-\frac{\xi_k}{\epsilon}\alpha(z)\big),$$

where C is a constant depending only on the boundary data and $\alpha(z)$ is given in (4.38). From the definition of $\tilde{\sigma}_T$ in (3.1), $\tilde{\alpha}(z)$ (4.15), $\alpha(z)$ (4.38), we have $\sigma_T(z)=\partial_z\alpha(z)$, $\tilde{\sigma}_T(z)=\partial_z\tilde{\alpha}(z)$ and $\tilde{\alpha}(z_j)=\alpha(z_j),\forall j$. Thus,

$$\begin{aligned}
&-\int_{z_i}^{z_{i+1}}\exp\Big(\frac{\tilde{\alpha}(z)-\tilde{\alpha}(z_{i+1})}{\epsilon}\xi_k\Big)\frac{\sigma_T-\tilde{\sigma}_T}{\epsilon}\xi_k\sum_{m\in V} w_m\mu_m l_m^{(k)}\gamma_{Lm}dz\\
&=-C\exp\Big(-\frac{\tilde{\alpha}(z_{i+1})}{\epsilon}\xi_k\Big)\int_{z_i}^{z_{i+1}}\frac{\sigma_T-\tilde{\sigma}_T}{\epsilon}\xi_k\exp\Big(\frac{\tilde{\alpha}(z)-\alpha(z)}{\epsilon}\xi_k\Big)dz\\
&=C\exp\Big(-\frac{\tilde{\alpha}(z_{i+1})}{\epsilon}\xi_k\Big)\Big(\exp\Big(\frac{\tilde{\alpha}(z_{i+1})-\alpha(z_{i+1})}{\epsilon}\xi_k\Big)-\exp\Big(\frac{\tilde{\alpha}(z_i)-\alpha(z_i)}{\epsilon}\xi_k\Big)\Big)\\
&=0. \qquad (4.42)
\end{aligned}$$

Note the cancelation in (4.42) is exactly due to the definition of $\tilde{\sigma_T}(x)$ as in (3.1). From Lemma 4.2, we have

$$\int_{z_i}^{z_{i+1}}\exp\Big(\frac{\tilde{\alpha}(x)-\tilde{\alpha}(z_{i+1})}{\epsilon}\xi_k\Big)\sum_{m\in V} w_m l_m^{(k)}\mathfrak{Q}_m dx\sim O(\delta z^3)\,, \qquad (4.43)$$

by using (4.42). The same results hold for $-M+1\le k\le -1$.

Moreover, we also need to estimate

$$\int_{z_i}^{z_{i+1}}\sum_{m\in V} w_m\mu_m\mathfrak{Q}_m dsdx,\qquad \int_{z_i}^{z_{i+1}}\sum_{m\in V} w_m\mathfrak{Q}_m dsdx$$

that appear in the integrals in $\Lambda_M^{(2)}$ and Θ_M when confined in $[z_i,z_{i+1}]$. Multiplying both sides of (4.37a) by w_m and summing over V gives

$$\partial_x\Big(\sum_{m\in V} w_m\mu_m\gamma_m\Big)=0.$$

Then, since γ_m decays exponentially to zero, $\sum_{m\in V} w_m\mu_m\gamma_m=0$ over $[0,\infty)$, thus, from (4.41),

$$\begin{aligned}
&\int_{z_i}^{z_{i+1}}\sum_{m\in V} w_m\mu_m\mathfrak{Q}_m dsdx\\
&=\int_{z_i}^{z_{i+1}}\Big(\frac{\sigma_T-\tilde{\sigma}_T}{3\sigma_T}\frac{d\phi}{dz}-\epsilon(\sigma_T-\tilde{\sigma}_T)\sum_{m\in V} w_m\mu_m g_m\Big)dsdx\sim O(\Delta z^3), \qquad (4.44)
\end{aligned}$$

$$
\begin{aligned}
&\int_{z_i}^{z_{i+1}} \sum_{m\in V} w_m \mathfrak{Q}_m ds dx \\
=&\int_{z_i}^{z_{i+1}} \Big((\sigma_a-\tilde{\sigma}_a)\Big(\phi+\epsilon^2\sum_{n\in V} w_n g_n\Big)-(q-\tilde{q})\Big) ds dx \sim \epsilon O(\Delta z^3). \qquad (4.45)
\end{aligned}
$$

