Math 837 Homework Assignment No. 1

1. Give the level set equation for the following Hamilton-Jacobi equation

$$S_t + H(x, \nabla S) = 0\,, \qquad S \in R, x \in R^n$$

that allows you to compute the multivalued solution to S. The definition of the level set function is

$$\phi(t, x, p, q) = 0 \quad \text{when} \quad p = \nabla S, \quad q = S\,.$$

2. For the shallow-water system

$$\begin{aligned} &h_t + (hu)_x = 0 \\ &(hu)_t + (hu^2 + gh^2/2)_x = sgh - ru^2 \end{aligned}$$

where h is the height, u the velocity, g the gravitational constant, s the slope of the river bottom, r the friction coefficients of the bottom, scale it to long time and large space, and then carry out the chapman-Enskog expansion to find the "Navier-Stokes" limit. Under what condition the viscosity coefficient is positive?

3. Consider the following transport equation

$$W_t + k \cdot \nabla_x W - \nabla_x V \cdot \nabla_k W = \int_{|\xi'|=1} W(t, x, |k|, \xi') d\xi' - 4\pi W$$

where $W(t, x, k) > 0$ is the probability density distribution, $V(x)$ is the potential, $x, k \in R^3$, $k = |k|\xi$ for $\xi \in R^3$ and $|\xi| = 1$. V is defined by

$$V = \int W dk.$$

Scale the equation properly and find its diffusion approximation.