1. The polynomial \( p(x) = x^3 - x^2 - x - 1 \) has its only positive root near \( \xi \approx 1.839 \ldots \). Without using \( f'(x) \), construct an iteration function \( \Phi(x) \) having the fixed point \( \xi = \Phi(\xi) \) and having the property that the iteration converges for any starting point \( x_0 > 0 \). Write a code to implement this method and output your numerical results for several different starting points.

2. Let \( f : R \to R \) have a single, simple zero \( x \). Show that, if \( \Phi(x) = x - f(x) \) and the recursion:

\[
y_i = \Phi(x_i), \quad z_i = \Phi(y_i) \quad i = 0, 1, 2, \ldots
\]

\[
x_{i+1} = x_i - \frac{(y_i - x_i)^2}{z_i - 2y_i + x_i}
\]

are used, the result is the quasi-Newton method

\[
x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n) - f(x_n - f(x_n))}, \quad n = 0, 1, \ldots
\]

Show that this iteration converges at least quadratically to simple zeros and linearly to multiple zeros.

3. Consider the boundary value problem

\[
\frac{d^4u}{dx^4} = f, \quad 0 < x < 1
\]

\[
u(0) = u'(0) = u(1) = u'(1) = 0.
\]

(a) Show that the problem (1) can be given the following variational formulation: Find \( u \in W \) such that

\[
(u'', v'') = (f, v) \quad \forall v \in W,
\]

where \( W = \{v : v \text{ and } v' \text{ are continuous on } [0, 1], v'' \text{ is piecewise continuous and } v(0) = v'(0) = v(1) = v'(1) = 0\} \).

(b) For \( I = [a, b] \) an interval, define \( P_3(I) = \{v : v \text{ is a polynomial of degree 3 on } I\} \). Show that \( v \in P_3(I) \) is uniquely determined by the values \( v(a), v'(a), v(b), v'(b) \). Find the corresponding basis functions (the basis function corresponding to the value \( v(a) \) is the cubic polynomial \( v \) such that \( v(a) = 1, v'(a) = 0, v(b) = v'(b) = 0 \), etc).

(c) Starting from (b) construct a finite-dimensional subspace \( W_h \) of \( W \) consisting of piecewise cubic functions. Specify suitable parameters to describe the functions in \( W_h \) and determine the corresponding basis functions.
(d) Formulate a finite element method for (1) based on the space $W_h$. Find the corresponding linear system of equations in the case of a uniform partition. Determine the solution in e.g. the case of two intervals and $f$ constant. Compare with the exact solution.

4. Let $\Omega$ be a square with side 1. Show that

$$\left( \int_{\Omega} v^2 \, dx \right) \leq \frac{1}{2} \left( \int_{\Omega} |\nabla v|^2 \, dx \right) \quad \forall v \in H^1_0(\Omega).$$

5. Show (formally) that $u$ is the solution of the variational problem

$$\min_{v \in H^1_0(I)} \left[ \frac{1}{2} \int_I k(x)v''^2 \, dx - \int_I v \, dx \right],$$

where $I = (0,1)$, and

$$k(x) = \begin{cases} 1 & x \in I_1 = (0,1/2), \\ 1/2 & x \in I_2 = (1/2,1), \end{cases}$$

if and only if $u$ satisfies

$$-k(x)u''(x) = 1 \quad \text{in } I_1 \text{ and } I_2,$$

$$u_1 = u_2, 2 \frac{du_1}{dx} = \frac{du_2}{dx} \quad \text{for } x = 1/2,$$

$$u(0) = u(1) = 0,$$  \hspace{1cm} (2)

where $u_i = u|_{I_i}, i = 1,2$. Then formulate a finite element method for (2) using piecewise linear functions. Determine the corresponding linear system in the case of a uniform partition and give an interpretation of this system as a difference method for (2).