In summary $\Lambda_k^{(2)}$ is $O(\Delta z^2)$ for $1\le |k|\le M$, namely, $\|\Lambda^{(2)}\|_\infty \sim O(\Delta z^2)$. And Lemma 4.3 implies $\mathfrak{L}_{Lm}\Lambda^{(2)} \sim \Delta z^2$, $\mathfrak{L}_{Rm}\Lambda^{(2)} \sim \Delta z^2$. When $z\in[z_j, z_{j+1}]$, the analogous estimate of $\int_{z_j}^{z} \sim O(\Delta z^2)$ or $\epsilon O(\Delta z^2)$ is easy to obtain. Then by using (4.43)-(4.45), a similar discussion shows that the integrals appearing in the definition of Θ_k in (4.27) are also of order Δz^2. Therefore we prove our result. □

Now we are ready to prove Theorem 4.1. From the definition of Ω, Π, $\Lambda^{(1)}$ and Lemma 4.3, it is obvious that

$$
\|\Omega\|_\infty < C_\omega, \qquad \|\Pi^{-1}\|_\infty < C_\pi \qquad (4.46)
$$

where C_ω, C_π are constants independent of Δz and ϵ. (4.31) implies

$$
\begin{aligned}
\big\|\tilde{e}^{(i)}(z)\big\|_{L^\infty([z_L,z_R]\times V)} &= \epsilon^i \big\|(\Pi^{-1}\Omega)^i\Theta(z)\big\|_{L^\infty([z_L,z_R]\times V)} \\
&\le \epsilon^i \|\Pi^{-1}\Omega\|_\infty^i \|\Theta\|_{L^\infty([z_L,z_R]\times V)}.
\end{aligned}
$$

Assume when $\epsilon<\epsilon_0$, $\epsilon\|\Pi^{-1}\Omega\|_\infty \le \epsilon\|\Pi^{-1}\|_\infty\|\Omega\|_\infty < 1$. Using the Neumann series (4.30) and Lemma 4.5, one gets

$$
\begin{aligned}
\|\tilde{\mathbf{e}}\|_\infty < \sum_{i=0}^{\infty}\|\tilde{\mathbf{e}}_{(i)}\|_\infty < \sum_{i=0}^{\infty}\Big(\|\Pi^{-1}\|_\infty C_\omega\epsilon\Big)^i\|\Theta\|_\infty &= \frac{1}{1-\|\Pi^{-1}\|_\infty C_\omega \epsilon}\|\Theta\|_\infty \\
&< \frac{C_\theta}{1-\|\Pi^{-1}\|_\infty C_\omega\epsilon}\Delta z^2.
\end{aligned}
$$

Then, following the fact that the coefficients in (4.8) are independent of ϵ, we complete our proof. □

Remark: The result stated in Theorem 4.1 is not only an improvement of the result of [14] for the convergence rate. Moreover, it shows that even if the boundary layer is not numerically resolved, we still have a second order convergence. This latter property is the unique feature of the new method due to the use of the exact solution in each cell.

5. Numerical examples. In this section, we present numerical results for several problems chosen to verify the performance of the scheme described above. In all the test problems we have used the S_{16} standard Gaussian quadrature set for the velocity space. As usual, the "exact" solutions were obtained using a much finer, resolved grid in space. In all the figures we plot and errors in the tables we display the average density defined by

$$
\rho(z) = \sum_{m\in V} w_m \psi_m(z).
$$

We shall consider three transport problems in slab geometry.

