Math 837 Homework Assignment No. 2

Consider the following neutron transport equation

$$\mu\partial_x\Psi(x,\mu)+\frac{\sigma_T(x)}{\epsilon}\Psi(x,\mu)=\left(\frac{\sigma_T(x)}{\epsilon}-\epsilon\,\sigma_A(x)\right)\overline{\Psi}(x)+\epsilon Q(x)\,, \tag{1}$$

where

$$\overline{\Psi}(x)=\frac{1}{2}\int_{-1}^{1}\Psi(x,\mu)d\mu\,.$$

Here $\Psi(x,\mu)$ is the density distribution of particles at position x, and $\mu\in(-1,1)$ is the cosine of the angle between the x-axis and particle velocity, $\sigma_T(x)$ and $\sigma_A(x)$ are scattering and absroption cross-sections, ϵ is the mean free path, $Q(x)$ is the source term. Consider the boundary value problem

$$\begin{aligned}\Psi(0,\mu)&=F_L(\mu)\,,\qquad \text{for } \mu\in(0,1]\,,\\ \Psi(1,-\mu)&=F_R(\mu)\,,\qquad \text{for } \mu\in(0,1]\,.\end{aligned}$$

1) Find the diffusion limit as $\epsilon\to 0$.

2) The discrete-ordinate method is a semi-discrete version of the transport equation where only the angle variable μ is discretized. In this method, the variable μ is discretized by a set of quadrature points $\mu_m\in(-1,1)$, symmetric with respect to 0, with quadrature weights $w_m>0$. Let $\psi_m(x)=\Psi(x,\mu_m)$. Approximate the integration in μ by the quadrature rule. Carry out the Chapman-Enskog expansion on the discrete-ordinate equation. What are the conditions on the quadrature set in order for the discrete-ordinate method to have the same diffusion limit as the original transport equation?

3) Assume the discrete ordinate equation is discretized in x by a) the upwind method and b) the centered difference scheme. Use the Gauss-Legendra quadrature rule with 8 points for μ. Perform the following numerical tests:

$$F_L=1,\quad F_R=0,\quad \sigma_T=\sigma_A=1,\quad Q=0,\quad \epsilon=0.01$$

Solve the discrete-ordinate equation using methods a) and b), for different ratio of $\Delta x/\epsilon$ (say $10,1,$ and 0.1), and compare the solution with that of the diffusion equation with the same boundary conditions. What can you observe? Can you explain your numerical observation?