Math 837 Homework Assignment No. 1

1. Give the level set equation for the following Hamilton-Jacobi equation

\[ S_t + H(x, \nabla S) = 0, \quad S \in \mathbb{R}, x \in \mathbb{R}^n \]

that allows you to compute the multivalued solution to \( S \). The definition of the level set function is

\[ \phi(t, x, p, q) = 0 \quad \text{when} \quad p = \nabla S, \quad q = S. \]

2. For the shallow-water system

\[
\begin{align*}
ht + (hu)_x &= 0 \\
(hu)_t + (hu^2 + gh^2/2)_x &= sgh - ru^2
\end{align*}
\]

where \( h \) is the height, \( u \) the velocity, \( g \) the gravitational constant, \( s \) the slope of the river bottom, \( r \) the friction coefficients of the bottom, scale it to long time and large space, and then carry out the Chapman-Enskog expansion to find the “Navier-Stokes” limit. Under what condition the viscosity coefficient is positive?

3. Consider the following transport equation

\[
W_t + k \cdot \nabla_x W - \nabla_x V \cdot \nabla_k W = \int_{|\xi'|=1} W(t, x, |k|, \xi') d\xi' - 4\pi W
\]

where \( W(t, x, k) > 0 \) is the probability density distribution, \( V(x) \) is the potential, \( x, k \in \mathbb{R}^3 \), \( k = |k|\xi \) for \( \xi \in \mathbb{R}^3 \) and \( |\xi| = 1 \). \( V \) is defined by

\[ V = \int W dk. \]

Scale the equation properly and find its diffusion approximation.