A stochastic Galerkin method for Hamilton-Jacobi equations with uncertainty

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October 8, 2014

Abstract

We develop a class of stochastic numerical schemes for Hamilton-Jacobi equations with random inputs in initial data and/or the Hamiltonians. Since the gradient of the Hamilton-Jacobi equations gives a symmetric hyperbolic system, we utilize the generalized polynomial chaos (gPC) expansion with stochastic Galerkin procedure in random space and the Jin-Xin relaxation approximation in physical space for shock capturing. We show that our numerical formulation preserves the symmetry and hyperbolicity of the underlying system, which allows one to efficiently quantify the uncertainty of the Hamilton-Jacobi equations due to random inputs, as demonstrated by the numerical examples.

Key words. Uncertainty quantification, Hamilton-Jacobi equations, random input, relaxation schemes, generalized polynomial chaos.

AMS subject classifications.

1 Introduction

In recent years there is a growing interest in developing efficient and robust numerical methods for uncertainty quantification (UQ) for partial differential equations (PDE) with random inputs, where the randomness can enter through initial conditions, boundary conditions, coefficients of the equations, etc. Among the most popular methods for UQ, the generalized polynomial chaos (gPC) approach [31] has received intensive attention. Combined with stochastic Galerkin (SG) method, it has been successfully applied to many physical or engineering problems, where fast convergence can be observed if the underlying solution is sufficiently smooth. See, for example, the overviews in [11, 30].

The application of the gPC-Galerkin approach to quasilinear transport equations, in which the solutions could develop singularities, is, however, quite limited. The existing work mostly focus on linear systems and scalar nonlinear problems with uncertain inputs, where the gPC-SG approach yields larger deterministic hyperbolic systems of equations ([12, 23, 24]). For nonlinear

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system of hyperbolic conservation laws, the lack of solution regularity not only prevents one from achieving high order accuracy, but also poses a more serious difficulty in the loss of hyperbolicity of the resulting gPC-SG system of equations [10]. Consequently, extra efforts are needed in order to obtain well-behaved discrete systems. One approach is to use gPC on entropy variables [22]. However, the entropy variables need be obtained by a functional minimization procedure at every mesh point and time step. This can be computationally expensive for large scale problems. It was also shown in [22] different entropy variables could lead to different levels of numerical oscillations due to Gibb’s phenomenon in the random space.

In this paper, we identify a special class of nonlinear systems of hyperbolic conservation laws that are suitable for the gPC-SG framework — the systems from the gradients of the Hamilton-Jacobi (H-J) equations. These equations arise in a variety of applications, such as optimal control, game theory, geometric optics, front propagation and semiclassical limit of quantum dynamics. In this paper, we consider a general Hamilton-Jacobi equation in $n$ space dimensions with $d$ dimensional random inputs

$$
\partial_t u + H(\nabla_x u, x, z) = 0, \quad (x, z, t) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}_+ \tag{1.1}
$$

with initial data:

$$u(x, z, 0) = u_0(x, z).$$

Here $t \geq 0$ is time, $H$ is a convex (in the first variable) Hamiltonian, $\nabla_x = (\partial_{x_1}, \cdots, \partial_{x_n})$ is the gradient with respect to $x = (x_1, \cdots, x_n)$, and $z \in \mathbb{R}^d$ is a random variable. The random variable $z$ is used to characterize the random inputs. They can be random system parameters in the equations and/or hyper-parameters parameterizing certain random input processes. For deterministic problems without the $z$ variable, this equation has been well studied both mathematically, in the framework of viscosity solution [8, 6, 7], and numerically using the shock capturing techniques [9, 19, 20]. The goal of this paper is to develop a robust and efficient numerical method for the random Hamilton-Jacobi equations subject to uncertain inputs, which, to the authors’ best knowledge, has not been studied before.

Other than random initial data and boundary conditions, there are many applications of Hamilton-Jacobi equations where the Hamiltonian can depend on random inputs. For example, in the semiclassical limit of the Schrödinger equations, the Hamiltonian is defined by

$$H(p, x, z) = \frac{1}{2} |p|^2 + V(x, z), \tag{1.2}$$

where $V$, the potential, is often obtained empirically by mean field calculations that naturally introduce uncertainty. In the case of geometric optics or level set equations, the Hamiltonian takes the form of

$$H(p, x, z) = c(x, z)|p|, \tag{1.3}$$

where the local wave speed $c$, the reciprocal of the diffraction index of the propagating media, is often uncertain or random. Consequently, it is of significant interest to study such problems and develop robust numerical methods to quantify these uncertainties.

