1. Consider the following method for solving the heat equation \( u_t = u_{xx} \):

\[
\frac{U_{n+1}^j - U_n^j}{\Delta t} = \theta \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2} + (1 - \theta) \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}
\]

for \( \theta \in [0, 1] \). Find how the truncation error of this scheme depends on \( \theta \) and how \( \theta \) affects the accuracy of the scheme. Conduct the von Neumann analysis to find the stability condition of the scheme. For what range of \( \theta \) is the scheme unconditionally stable?

2. Study the stability of the ADI method for three-dimensional heat equation:

\[ u_t = u_{xx} + u_{yy} + u_{zz} \]

3. Applying the Crank-Nicolson scheme to solve the linear heat equation

\[
\frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0 \quad 0 < x < 1, \quad t > 0
\]

subject to boundary conditions

\[ u(0, t) = u(1, t) = 0 \]

and initial condition

\[ u(x, 0) = 10 \sin \pi x, \quad 0 \leq x \leq 1. \]

The solution of this problem is

\[ u(x, t) = 10e^{-\pi^2 t} \sin \pi x. \]

Since the scheme is implicit you will need to write a tridiagonal solver. By using 100 points and a suitable time step for second order accuracy, output the numerical results at various times along with the exact solution and plot the error vs grid point (in log – log) to verify the second order accuracy.

4. Consider the following reaction-diffusion equation

\[
\partial_t u = \partial_{xx} u + \partial_{yy} u + \frac{1}{\epsilon} u(1 - u^2).
\]

Devise a suitable scheme (second order) for this equation, with a stability condition independent of \( \epsilon \). Let \( \Omega \) be an ellipse within the domain \([0, 1] \times [0, 1] \), and define

\[ u(0, x) = 1 \quad \text{if} \ x \ \text{inside} \ \Omega; \quad u(0, x) = -1 \quad \text{if} \ x \ \text{outside} \ \Omega. \]
For the boundary condition you may set $u = -1$. Let $\epsilon = 0.05$. For a suitably small $\Delta x$ (compared to $\epsilon$), observe the evolution of the zero level set of $u$. Plot the zero level curve at some representative time to see its evolution. What happens if $\Delta x$ is getting larger and larger--do you still observe the desired evolution? (Hint: an ellipse, actually any close curve, should eventually become a circle and then disappear. This phenomenon is known as “motion by mean curvature”. You may try different initial curve).