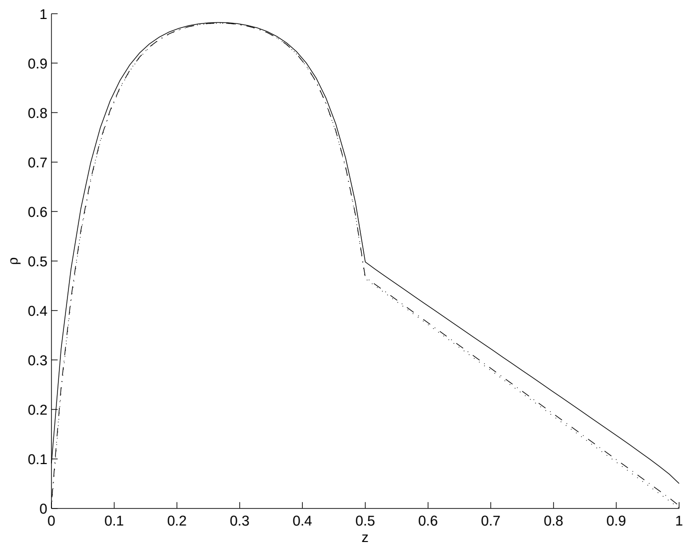

FIG. 5.1. *Example 1. The numerical solutions of* $\rho(z)$ *for* $\epsilon = 0.1, 0.01, 0.001$ *are plotted by the solid line, dashed line and dotted line respectively.*

Example 1

$$\begin{array}{lllll} z \in [0,1]. & \psi_m(0) = 0, \quad \mu_m > 0; & \psi_m(1) = 0, \quad \mu_m < 0, & & \\ \sigma_T = 10, & \sigma_a = 10, & q = 10, & \epsilon = 0.1, & z \in [0, 0.5] \\ \sigma_T = 1, & \sigma_a = 0, & q = 0, & \epsilon = 0.1. & z \in [0.5, 1] \end{array}$$

In this problem we have a slab with a flat interior source adjoining a purely scattering slab with no interior source. It is obvious that the method gives the exact solution for this example since σ_T, σ_a, q are piecewise constants. The results of $\epsilon = 0.1, 0.01, 0.001$ are given in Figure 5.1. One can see that there is no exponential change near the interface as $\epsilon \to 0$, indicating no interface layer on both sides of the interface (which is what we showed in Section 2). Note that the solutions of $\epsilon = 0.01$ and $\epsilon = 0.001$ are quite close, since they are both close to the diffusion limit. One can also see that the solution is continuous while the first derivatives jump at the interface.

Example 2

$$\begin{array}{lllll} z \in [0,2]\,. & \psi_m(0) = 0, \quad \mu_m > 0; & \psi_m(2) = 0, \quad \mu_m < 0 & & \\ \sigma_T = 10z + 1, & \sigma_a = 1 + z, & q = 10, & \epsilon = 0.01, & z \in [0,1]\,; \\ \sigma_T = z, & \sigma_a = 1, & q = 0, & \epsilon = 0.01. & z \in [1,2]\,. \end{array}$$

In this problem, the source term and the scattering cross section depend on z. The reference 'exact' solution is obtained by the new method with a very fine mesh, $\Delta z = 1/256$. We can see from Figure 5.2 that accurate numerical results can be obtained by coarse meshes. The numerical errors of this method for different Δz are listed in Table 5.1. It is obvious that the convergent order of this method is two, which agrees with our error estimate. The L^∞ norm of the errors for different ϵ and

