1. (a) Find a polynomial \( p(x) \) of degree \( \leq 2 \) that satisfies

\[
p(x_0) = y_0, \quad p'(x_0) = y'_0, \quad p'(x_1) = y'_1
\]

Give a formula in the form

\[
p(x) = y_0 l_0(x) + y'_0 l_1(x) + y'_1 l_2(x).
\]

by finding \( l_0, l_1, l_2 \).

(b) Find a formula for the following polynomial interpolation problem. Let \( x_i = x_0 + ih, i = 0, 1, 2 \). Find a polynomial \( p(x) \) of degree \( \leq 4 \) for which

\[
p(x_i) = y_i \quad i = 0, 1, 2
\]

\[
p'(x_0) = y'_0 \quad p'(x_2) = y'_2
\]

with the \( y \) value given. (Hint: try to construct basis functions).

2. (a) Let \( f(x) \) be three times continuously differentiable on \([-\alpha, \alpha] \) for some \( \alpha > 0 \), and consider approximating it by the rational function

\[
R(x) = \frac{a + bx}{1 + cx}
\]

To generalize the idea of the Taylor series, choose the constant \( a, b \) and \( c \) so that

\[
R^{(j)}(0) = f^{(j)}(0) \quad j = 0, 1, 2
\]

Is it always possible to find such an approximation \( R(x) \)? The function \( R(x) \) is an example of a Pade approximation to \( f(x) \).

(b) Apply the above Pade approximation to the case \( f(x) = e^x \), and give the resulting approximation \( R(x) \). Analyze its error on \([-1, 1]\). Plot \( f(x), R(x) \) and the quadratic Taylor polynomial. Compare the errors of these two different approximations.

3. Solve analytically the following minimization problems and determine whether there is a unique value of \( \alpha \) that gives the minimum. In each case, \( \alpha \) is allowed to range over all real numbers. We are approximating the function \( f(x) = x \) with polynomials of the form \( \alpha x^2 \).

\[
(a) \quad \min_{\alpha} \int_{-1}^{1} [x - \alpha x^2]^2 dx
\]

\[
(b) \quad \min_{\alpha} \int_{-1}^{1} |x - \alpha x^2| dx
\]

\[
(c) \quad \min_{\alpha} \max_{-1 \leq x \leq 1} |x - \alpha x^2|
\]
Plot $x$ and your choices of $\alpha x^2$ in $[-1, 1]$ for all three cases.

4. Let $f(x) = \cos^{-1}(x)$ for $-1 \leq x \leq 1$ (the principal branch $0 \leq f \leq \pi$). Find the polynomial of degree two $p(x)$ which minimizes

$$\int_{-1}^{1} \frac{[f(x) - p(x)]^2}{\sqrt{1-x^2}} \, dx$$