Unlike a general nonlinear system of hyperbolic conservation laws, the gradient of the H-J equations is a symmetric and strongly hyperbolic system, thus is suitable for the gPC-SG approach. Our approach is a combination of a relaxation approach, originally proposed by Jin and Xin [14], for deterministic Hamilton-Jacobi equations, and the gPC-SG approach. First, we
convert the original H-J equations (1.1) into a system of conservation laws by taking its gradient, and then use the Jin-Xin relaxation approximation to construct a non-oscillatory shock capturing scheme. The advantage of the Jin-Xin relaxation is that the underlying convection terms are linear, which provides a very natural building block for the gPC approximation since the gPC system as well as its relaxation approximation remains globally hyperbolic. The temporal and spatial discretizations can then be done in a natural way, similar to that of deterministic systems in [15]. Our approach inherits most of the advantages of the relaxation schemes for deterministic hyperbolic conservation laws and Hamilton-Jacobi equations: simplicity and generality. Theoretically, the hyperbolicity of the new stochastic Galerkin system is guaranteed for the general cases. Practically, the linear characteristics allow one to easily apply the standard upwind schemes without requiring a new Riemann or approximate Riemann solver (as was done in [26, 25, 21]), thus can be applied to general H-J equations as a black box.

The main goal of this paper is to demonstrate that the Hamilton-Jacobi equations are adequate for uncertainty quantification using the intrusive gPC formulation since they result in a globally hyperbolic and symmetric gPC system. This is a remarkable property not shared by general nonlinear hyperbolic systems of conservation laws. We shall not try to address the numerical difficulties of dealing with discontinuities in the random space using the global polynomial approximation. It is well known that piecewise approximations should be used in this case. These include multi-element gPC [28, 27, 13], wavelet basis [16, 17], Essentially-Non-Oscillatory (ENO) or Weighted Essentially-Non-Oscillatory (WENO) interpolations [2] which may also need to be coupled with local subspace recovery techniques [3], or multiresolution approaches [1, 26]. Interested readers are referred to these references.

The rest of the paper is organized as follows: In Section 2, we shall briefly present the H-J equations, its gradient system and the relaxation approximation. In Section 3, we briefly review the gPC-SG method. We then present the details of stochastic Galerkin scheme for the H-J equations with random inputs, based on the relaxation approximation in Section 4. Finally, we provide several numerical examples to illustrate the effectiveness of the methods in Section 5.

2 Hamilton-Jacobi equations and their basic properties

2.1 An equivalent system of hyperbolic conservation laws and the strong hyperbolicity

It is well-known that, in the deterministic case, the H-J equation (1.1) has an equivalent form, in any space dimension, to a gradient system of hyperbolic conservation laws [4, 14] in the sense of zero-viscosity limit. Define

\[ p = (p^{(1)}, \ldots, p^{(n)}) := \nabla_x u; \]

then \( p \) formally solves the \( n \)-dimensional system of conservation laws

\[ \partial_t p + \nabla_x H(p, x, z) = 0 \] (2.2)
or
\[
\partial_t p + \sum_{i=1}^{n} \begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\partial_{p_i} H & \cdots & \partial_{p_n} H \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\end{pmatrix} \partial_{x_i} p = 0 \tag{2.3}
\]
with initial condition
\[
p(x, z, 0) = \nabla_x u_0(x, z). \tag{2.4}
\]
Clearly, if \( p \) is known, one can recover \( u \) from \( p \) by integrating the ordinary differential equation
\[
\partial_t u + H(p, x, z) = 0. \tag{2.5}
\]

The system (2.3) appears to be a weakly hyperbolic system, since the Jacobian matrix in the \( i \)th space dimension could be a Jordan block. However, from (2.1),
\[
\partial_{x_i} p_j = \partial_{x_j} p_i \quad \text{for any } i, j, \tag{2.6}
\]
and by using this property in (2.3), then one gets an equivalent system
\[
\partial_t p + \sum_{i=1}^{n} \partial_{p_i} H \partial_{x_i} p = 0, \tag{2.7}
\]
which is clearly a strongly and symmetric hyperbolic system [29]. Although in each space dimension the eigenvalues are identical, the corresponding eigenspace has a full set of linearly independent eigenvectors.

### 2.2 A relaxation approximation

We adopt the Jin-Xin relaxation system [14]:
\[
\begin{align*}
\partial_t p + \nabla_x w &= 0, \\
\partial_t w + \alpha \nabla_x \cdot p &= -\frac{1}{\epsilon} (w - H(p, x, z)), \\
\partial_t u + w &= 0,
\end{align*} \tag{2.8}
\]
where \( \alpha \) is a positive constant satisfying the so-called subcharacteristic condition ([14]):
\[
\sqrt{\alpha} > |\nabla_p H|. \tag{2.9}
\]
As \( \epsilon \to 0^+ \), the two systems differ from each other by \( O(\epsilon) \). The main advantage of this relaxation system over the original system lies in the linear characteristics of the relaxation system, which will retain the hyperbolicity of the resulting stochastic Galerkin system and therefore avoids the use of sophisticated and time-consuming Riemann solvers.