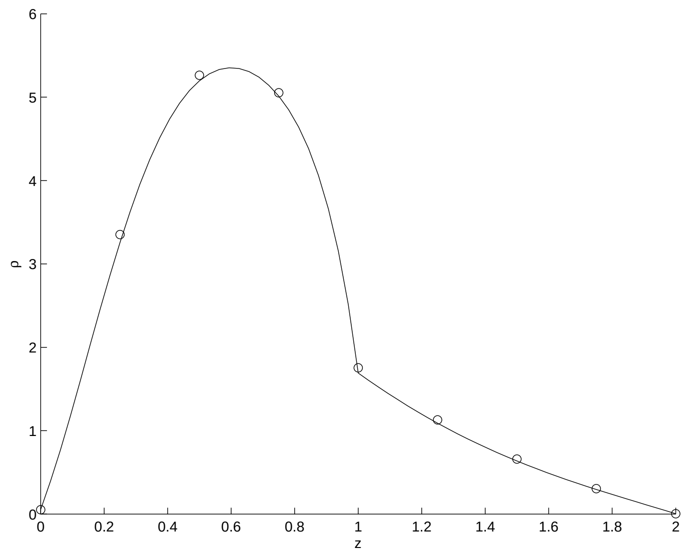

FIG. 5.2. *Example 2. Numerical solutions with $\Delta z = 1/4$ are represented by circles, while the solid line is the 'exact' solution which is calculated by the same numerical method with $\Delta z = 1/256$*

Δz	$\|\text{Error}\|_\infty$
2^{-2}	0.0945
2^{-3}	0.0238
2^{-4}	0.0059
2^{-5}	0.0014

TABLE 5.1

Example 2. The error between the "exact" solution and the numerical solutions computed with different Δz. Here the 'exact solution' is given by the same numerical method with $\Delta z = 1/256$.

Δz are listed in Table 5.2. The uniform quadratic convergence is easy to see.

Example 3

$$z \in [0,1]. \qquad \psi_m(0) = 5\mu_m, \quad \mu_m > 0; \qquad \psi_m(1) = 0, \quad \mu_m < 0$$
$$\sigma_T = 1 + (10z)^2, \qquad \sigma_a = 1 + (10z)^2, \qquad q = 0, \qquad \varepsilon_1 = 0.01, \qquad z \in [0, 0.2]\,;$$
$$\sigma_T = 1, \qquad \sigma_a = z, \qquad q = 2, \qquad \varepsilon_2 = 1. \qquad z \in [0.2, 1]\,.$$

In this example the mean free paths are different at the two sides of the interface. We can see clearly in Figure 5.3 that this method still gives a good result and a good description of the internal layer and of the boundary layer, *even if the boundary layer is not resolved numerically.* The error between the exact solution and the numerical solutions computed by this method with different Δz for different ϵ are shown in the Table 3. Here the 'exact' solution refers to the result computed by this numerical method when $\Delta z = 1/200$.

6. Conclusion. In this paper, we study a numerical method for the discrete-ordinate linear transport equation with boundary and interface. The main idea of the

Δz \ ϵ	1	0.1	0.01
2^{-2}	0.27615	0.00776	0.0945
2^{-3}	0.0865	0.0197	0.0238
2^{-4}	0.0230	0.0050	0.0059
2^{-5}	0.0056	0.0012	0.0014

TABLE 5.2

Example 2. The L^∞ norm of the errors between the "exact" solution and the numerical solutions computed with different Δz for different ϵ, where the 'exact solution' is calculated by the same numerical method with $\Delta z = 1/256$.

Δz	$\|\text{Error}\|_\infty$
1/5	$2.854 * 10^{-2}$
1/10	$7.268 * 10^{-3}$
1/20	$1.813 * 10^{-3}$
1/40	$4.400 * 10^{-4}$

TABLE 5.3

Example 3. The error between the "exact" solution and the numerical solutions computed by the numerical method with different Δz , where the 'exact' solution refers to the result computed with $\Delta z = 1/200$

method is to first approximate the scattering coefficients using their cell averages, and then solve the local equation in each cell *excatly* [12]. Numerical solutions at different cells are pieced together using the interface condition which requires the density distribution to be continuous. This method is shown to be asymptotic-preserving, thus captures the correct diffusion limit with the correct interface condition. Moreover, we prove the this method *converges quadratically, uniformly in the mean free path.* This result is sharp, in constract to the previous analysis [14] in the same direction. Moreover, the method *converges even if the boundary layer is not numerically resolved.*

In the future we will study this method in two space dimensions [3], and some other related transport equations.

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Appendix A.

