

## Math 712 Homework Assignment No. 5

Due Dec 6, 2004 at 12 noon in my office or my mbox  
Do not accept late homework

1. Consider the level set equation

$$\partial_t u - \kappa |\nabla u| = 0,$$

where  $\kappa$  is the mean curvature which can be defined by  $u$ . Using center difference for space and forward Euler for time. Periodic boundary condition can be used. Consider a seven-point star

$$\gamma(s) = (0.1 + (0.065) \sin(7 \cdot 2\pi s))(\cos(2\pi s), \sin(2\pi s)) \quad s \in [0, 1]$$

as the initial curve. The initial condition for  $u$  can be  $u(x, 0) = 1 \pm d^2$ , where  $d$  is the signed distance from the front (note the front now should be points where  $u \equiv 1$  instead of the zero level curve). The computational domain is a square centered at the origin of side length  $1/2$ . Use 100 mesh points per side and a suitable time step for numerical stability. At any time  $n\Delta t$ , the front can be plotted by the contour of  $u = 1$ . Output a few snapshots to observe the evolution of the front.

2. Consider the Schrödinger equation

$$\begin{aligned} i\varepsilon u_t^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon - V(\mathbf{x})u^\varepsilon &= 0 \quad t > 0, \mathbf{x} \in R^d \\ u^\varepsilon(\mathbf{x}, 0) &= u_0^\varepsilon(\mathbf{x}). \end{aligned} \tag{1}$$

Here  $u(\mathbf{x}, t)$  is the complex valued wave function,  $t$  is time,  $\mathbf{x}$  is the space variable,  $V(\mathbf{x})$  is the potential function,  $i = \sqrt{-1}$ .  $d$  is the space dimension. Define the position density

$$n^\varepsilon(\mathbf{x}, t) = |u^\varepsilon(\mathbf{x}, t)|^2.$$

1) Assume the  $u$  vanishes at the boundary. Prove that the position density is conserved in time, namely,

$$\partial_t \int_{R^d} n^\varepsilon d\mathbf{x} = 0.$$

2) For  $d = 1$ , use the operator splitting method. First solve

$$\mathcal{S}_1 : \quad i\varepsilon u_t^\varepsilon + \frac{\varepsilon^2}{2} u_{xx}^\varepsilon = 0$$

by the Fourier spectral method (note in the Fourier space the ODE can be integrated in time, so the procedure is, first take the Fourier transform of this equation, then integrate the ODE in time exactly in the phase space, followed by inverse Fourier transform), and then solve the ODE

$$\mathcal{S}_2 : \quad i\varepsilon u_t^\varepsilon = V(x)u^\varepsilon$$

exactly. Namely, from  $t^n$  to  $t^{n+1}$ ,

$$U^{n+1} = \mathcal{S}_2(\Delta t)\mathcal{S}_1(\Delta t)U^n.$$

Let the numerical solution be  $\mathbf{u} = (U_0, \dots, U_{M-1})$ . Define the the  $l^2$  norm as

$$\|\mathbf{u}\|_{l^2} = \sqrt{\Delta x \sum_{j=0}^{M-1} |U_j|^2}.$$

Prove the stability result:

$$\|\mathbf{u}^n\|_{l^2} = \|\mathbf{u}^0\|_{l^2}.$$

3) Take the initial condition  $u_0(x) = A_0(x)e^{iS_0(x)/\varepsilon}$  where

$$A_0(x) = e^{-25(x-0.5)^2}, \quad S_0(x) = -\frac{1}{5} \ln \left( e^{5(x-0.5)} + e^{-5(x-0.5)} \right), \quad x \in R$$

Let  $V(x) = 10$ . Solve the problem over interval  $[0, 1]$  with periodic boundary conditions. Use the above splitting-spectral method to solve this problem and output your numerical  $n(x, t)$  at  $t=0.54$ . Use the following sets of parameters, (a)  $\varepsilon = 0.0256, \Delta x = 1/16$ ; (b)  $\varepsilon = 0.0064, \Delta x = 1/64$ ; (c)  $\varepsilon = 0.0008, \Delta x = 1/512$ . What can you observe numerically as we let  $\varepsilon \rightarrow 0$ . Can you predict a weak (or average) limit for  $u$  with  $\varepsilon \rightarrow 0$ ? You may draw the conjectured weak limit by hand on the pictures. What happens if you choose  $\Delta x \gg \varepsilon$ , say  $\varepsilon = 0.0008, \Delta x = 1/32$ ? Do you still get the right weak limit?