### 3 GPC Galerkin method

We now briefly review the gPC method to serve a building block for developing the proposed scheme. In the gPC expansion, one approximates the solution of a stochastic problem via an
orthogonal polynomial series ([31]). That is, for random variable \( z \in \mathbb{R}^d \), one seeks

\[
p(x, z, t) \approx p_N(x, z, t) = \sum_{m=1}^{M} \hat{p}_m(t, x)\Phi_m(z), \quad M = \left( d + N \right),
\]

(3.1)

where \( \{ \Phi_m(z) \} \) are from \( \mathbb{P}^{d}_N \), the \( d \)-variate orthogonal polynomials of degree up to \( N \geq 1 \), and orthonormal:

\[
\int \Phi_i(z)\Phi_j(z)\pi(z) \, dz = \delta_{ij}, \quad 1 \leq i, j \leq M = \dim(\mathbb{P}^{d}_N).
\]

(3.2)

Here \( \pi(z) \) is the probability distribution function of \( z \) and \( \delta_{ij} \) the Kronecker delta function. The orthogonality with respect to \( \pi(z) \) then defines the orthogonal polynomials. For example, Gaussian distribution defines Hermite polynomials; uniform distribution defines Legendre polynomials, etc. Note that when the random dimension \( d > 1 \), \( \{ \Phi_m(z) \} \) are multi-dimensional polynomials of \( z = (z_1, \ldots, z_d) \). An ordering scheme for multiple index is required to re-order the polynomials into a single index \( m \) here. Typically, the graded lexicographic order is used, see, for example, Section 5.2 of [30].

## 4 The gPC-Galerkin relaxation scheme

In the gPC Galerkin scheme for system (2.8), one seeks polynomial approximations of the following form:

\[
p_N(x, z, t) = \sum_{m=1}^{M} \hat{p}_m(t, x)\Phi_m(z),
\]

\[
w_N(x, z, t) = \sum_{m=1}^{M} \hat{w}_m(t, x)\Phi_m(z),
\]

\[
H_N(x, z, t) = \sum_{m=1}^{M} \hat{H}_m(t, x)\Phi_m(z),
\]

\[
u_N(x, z, t) = \sum_{m=1}^{M} \hat{u}_m(t, x)\Phi_m(z).
\]

(4.1)

Let

\[
\hat{p} = (\hat{p}_1, \ldots, \hat{p}_M)^T, \quad \hat{w} = (\hat{w}_1, \ldots, \hat{w}_M)^T, \quad \hat{H} = (\hat{H}_1, \ldots, \hat{H}_M)^T, \quad \hat{u} = (\hat{u}_1, \ldots, \hat{u}_M)^T.
\]

be the coefficient vectors. Let

\[
r = (\Phi_1, \ldots, \Phi_M)^T.
\]

One can insert (4.1) into the system (2.2) or (2.3) and then conduct the Galerkin projection on the governing equation to get

\[
\partial_t \hat{p}_i + \sum_{l=1}^{n} \int \pi(z)\partial_{p_l}H(p_N, x, z)rr^T \, dz \partial_{x_i} \hat{p}_l = 0, \quad i = 1, \ldots, M
\]

(4.2)

where \( 0 \) is the \( M \)-dimensional zero vector. Clearly, in the above equation, the Jacobian matrices

\[
\int \pi(z)\partial_{p_l}H(p_N, x, z)rr^T \, dz
\]

(4.3)
for all \(1 \leq l \leq n\) are symmetric (here the integration over \(z\) is taken entry-wise for the matrices), thus the gPC-SG system (4.2) is *symmetric hyperbolic*, which gives a well-posed system allowing one to use the standard shock capturing methods to solve it.

One could also apply the gPC expansion directly to the original H-J equation (1.1) and do the Galerkin projection to obtain a system that can be solved by some shock capturing methods (such methods are usually constructed via its connection with the gradient system (2.2) as well).

### 4.1 The relaxation scheme

While there are many possible shock capturing methods that can be used to solve (4.2), here we employ the relaxation approach, not only for its simplicity (no Riemann or approximate solver is needed) but also for its desirable theoretical property — under the semilinear relaxation approximation the gPC system takes the same (but vectorized) form as the original relaxation system (2.8), thus retaining the global hyperbolicity under the relaxation approximation. Furthermore, in many applications, e.g. in semiclassical approximation of the Schrödinger equation, \(u\) represents the phase while \(p\) represents the velocity, both of which are of physical importance, thus it is advantageous to have a method that provides both quantities. The relaxation scheme offers such a possibility.

By inserting (4.1) into the relaxation system (2.8) and then conducting the Galerkin projection on the governing equation, we obtain

\[
\begin{align*}
  \partial_t \hat{p} + \nabla_x \hat{w} &= 0, \\
  \partial_t \hat{w} + \alpha \nabla_x \cdot \hat{p} &= -\frac{1}{\epsilon} (\hat{w} - \hat{H}), \\
  \partial_t \hat{u} + \hat{w} &= 0.
\end{align*}
\]

(4.4)

The hyperbolicity of the stochastic Galerkin systems (4.4) is straightforward.