THEOREM A.1. *Consider the equation*

$$\sum_{n\in V} \frac{w_n}{1-\mu_n \xi} = \frac{1}{r}. \tag{A.1}$$

(i) *When $r<1$, (A.1) has $2M$ simple roots occur in positive/negative pairs, let them be ξ_n $(1 \le |n| \le M)$. Assume $l_m^{(n)} = \frac{1}{1-\mu_m \xi_n}$, we have*

$$\sum_{m\in V} w_m \mu_m l_m^{(k)} l_m^{(n)} = \begin{cases} 0 & n \neq k \\ c^{(k)} & n = k \end{cases}, \tag{A.2}$$

where $c^{(k)}$ satisfy

$$\sum_{k\in V} \frac{1}{c^{(k)}} l_m^{(k)} l_n^{(k)} = \begin{cases} 0 & m \neq n \\ \frac{1}{w_n \mu_n} & m = n \end{cases}. \tag{A.3}$$

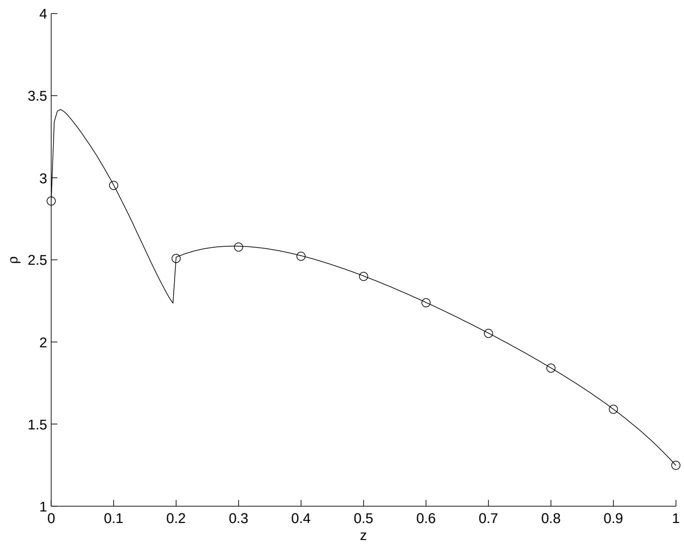

FIG. 5.3. *Example 3. Numerical solutions with* $\Delta z = 1/10$ *are represented by circles, while the "exact" solution is represented by the solid lines.*

(ii) *When* $r = 1$, *(A.1) has* $2M-2$ *simple roots appear in positive/negative pairs while 0 is a double root. Assume* ξ_n $(1 \le n \le M-1)$ *is the unique (positive, simple) root in* $(1/\mu_{n+1}, 1/\mu_n)$, $\xi_{-n} = -\xi_n$ *and* $l_m^{(n)} = \frac{1}{1-\mu_m \xi_n}$. *(A.2) still holds for* $k, n \in \{-M+1, \cdots, M-1\}$. *Moreover, defining* $c^{(-M)} = c^{(M)} = \sum_{m\in V} w_m \mu_m^2 = 1/3$, *we have*

$$\sum_{1\le|k|\le M-1} \frac{1}{c^{(k)}} l_n^{(k)} l_m^{(k)} + \frac{\mu_n}{c^{(-M)}} + \frac{\mu_m}{c^{(M)}} = \begin{cases} 0 & m \neq n \\ \frac{1}{w_n \mu_n} & m = n \end{cases}, \tag{A.4}$$

and

$$\sum_{m\in V} \mu_m w_m l_m^{(k)} = 0, \qquad \sum_{m\in V} \mu_m^2 w_m l_m^{(k)} = 0. \tag{A.5}$$

Proof. Let

$$W = \begin{pmatrix} w_{-M} & \cdots & w_M \\ \vdots & \vdots & \vdots \\ w_{-M} & \cdots & w_M \end{pmatrix}, \qquad U = \begin{pmatrix} \mu_{-M} & & \\ & \ddots & \\ & & \mu_M \end{pmatrix},$$

$$A = U^{-1}(I - rW),$$

where I is the identity matrix of order $2M$. Assume $l^{(n)}$ is the eigenvector of A corresponding to ξ_n. Then

$$U^{-1}(I - rW) l^{(n)} = \xi_n l^{(n)}$$

holds and

$$l_m^{(n)} = \frac{r \sum_{k\in V} w_k l_k^{(n)}}{1 - \xi_n \mu_m}, \qquad m \in V. \tag{A.6}$$

