1. Study the stability of the ADI method for three-dimensional heat equation:

\[ \frac{\partial u}{\partial t} = u_{xx} + u_{yy} + u_{zz} . \]

2. Applying the Crank-Nicolson scheme to solve the linear heat equation

\[ \frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0 \quad 0 < x < 1 , \quad t > 0 \]

subject to boundary conditions

\[ u(0, t) = u(1, t) = 0 \]

and initial condition

\[ u(x, 0) = 10 \sin \pi x , \quad 0 \leq x \leq 1 . \]

The solution of this problem is

\[ u(x, t) = 10e^{-\pi^2 t} \sin \pi x . \]

Since the scheme is implicit you will need to write a tridiagonal solver. By using 100 points and a suitable time step for second order accuracy, output the numerical results at various times along with the exact solution and plot the error vs grid point (in log − log) to verify the second order accuracy.

3. Consider the following reaction-diffusion equation

\[ \frac{\partial}{\partial t} u = \partial_{xx} u + \partial_{yy} u + \frac{1}{\epsilon} u(1 - u^2) . \]

Devise a suitable scheme (second order) for this equation, with a stability condition independent of \( \epsilon \). Let \( \Omega \) be an ellipse within the domain \([0, 1] \times [0, 1]\), and define

\[ u(0, x) = 1 \quad \text{if } x \text{ inside } \Omega ; \quad u(0, x) = -1 \quad \text{if } x \text{ outside } \Omega . \]
For the boundary condition you may set $u = -1$. Let $\epsilon = 0.05$. For a suitably small $\Delta x$ (compared to $\epsilon$), observe the evolution of the zero level set of $u$. Plot the zero level curve at some representative time to see its evolution. What happens if $\Delta x$ is getting larger and larger—do you still observe the desired evolution? (Hint: an ellipse, actually any close curve, should eventually become a circle and then disappear. This phenomenon is known as “motion by mean curvature”. You may try different initial curve).