The relaxation scheme consists of the following two steps:

- **Convection step:**

  \[
  \begin{align*}
    \partial_t \hat{p} + \nabla_x \hat{w} &= 0, \\
    \partial_t \hat{w} + \alpha \nabla_x \cdot \hat{p} &= 0, \\
    \partial_t \hat{u} + \hat{w} &= 0.
  \end{align*}
  \]

  (4.5)

- **Relaxation step:**

  \[
    \hat{w} = \hat{H}.
  \]

(4.6)

**Remark 4.1** The definition of \(\hat{H}\) will depend on the specific form of \(H\), as will be discussed later.

**Remark 4.2** Since \(\hat{H}\) in (4.4) is not the same matrix as \(\nabla_p H\), one might need a different constant \(\alpha\) in (4.4) from that in (2.8). However, since the Jacobian matrix (4.3) has eigenvalues that can be bounded by the maximum of \(|\partial_p H(p_N, x, z)|\) (\([32]\)), one can use the same \(\alpha\). Since \(\alpha\) contributes to the numerical viscosity, one can make \(\alpha\) space and time dependent by satisfying the subcharacteristic condition (2.9) locally. We do not elaborate this in the current paper.
4.2 A finite volume MUSCL spatial discretization

For clear exposition purpose we employ the second order MUSCL discretization for the convection step (4.5). We will only present the two space dimension case. For details, see [14].

The two dimensional system of (4.5) is

\[
\begin{align*}
\partial_t \mathbf{p}^{(1)} + \partial_x \mathbf{w} &= 0, \\
\partial_t \mathbf{p}^{(2)} + \partial_y \mathbf{w} &= 0, \\
\partial_t \mathbf{w} + \alpha \partial_x \mathbf{p}^{(1)} + \alpha \partial_y \mathbf{p}^{(2)} &= 0, \\
\partial_t \mathbf{u} + \mathbf{w} &= 0.
\end{align*}
\]

(4.7)

For spatial discretization, we choose the spatial grids \((x_{i+1/2}, y_{j+1/2})\) with a uniform mesh size \(\Delta x = x_{i+1/2} - x_{i-1/2}, \Delta y = y_{j+1/2} - y_{j-1/2}\). We denote the cell average in \(c_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]\) as

\[
\mathbf{u}_{ij} = \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{u}(x, y) dx dy.
\]

Integrating (4.7) over cell \(c_{ij}\), we obtain

\[
\begin{align*}
\partial_t \mathbf{p}^{(1)}_{ij} + \frac{1}{\Delta x}(\hat{\mathbf{w}}_{i+1/2,j} - \hat{\mathbf{w}}_{i-1/2,j}) &= 0, \\
\partial_t \mathbf{p}^{(2)}_{ij} + \frac{1}{\Delta y}(\hat{\mathbf{w}}_{i,j+1/2} - \hat{\mathbf{w}}_{i,j-1/2}) &= 0, \\
\partial_t \hat{\mathbf{w}}_{ij} + \frac{\alpha}{\Delta x}(\hat{\mathbf{p}}^{(1)}_{i+1/2,j} - \hat{\mathbf{p}}^{(1)}_{i-1/2,j}) + \frac{\alpha}{\Delta y}(\hat{\mathbf{p}}^{(2)}_{i,j+1/2} - \hat{\mathbf{p}}^{(2)}_{i,j-1/2}) &= 0, \\
\partial_t \hat{\mathbf{u}}_{i+1/2,j} + \hat{\mathbf{H}}_{i+1/2,j} &= 0,
\end{align*}
\]

(4.8)

where the numerical flux \(\hat{\mathbf{w}}_{i+1/2,j}\) is the cell average in the y variable and pointwise value at \(x_{i+1/2}\) (similarly for the other fluxes), defined by the MUSCL scheme [18]:

\[
\hat{\mathbf{w}}_{i+1/2,j} = \frac{1}{2} (\mathbf{w}_{ij} + \mathbf{w}_{i+1,j}) - \frac{\sqrt{\alpha}}{2} (\mathbf{p}^{(1)}_{i+1/2,j} - \mathbf{p}^{(1)}_{i,j})
\]

\[
+ \frac{\sqrt{\alpha}}{4} \Delta x (\sigma_{ij}^{x,+} + \sigma_{ij}^{x,-})
\]

with the limited slope

\[
\sigma_{ij}^{x,\pm} = \frac{1}{\Delta x} \left( \mathbf{p}^{(1)}_{i+1,j} \pm \frac{1}{\sqrt{\alpha}} \mathbf{w}_{i+1,j} - \mathbf{p}^{(1)}_{i,j} \mp \frac{1}{\sqrt{\alpha}} \mathbf{w}_{i,j} \right) \phi(\theta_{ij}^{x,\pm})
\]