Multiplying $l_m^{(n)}$ by w_m and summing over V give the eigenfunction as follows:

$$\sum_{m\in V}\frac{w_m}{1-\xi\mu_m}=\frac{1}{r},$$

which is just (A.1). The properties of (A.1)'s roots when $r<1$ and $r=1$ as shown in the theorem have already been studied in [8]. From (A.6), let

$$l_m^{(n)}=\frac{1}{1-\xi_n\mu_m},\quad l^{(n)}=(l_{-M}^{(n)},\cdots,l_M^{(n)}) \tag{A.7}$$

is the eigenvector corresponding to ξ_n.

(i) When $r<1$, A is diagonalizable. Let the eigenvalues satisfy $\xi_n=\xi_{-n}$, for $n=1,\cdots,M$. Assume $A=P^{-1}DP$, where

$$D=\mathrm{diag}\{\xi_{-M},\cdots,\xi_{-1},\xi_1,\cdots,\xi_M\}.$$

Then the nth column of P^{-1} is the eigenvector corresponding to ξ_n and P^{-1} can be written as

$$\Big[\big(P^{-1}\big)_{mn}\Big]=\Big[\Big(\frac{1}{1-\xi_n\mu_m}\Big)_{mn}\Big]\mathrm{diag}\{c_{-M},\cdots,c_{-1},c_1,\cdots,c_M\}\,. \tag{A.8}$$

Here c_i are some constants, $[(a_{ij})_{mn}]$ denotes the matrix whose element in row m and column n is a_{ij}. Now consider A^T, the eigenvalues of A^T are the same as A. Assuming $\iota^{(n)}$ is the eigenvector of A^T corresponding to ξ_n, we have

$$(I-rW^T)U^{-1}\iota^{(n)}=\xi_n\iota^{(n)}.$$

Thus

$$\iota_m^{(n)}=\frac{r\mu_m w_m}{1-\mu_m\xi_n}\sum_{k\in V}\frac{\iota_k}{\mu_k},$$

and

$$\iota_m^{(n)}=\frac{\mu_m w_m}{1-\mu_m\xi_n}$$

is an eigenvector corresponding to ξ_n. Since $A^T=P^TD(P^T)^{-1}$, the nth column of P^T is the eigenvector of A^T corresponding to ξ_n and P can be written as

$$\Big[(P)_{mn}\Big]=\mathrm{diag}\{c'_{-M},\cdots,c'_M\}\Big[\Big(\frac{1}{1-\xi_m\mu_n}\Big)_{mn}\Big]\mathrm{diag}\{w_{-M}\mu_{-M},\cdots,w_M\mu_M\}. \tag{A.9}$$

From (A.8)(A.9),

$$\Big[\big(l_n^{(m)}\big)_{mn}\Big]\mathrm{diag}\{w_{-M}\mu_{-M},\cdots,w_M\mu_M\}\Big[\big(l_m^{(n)}\big)_{mn}\Big]$$

is equal to an diagonal matrix

$$\mathrm{diag}\Big\{\frac{1}{c_{-M}c'_{-M}},\cdots,\frac{1}{c_{-1}c'_{-1}},\frac{1}{c_1c'_1},\cdots,\frac{1}{c_Mc'_M}\Big\}.$$

Let $c^{(k)} = \frac{1}{c_k c'_k}$,

$$\Big[(l_n^{(m)})_{mn}\Big]\mathrm{diag}\{w_{-M}\mu_{-M},\cdots,w_M\mu_M\}\Big[(l_m^{(n)})_{mn}\Big]\mathrm{diag}\Big\{\frac{1}{c^{(-M)}},\cdots,\frac{1}{c^{(M)}}\Big\},$$

$$\Big[(l_m^{(n)})_{mn}\Big]\mathrm{diag}\Big\{\frac{1}{c^{(-M)}},\cdots,\frac{1}{c^{(M)}}\Big\}\Big[(l_n^{(m)})_{mn}\Big]\mathrm{diag}\{w_{-M}\mu_{-M},\cdots,w_M\mu_M\}$$

are identity matrices. This gives (A.2)(A.3).

