1. Give exact ways of avoiding loss-of-significance errors in the following computations.

\[
\begin{align*}
(a) & \quad \log(x+1) - \log(x) \quad \text{large } x; \\
(b) & \quad \frac{1 - \cos(x)}{x^2} \quad x \approx 0 \\
(c) & \quad \frac{\log(1-x) + xe^{x/2}}{x^3} \quad x \approx 0 \\
(d) & \quad \frac{e^x - e^{-x}}{2x} \quad x \approx 0
\end{align*}
\]

2. Write a program to compute experimentally

\[
\lim_{p \to \infty} (x^p + y^p)^{1/p}
\]

where \(x\) and \(y\) are positive numbers. First do the computation in the form just shown. Second, can you propose a different way of computing the limit that can avoid the loss of significant digits? Run the program for a variety of large and small values of \(x\) and \(y\), for example, \(x = y = 10^{10}\) and \(x = y = 10^{-10}\).

3. Consider finding a rational function \(p(x) = (a + bx)/(1 + cx)\) that satisfies

\[p(x_i) = y_i \quad i = 1, 2, 3\]

with \(x_1, x_2, x_3\) distinct. Does such a function \(p(x)\) exists, or are additional conditions needed to ensure the existence and uniqueness of \(p(x)\)?

4. For \(f(x) = 1/(1 + x^2), -5 \leq x \leq 5\), produce \(p_n(x)\) using \(n + 1\) evenly spaced nodes on \([-5, 5]\). Calculate \(p_n(x)\) at a large number of points, and graph it or its error on \([-5, 5]\).

5. Write a program to solve the system

\[
\begin{align*}
x^2 + xy^3 &= 9 \\
3x^2y - y^3 &= 4
\end{align*}
\]

using Newton’s method for nonlinear systems. Use each of the initial guesses \((x_0, y_0) = (1.2, 2.5), (-2, 2.5), (-1.2, -2.5), (2, -2.5)\). Observe which root to which the method converges, the number of iterates required, and the speed of convergence.
6. Consider Newton’s method for finding the positive square root for \( a > 0 \). Derive the following results, assuming \( x_0 > 0, x_0 \neq \sqrt{a} \).

(a) \[ x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \]

(b) \[ x_{n+1}^2 - a = \left[ \frac{x_n^2 - a}{2x_n} \right]^2 \quad n \geq 0, \quad \text{and thus} \quad x_n > \sqrt{a} \quad \text{for all} \quad n > 0 \]

(c) The iterates \( x_n \) are a strictly decreasing sequences for \( n \geq 1 \).

\[ \text{Hint: Consider the sign of} \quad x_{n+1} - x_n. \]

(d) \[ e_{n+1} = -\frac{e_n^2}{2x_n}, \quad \text{with} \quad e_n = \sqrt{a} - x_n, \]

\[ \text{Re}(x_{n+1}) = -\frac{\sqrt{a}}{2x_n} [\text{Rel}(x_n)]^2 \quad n \geq 0 \quad \text{with} \ \text{Rel}(x_n) \ \text{the relative error in} \ x_n. \]

(e) If \( x_0 \geq \sqrt{a} \) and \( |\text{Rel}(x_0)| \leq 0.1 \), bound \( \text{Rel}(x_4) \).