\[
\theta_{ij}^{x,\pm} = \frac{\mathbf{p}^{(1)}_{i+1,j} \pm \frac{1}{\sqrt{\alpha}} \mathbf{w}_{i+1,j} - \mathbf{p}^{(1)}_{i,j} \mp \frac{1}{\sqrt{\alpha}} \mathbf{w}_{i,j}}{\mathbf{p}^{(1)}_{i+1,j} \pm \frac{1}{\sqrt{\alpha}} \mathbf{w}_{i+1,j} - \mathbf{p}^{(1)}_{i,j} \mp \frac{1}{\sqrt{\alpha}} \mathbf{w}_{i,j}}
\]

using the van Leer slope limiter

\[
\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}.
\]

5 Numerical Examples

In this section, we present several numerical examples to illustrate the effectiveness of our methods. The randomness/uncertainty can enter the problem through either initial conditions
or Hamiltonians. For simplicity, we will always assume a 1D random variable $z$ obeying the uniform distribution on $[-1, 1]$, thus the Legendre polynomial chaos are used as gPC basis. Our examples include both one-dimensional and two-dimensional nonlinear systems. The purpose is to clearly demonstrate the effectiveness of our method — the strong hyperbolicity of the gPC Galerkin systems. Therefore, we do not pursue more complicated examples involving multiple random variables, as it can be done in a very straightforward manner.

In the following, the first-order scheme refers to upwind in space and forward Euler in time; the second-order scheme refers to MUSCL scheme (with the van Leer limiter) in space and Heun’s method in time. The initial condition is chosen consistently as $p_0(x, z) = \nabla_x u_0(x, z)$ and $w_0(x, z) = H(p_0, x, z)$. The periodic boundary condition is assumed except for the first example. Two metrics are typically used to quantify the difference in the numerical solution $u^h(x, z, t)$ and the reference solution $u(x, z, t)$: the difference in mean and in standard deviation, with $l^1$-norm in $x$:

$$e_{\text{mean}}(t) = \|E[u^h] - E[u]\|_{l^1},$$
$$e_{\text{std}}(t) = \|\sigma[u^h] - \sigma[u]\|_{l^1}. \quad (5.1)$$

### 5.1 1D examples

#### Example 1. 1D Burgers’ equation with random Riemann initial data

Consider

$$u_t + H(u_x) = 0, \quad H(p) = \frac{1}{2} p^2, \quad x \in [0, 1], \quad (5.2)$$

with randomly perturbed initial data

$$u_0(x, z) = \begin{cases} (u_l + \sigma z)x, & x \leq 0.5, \\ (u_r + \sigma z)x + 0.5(u_l - u_r), & x > 0.5, \end{cases} \quad \sigma = 0.1. \quad (5.3)$$

This Hamiltonian, in the form of (1.2), arises in the semiclassical limit of the linear Schrödinger equation. The derivative of (5.2) is clearly the Burgers’ equation.

Here we need to handle the nonlinear Hamiltonian $H$ in the context of stochastic Galerkin approach. The corresponding gPC approximation based on $p_N$ is given by

$$H(p) \approx \frac{1}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{p}_i \hat{p}_j S_{ijk} \Phi_i(z) \Phi_j(z), \quad (5.4)$$

which is not in $\mathbb{P}_N$, but always in $L^2_{\pi(z)}$, where

$$L^2_{\pi(z)} = \{ f : I_z \rightarrow \mathbb{R} \mid E[f^2] = \int_{I_z} f^2(z) \pi(z) \, dz < \infty \}. \quad (5.5)$$

We then approximate $H(p)$ by its orthogonal projection $H_N$ on $\mathbb{P}_N$, whose gPC coefficients are defined as follows:

$$\hat{H}_k = \frac{1}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{p}_i \hat{p}_j S_{ijk}, \quad 0 \leq k \leq N, \quad (5.6)$$

where

$$S_{ijk} = \int \Phi_i(z) \Phi_j(z) \Phi_k(z) \pi(z) \, dz, \quad 0 \leq i, j, k \leq N. \quad (5.7)$$
We choose $\alpha = 1.5$, $N = 7$ (highest degree in gPC expansion), $N_x = 200$, CFL = 0.8 for the first-order scheme, CFL = 0.4 for the second-order scheme. We consider two cases and compare the numerical solutions with the reference solutions at $t = 0.3$ (20 Gauss-Legendre quadrature points are used to evaluate the mean and standard deviation of the exact solution).