(ii) When $r = 1$ in A, A is singular. We have to consider its Jordan Canonical Form. Suppose $A = P^{-1}JP$, where

$$J = \begin{pmatrix} \begin{pmatrix} \xi_{-M+1} & & \\ & \ddots & \\ & & \xi_{M-1} \end{pmatrix} & \\ & \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} \end{pmatrix}, \tag{A.10}$$

The eigenvector corresponding to ξ_n, $1 \le |n| \le M-1$, is still $l^{(n)}$ defined in (A.7). Let $\mathbf{i} = (1,\cdots,1)^T_{2M\times 1}$ and $\mathbf{u} = (\mu_{-M},\cdots,\mu_{-1},\mu_1,\cdots,\mu_M)^T$. It is easy to check that $A\mathbf{i} = 0$ and $A\mathbf{u} = \mathbf{i}$, then P^{-1} can be written as

$$P^{-1} = [l^{(-M+1)},\cdots,l^{(M-1)},\mathbf{u},\mathbf{i}].$$

As for A^T, we can get

$$P = \mathrm{diag}\{c^{(-M+1)},\cdots,c^{(M-1)},c^{(-M)},c^{(M)}\}[l^{(-M+1)},\cdots,l^{(M-1)},\mathbf{i},\mathbf{u}]^T \\ \times\mathrm{diag}\{w_{-M}\mu_{-M},\cdots,w_M\mu_M\}$$

by a similar discussion. Since $P^{-1}P$, PP^{-1} are identity matrices, (A.4)(A.5) are obtained. □

REFERENCES

[1] G. Bal, L. Ryzhik, *Diffusion approximation of radiative transfer problems with interfaces*, SIAM J. App. Math. ,60(2000), pp. 1887–1912

[2] C. Bardos, R. Santos and R. Sentis,*Diffusion approximation and computation of the critical size*, Trans. Amer. Math. Soc., 284(1984), pp. 617–649

[3] R. C. De Barros, F. C. Da Silva and H. A. Filho, *Recent advances in spectral nodal methods for x,y-geometry discrete ordinates deep penetration and eigenvalue problems*, Progress in Nuclear Energy, 35(1999), pp. 293–331

[4] A. E. Berger,J. M. Solomon and M. Ciment, *An analysis of a uniformly accurate difference Method for a singular perturbation problem*, Math. of Comput., 37(155)(1981),pp. 79–94

[5] A. E. Berger, H. Han and R. B. Kellogg, *A priori estimates and analysis of a numerical method for a turning point problem*, Math. of Comput., 42(166)(1984), pp. 465–492

[6] K. M. Case and P. F. Zweifel, *Existence and uniqueness theorems for the neutron transport equation*, J. Math. Phys., 4(11)(1963)

[7] R. Caflisch, S. Jin and G. Russo, *Uniformly accurate schemes for hyperbolic systems with relaxations*, SIAM J. Num. Anal., 34(1997), pp. 246–281

[8] S. Chandrasekhar, *Radiative Transfer*,Dover, New York, (1960)

[9] P. Crispel, P. Degond and M.-H. Vignal, *An asymptotic preserving scheme for the two-fluid Euler-Poisson model in the quasineutral limit*, J. Comp. Phys., 223(2007), pp. 208–234,

[10] P. Degond, S. Jin and J. G. Liu, *Mach-number uniform asymptotic-preserving gauge schemes for compressible flows*, Bulletin of the Institute of Mathematics, Academia Sinica, New Series, 2(4)(2007), pp. 851–892,

[11] P. Degond, S. Jin and M. Tang, *On the time-splitting spectral method for the complex Ginzburg-Landau equation in the large time and space scale limit*. SIAM J. Sci. Comp., to appear.

