1. Consider the equation
\[ q_t + bq_x = aq, \quad q(x,0) = q_0(x), \]
where \( b > 0 \).
(a) Write down the exact solution to this problem.
(b) Show that the method
\[ Q_{i+1}^n = Q_i^n - \frac{b\Delta t}{\Delta x} (Q_i^n - Q_{i-1}^n) + \Delta taQ_i^n \]
is first order accurate for this equation.
(c) A scheme \( Q_{n+1}^n = \mathcal{N}Q^n \) is called Lax-Richtmyer-stable if there exists a constant \( \alpha > 0 \) such that
\[ \|N(E^n)\| \leq (1 + \alpha \Delta t)\|E^n\| \]
for any grid function \( E^n \). Show that the method in part (b) has this stability in the 1-norm under a suitable condition, and give this stability condition.
(d) Is the scheme in part (b) TVB? TVD?

2. For a linear hyperbolic system
\[ \partial_t U + A \partial_x U = 0, \quad U \in \mathbb{R}^n \]
where \( A \) is an \( n \times n \) real matrix with \( n \) distinct real eigenvalues
\[ \lambda_1 < \lambda_2 < \cdots < \lambda_n \]
with the corresponding right eigenvectors \( r_1, r_2, \ldots, r_n \), finds its solution (as a function of \( x/t \)) to the Riemann problem with initial data \( (U_l, U_r) \). (You can diagonalize the system and then use the method of characteristics)

3. Consider the isothermal flow equations:
\[ \begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + \rho) &= 0
\end{align*} \]
Here \( \rho \) is the density, \( u \) is the velocity. Solve the following Riemann problem:
\[ \rho_L = 1.0, \quad \rho_R = 0.2; \quad u_L = u_R = 0.0, \]
using the Roe scheme and the local Lax-Friedrichs scheme. Solve the problem over domain \([-1, 1]\) and use simply the constant boundary conditions. Use 100 and 200 points respectively for space discretization, and the time step according to the CFL condition. Plot your numerical results at \( t = 0.25 \) against the “exact solution”, which can be obtained by using, say, 1000 grid points in space. What differences can you observe between these two schemes.