Case 1 (shock): $u_l = 1$, $u_r = -1$. The exact $p$ and $u$ are given by

\[ p(x, z, t) = \begin{cases} 
  u_l + \sigma z, & x \leq 0.5 + \sigma z t, \\
  u_r + \sigma z, & x > 0.5 + \sigma z t;
\end{cases} \]  

(5.8)

\[ u(x, z, t) = \begin{cases} 
  \left(u_l + \sigma z\right)x - \frac{(u_l + \sigma z)^2}{2}t, & x \leq 0.5 + \sigma z t, \\
  \left(u_r + \sigma z\right)x + 0.5(u_l - u_r) - \frac{(u_r + \sigma z)^2}{2}t, & x > 0.5 + \sigma z t.
\end{cases} \]  

(5.9)

The solutions are plotted in Figure 1. In this case, there is a shock in $p$ along the $x$ direction. After taking the average in $z$, the sharp discontinuity is smoothed out.

Figure 1: Example 1, case 1 (shock). Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Top: mean and standard deviation of $u$; bottom: mean and standard deviation of $p = u_x$. All solutions are zoomed into $x \in [0.3, 0.7]$. 
Case 2 (rarefaction): \( u_l = -1, u_r = 1 \). The exact \( p \) and \( u \) are given by

\[
p(x, z, t) = \begin{cases} u_l + \sigma z, & x \leq 0.5 + (u_l + \sigma z)t, \\ u_r + \sigma z, & x > 0.5 + (u_r + \sigma z)t; \end{cases}
\]

\[ u(x, z, t) = \begin{cases} \frac{x-0.5}{t}, & 0.5 + (u_l + \sigma z)t < x \leq 0.5 + (u_r + \sigma z)t, \\ u_l + \sigma z, & x > 0.5 + (u_r + \sigma z)t; \end{cases} \]

\[ (u_l + \sigma z)x - \frac{(u_l + \sigma z)^2}{2}t, & x \leq 0.5 + (u_l + \sigma z)t, \\ 0.5(u_l + \sigma z) + \frac{1}{2t}(x - 0.5)^2, & 0.5 + (u_l + \sigma z)t < x \leq 0.5 + (u_r + \sigma z)t \]

\[ (u_r + \sigma z)x + 0.5(u_l - u_r) - \frac{(u_r + \sigma z)^2}{2}t, x > 0.5 + (u_r + \sigma z)t. \]

The solutions are plotted in Figure 2. In this case, a rarefaction wave will form in \( p \) along the \( x \) direction.

![Graphs of Eu, Su, Ep, and Sp](image)

Figure 2: Example 1, case 2 (rarefaction). Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Top: mean and standard deviation of \( u \); bottom: mean and standard deviation of \( p = u_x \).

In either case, we obtain good agreement between the numerical solutions and the reference solutions. We also conduct the convergence test with respect to the gPC order: in the rarefaction case (Figure 3, right), the error decays very rapidly in \( N \) and then saturates once the spatial error dominates. Note that the error in \( u \) is smaller than in \( p \) since the former has better regularity. In the shock case (Figure 3, left), the error decreases very slowly because of the discontinuities.
in the solution. We then fix a relatively large $N$ and test the convergence regarding the spatial discretization. The results for the rarefaction case are shown in Figure 4. Due to the finite regularity of the solution, the second-order scheme can only achieve the first-order accuracy.

**Remark 5.1** The convergence of the global gPC suffers severely for the shock case and is not shown. This is a well known case where piecewise basis shall be adopted, as discussed in the Introduction. Since this is not the focus of this paper, we leave it for future work.

Figure 3: Example 1, convergence test. Error vs. gPC order $N$. Second-order scheme is used with $\Delta x = 6.25e^{-4}$, $\Delta t = 2e^{-4}$. Left: case 1 (shock); right: case 2 (rarefaction).

Figure 4: Example 1, case 2 (rarefaction). Convergence test. Error vs. spatial discretization $\Delta x$. $N = 9$. $\Delta t = 2e^{-4}$. Left: first-order scheme; right: second-order scheme.

**Example 2. 1D Burgers’ equation with random smooth initial data**

Consider

$$u_t + H(u_x) = 0, \quad H(p) = \frac{1}{2}p^2, \quad x \in [0, 2\pi],$$

(5.12)

with random smooth initial data

$$u_0(x, z) = \sin(x + \sigma z), \quad \sigma = 0.6.$$ (5.13)
The deterministic version of this problem (when \( \sigma = 0 \)) was used as a test example in [5].

The solution of \( u \) develops a caustic at time \( t = 1 \). When \( t = 0.5 \), it is still smooth, and we can construct the exact solution by the method of characteristics: solve \( x = x_0 + \cos(x_0 + \sigma z)t \) for \( x_0 = x_0(x, z, t) \), then \( p(x, z, t) = \cos(x_0 + \sigma z) \) and \( u(x, z, t) = \sin(x_0 + \sigma z) + \frac{\cos^2(x_0 + \sigma z)}{2}t \).

We did the similar convergence test as before to verify the accuracy of the scheme. The results are gathered in Figure 5 and Figure 6, from which we clearly observe the first and second order convergence in space, and spectral convergence with respect to gPC order (the error saturates after \( N = 6 \) due to the numerical error in spatial and temporal discretizations).