[12] R. C. De Barros and E. W. Larsen, *A numerical method for one-group slab-geometry discrete ordinates problems with no spacial truncation error*, Nucl .Sci. Eng., 104(1990), pp. 199–208

[13] P. Edstrom,*A fast and stable solution method for the radiative transfer problem*, SIAM Rev. 47(3)(2005), pp. 447–468

[14] F. Golse, S. Jin and C. D. Levermore, *The convergence of numerical transfer schemes in diffusive regimes I: discrete-ordinate method*, SIAM J. Numer. Anal. 36(5)(1999), pp. 1333–1369

[15] S.K. Godunov, *A finite difference method for the numerical computation and discontinuous solutions of the equations of fluid dynamics*, Mat. Sb., 47(1959), pp. 271

[16] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 3rd edition, (1996)

[17] S. Jin, *Runge-Kutta methods for hyperbolic conservation laws with stiff relaxation terms*, J. Comp. Phys. 122(1995), pp. 51–67.

[18] S. Jin, *Efficient Asymptotic-Preserving (AP) Schemes for Some Multiscale Kinetic Equations*, SIAM J. Sci. Comp., 21(1999), pp. 441–454

[19] S. Jin, C. D. Levermore, *The discrete-ordinate method in diffusive regimes*, Transport theory and statistical physics, 20(5)(1991), pp. 413–439

[20] S. Jin, C. D. Levermore, *Fully-discrete numerical transfer in diffusive regimes*, Transport theory and statistical physics, 22(6)(1993), pp. 739–791

[21] S. Jin and C. D. Levermore, *Numerical schemes for hyperbolic conservation laws with stiff relaxation terms*, J. Comp. Phys. 126(1996), pp. 449–467.

[22] S. Jin, L. Pareschi and G. Toscani, *Diffusive relaxation schemes for discrete-velocity kinetic equations*, SIAM J. Num. Anal. 35(1998), pp. 2405–2439

[23] S. Jin, L. Pareschi and G. Toscani, *Uniformly accurate diffusive relaxation schemes for multiscale transport equations*, SIAM. J. Numer. Anal. 38(3)(2001), pp. 913–936

[24] S. Jin, X. Liao and X. Yang, *Computation of interface reflection and regular or diffuse transmission of the planar symmetric radiative transfer equation with isotropic scattering and its diffusion limit*, SIAM J. Sci. Comp., SIAM J. Sci. Comp., 30, 1992-2017, 2008

[25] S. Jin, X. Yang and G.W. Yuan, *A domain decomposition method for a two-scale transport equation with energy flux conserved at the interface, Kinetic and Related Models*, 1(2008), pp. 65–84,

[26] R. B. Kellogg and Han Houde, *Numerical analysis of singular perturbation problems*, Hemisphere Publ Corp, USA. Distributed outside North America by Springer-Verlag, (1983) pp. 323–335

[27] A. Klar, *An asymptotic induced scheme for nonstationary transport equations in the diffusive limit*, SIAM J. Num. Anal. 35(3)(1998), pp. 1073–1094,

[28] P. Kubelka, *New contribution to the optics of intensely light-scattering materials. Part I*, J. Opt. Soc. Amer., 38(1931), pp. 448–457

[29] E. W. Larsen, *Unconditionally stable diffusion-synthetic acceleration methods for the slab geometry discrete ordinates equations. Part I: Theory*, Nuclear Science and Engineering, 82(1982), pp. 47–63

[30] E. W. Larsen, J. E. Morel and W. F. Miller Jr., *Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes*, J. Comput. Phys., 69(1987), pp. 283–324

[31] E. W. Larsen, J. E. Morel, *Asymptotic solutions of numerical transport problems in optically thick,diffusive regimes II*, J. Comput. Phys., 83(1989), pp. 212–236

[32] A. Schuster, *Radiation through a foggy atmosphere*, Astrophys. J., 21(1905), pp. 1–22