Math 714 Homework Assignment No. 3  
Due Nov 1

1. Show that the viscous Burgers equation \( u_t + uu_x = \epsilon u_{xx} \) has a traveling-wave solution of the form \( w^\epsilon(x, t) = w^\epsilon(x - st) \), by deriving an ODE for \( w \) and verifying that this ODE has solutions of the form

\[
w(\xi) = u_r + \frac{1}{2}(u_l - u_r) \left[ 1 - \tanh \left( \frac{(u_l - u_r) \xi}{4\epsilon} \right) \right],
\]

when \( u_l > u_r \), with the propagation speed \( s \) agreeing with the shock speed given by the Rankine-Hugoniot jump condition. Note that \( w(\xi) \rightarrow u_l \) as \( \xi \rightarrow -\infty \), and \( w(\xi) \rightarrow u_r \) as \( \xi \rightarrow \infty \). Sketch this solution and indicate how it varies as \( \epsilon \rightarrow 0 \).

What happens to this solution if \( u_l < u_r \), and why is there no traveling-wave solution with limiting values of this form?

2. Determine the exact solution to Burger’s equation

\[
u_t + \left( \frac{u^2}{2} \right)_x = 0
\]

for \( t > 0 \) with initial data

\[
u(x, 0) = 2, \quad \text{if } 0 < x < 1,
\]
\[
u(x, 0) = 0, \quad \text{otherwise}.
\]

Note that the rarefaction wave catches up to the shock at some time \( T_c \). For \( t > T_c \) determine the location of the shock by two different approaches:

(a) Let \( x_s(t) \) represent the shock location at time \( t \). Determine and solve an ODE for \( x_s(t) \) by using the Rankine-Hugoniot jump condition, which must hold across the shock at each time.

(b) For \( t > T_c \) the exact solution is triangular-shaped. Use conservation to determine \( x_s(t) \) based on the area of this triangle. Sketch the corresponding “over-turned” solution.

3. Repeat problem 2 with the initial data

\[
u(x, 0) = 2, \quad \text{if } 0 < x < 1,
\]
\[
u(x, 0) = 4, \quad \text{otherwise}.
\]

Note that in this case the shock catches up with the rarefaction wave.
4. Use the local Lax-Friedrichs scheme, the Lax-Wendruff Scheme, and the Godunov scheme to solve problems 2 and 3. Use 200 grid points in space, and a suitable time step which satisfies the CFL condition. Plot your numerical results before, at and after $T_c$ against the exact solutions. What do you observe if you double the number of grid points? Discuss the numerical performance of the three schemes.