After \( t = 1 \), a shock forms in \( p = u_x \). We choose \( \alpha = 1.1 \), \( N = 7 \), \( N_x = 200 \), CFL = 0.8 for the first-order scheme, CFL = 0.4 for the second-order scheme, and compare our solution with the reference solution at \( t = 1.5 \). The reference solution is computed based on the stochastic collocation using the second-order scheme with \( N_x = 800 \), CFL = 0.4, and 20 Gauss-Legendre quadrature points in the random space. See Figure 7. Note that the zig-zag behavior in the mean of \( p \) is due to the pitfall of high order polynomial interpolation of a discontinuity solution in the random space, as discussed (and fixed) in [3]. For reference, we also plot the deterministic solutions (i.e. \( \sigma = 0 \), no uncertainty in the initial condition) in Figure 8.

![Figure 5: Example 2, convergence test. Error vs. spatial discretization \( \Delta x \). \( N = 9 \). \( \Delta t = 0.001 \). Left: first-order scheme; right: second-order scheme.](image)

**Example 3. 1D Burgers’ equation with random potential**

Consider

\[
    u_t + H(u_x, x, z) = 0, \quad H(p, x, z) = \frac{1}{2}p^2 + V(x, z), \quad x \in [0, 2\pi],
\]

with random potential

\[
    V(x, z) = \frac{(1 + \sigma z)}{\pi^2}(x - \pi)^2, \quad \sigma = 0.3,
\]

and deterministic initial data

\[
    u_0(x) = \sin x.
\]

Here \( \tilde{H}_k \) can be computed the same as in (5.6) but with an extra term:

\[
    \tilde{H}_k = \frac{1}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} \tilde{p}_i \tilde{p}_j S_{ijk} + \hat{V}_k, \quad 0 \leq k \leq N,
\]
Figure 6: Example 2, convergence test. Error vs. gPC order $N$. Second-order scheme is used with $\Delta x = 0.0039$, $\Delta t = 0.001$.

Figure 7: Example 2. Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Top: mean and standard deviation of $u$; bottom: mean and standard deviation of $p = u_x$. 

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Figure 8: Example 2. $\sigma = 0$ (no uncertainty in initial condition). Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Left: $u$; right: $p = u_x$.

where $\hat{V}_k$ is the gPC expansion coefficient of the potential (5.15).

We choose $\alpha = 4$, $N = 7$, $N_x = 200$, CFL = 0.8 for the first-order scheme, CFL = 0.4 for the second-order scheme, and compare our solution with the reference solution at $t = 1.5$. The reference solution is computed based on the stochastic collocation using the second-order scheme with $N_x = 800$, CFL = 0.4, and 20 Gauss-Legendre quadrature points in the random space. See Figure 9.

Example 4. 1D Eikonal equation with random wave speed

Consider
\[ u_t + H(u_x, x, z) = 0, \quad H(p, x, z) = c(x, z)|p|, \quad x \in [0, 2\pi], \quad \text{ (5.18)} \]

with random wave speed
\[ c(x, z) = 1 + \sigma z, \quad \sigma = 0.2, \quad \text{ (5.19)} \]

and deterministic initial data
\[ u_0(x, z) = \sin x. \quad \text{ (5.20)} \]

In this and examples 6 and 7, the Hamiltonian $H$ is not a polynomial in $p$, so there is no simple way to evaluate its gPC coefficients. Therefore, we approximate the following integral directly using the Gauss quadrature rule:
\[ \hat{H}_k = \int_{I_z} H \left( \sum_{i=0}^N \hat{\rho}_i \Phi_i(z) \right) \Phi_k(z) \pi(z) dz. \quad \text{ (5.21)} \]

We choose $\alpha = 1.5$, $N = 7$, $N_x = 200$, CFL = 0.8 for the first-order scheme, CFL = 0.4 for the second-order scheme, and compare our solution with the reference solution at $t = 1$. The reference solution is computed based on the stochastic collocation using the second-order scheme with $N_x = 800$, CFL = 0.4, and 20 Gauss-Legendre quadrature points in the random space. See Figure 10.
Figure 9: Example 3. Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Top: mean and standard deviation of $u$; bottom: mean and standard deviation of $p = u_x$. 
Figure 10: Example 4. Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Top: mean and standard deviation of $u$; bottom: mean and standard deviation of $p = u_x$. 
5.2 2D examples

Example 5. 2D Burgers’ equation with random potential

Consider

\[ u_t + H(u_x, u_y, x, y, z) = 0, \quad H(p, q, x, y, z) = \frac{1}{2}(p+q+1)^2 + V(x, y, z), \quad (x, y) \in [-2, 2]^2, \] (5.22)

with random potential

\[ V(x, y, z) = \frac{(1 + \sigma z)}{8} (x^2 + y^2), \quad \sigma = 0.5, \] (5.23)

and deterministic initial data

\[ u_0(x, y) = -\cos \left( \frac{\pi}{2} (x + y) \right). \] (5.24)

Here the Hamiltonian is a quadratic function which can be computed similarly as those in 1D examples:

\[ \hat{H}_k = \frac{1}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{r}_i \hat{r}_j S_{ijk} + \hat{V}_k, \quad 0 \leq k \leq N, \] (5.25)

where \( \hat{r}_0 = \hat{\rho}_0 + \hat{\rho}_1 + 1, \hat{r}_i = \hat{\rho}_i + \hat{\rho}_i, i \neq 0, \) and \( \hat{V}_k \) is the gPC coefficient of the potential (5.23).

We choose \( \alpha = 50, N = 7, N_x = N_y = 40, \) CFL = 0.8 for the first-order scheme, CFL = 0.4 for the second-order scheme, and compare our solution with the reference solution at \( t = 1.5 \pi. \) The reference solution is computed based on the stochastic collocation using the second-order scheme with \( N_x = 160, \) CFL = 0.4, and 20 Gauss-Legendre quadrature points in the random space. The results are shown in Figure 11 and Figure 12.

Example 6. 2D Eikonal equation with random initial data

Consider

\[ u_t + H(u_x, u_y) = 0, \quad H(p, q) = -\sqrt{p^2 + q^2} + 1, \quad (x, y) \in [0, 1]^2, \] (5.26)

with random initial data

\[ u_0(x, y, z) = -\frac{1 + \sigma z}{4} \left[ \cos(2\pi x) - 1 \right] \left[ \cos(2\pi y) - 1 \right] + 1. \] (5.27)

The deterministic version of this problem was first considered in [19]. We choose \( \alpha = 1.1, N = 7, N_x = N_y = 40, \) CFL = 0.8 for the first-order scheme, CFL = 0.4 for the second-order scheme, and compare our solution with the reference solution at \( t = 0.3. \) The reference solution is computed based on the stochastic collocation using the second-order scheme with \( N_x = 160, \) CFL = 0.4, and 20 Gauss-Legendre quadrature points in the random space. The results are shown in Figure 13 and Figure 14.

Example 7. 2D Eikonal equation with random wave speed

Consider

\[ u_t + H(u_x, u_y, x, y, z) = 0, \quad H(p, q, x, y, z) = c(x, y, z) \sqrt{p^2 + q^2}, \quad (x, y) \in [-2, 2]^2, \] (5.28)

with random wave speed

\[ c(x, y, z) = 1 + \sigma \cos \left( \frac{\pi}{2} \right) z, \quad \sigma = 0.2, \] (5.29)
Figure 11: Example 5. Left: (reference) mean and standard deviation of $u$ by second-order collocation method with $N_x = N_y = 160$, CFL = 0.4; right: mean and standard variation of $u$ by second-order Galerkin method with $N_x = N_y = 40$, CFL = 0.4.

Figure 12: Example 5. A slice of the solution at $y = 0.5$. Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Left: mean of $u$; right: standard deviation of $u$. 
Figure 13: Example 6. Left: (reference) mean and standard deviation of $u$ by second-order collocation method with $N_x = N_y = 160$, CFL = 0.4; right: mean and standard deviation of $u$ by second-order Galerkin method with $N_x = N_y = 40$, CFL = 0.4.

Figure 14: Example 6. A slice of the solution at $y = 0.5$. Solid line: reference solution; circle: first-order scheme; star: second-order scheme. Left: mean of $u$; right: standard deviation of $u$. 
and deterministic initial data
\[ u_0(x, y) = |x| + |y| - 0.5. \] (5.30)

This problem was also modified based on the deterministic example in [19]. We use the second-order scheme with \( \alpha = 1.5, N = 7, N_x = N_y = 80, \) CFL= 0.4, and compute the solution to \( t = 0.9. \) For this problem, the initial zero level set of \( u \) is a square centered at the origin. The H-J equation describes a solution whose zero level set moves in the outward normal direction with speed \( c(x, y, z). \) Since the mean of \( c \) is a constant, the mean of \( u \) (Figure 15) is more or less the same as the deterministic case. However, if we check the variance, we can get a general idea about the impact of the random input (Figure 16).

![Figure 15: Example 7. Mean of \( u \) by second-order scheme. Left: 3-D plot at \( t = 0.9; \) right: zero level set at \( t = 0, 0.3, 0.6, 0.9. \)](image)

![Figure 16: Example 7. Standard deviation of \( u \) at \( t = 0.9 \) by second-order scheme. Left: 3-D plot; right: bird eye view.](image)
Acknowledgments

Shi Jin was partially supported by the NSF DMS grants no. 1114546, and no. 1107291: RNMS "KI-Net". D. Xiu was partially supported by AFOSR, NSF and DOE.

References


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