

Math 837 Homework Assignment No. 3
Due April 25, 2006

1. Consider the 1d compressible Euler equations

$$\begin{aligned}\partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0 \\ \partial_t E + \partial_x(u(E + p)) &= 0\end{aligned}$$

where ρ, u, p and E are respectively the density, velocity, pressure and total energy. For a polytropic gas, the equation of state is given by

$$p = (\gamma - 1)(E - \rho u^2/2)$$

Take $\gamma = 1.4$. Solve the following initial value problem

$$(\rho, u, E)|_{t=0} = (1, 0, 2.5) \quad \text{for } x < 0; \quad (\rho, u, E)|_{t=0} = (0.125, 0, 0.25) \quad \text{for } x > 0$$

You may choose the computational domain $[-0.5, 0.5]$ and compute the solution at $t = 0.1644$ by using the first and second order (with van Leer slope limiter) Roe schemes with 200 grid points. Choose a suitable time step that satisfies the CFL condition. Compare the solution with the exact solution (which may be obtained by using the second order Roe scheme with 1000 points).

2. Consider the two dimensional scalar constant coefficient convection equation

$$\partial_t u + \partial_x u + \partial_y u = 0$$

in domain $[-1, 1] \times [-1, 1]$, with initial condition

$$u_0(x, y) = 1, \quad \text{if } (x, y) \in S; \quad u_0(x, y) = 0, \quad \text{elsewhere;}$$

where

$$S = \{(x, y) : |x - y| < 2^{-1/2}, |x + y| < 2^{-1/2}\}.$$

The boundary condition is periodic. First give the exact solution at $t = 2, 8$. Solve the problem numerically by the first and second order upwind methods (with van Leer limiter) with dimensional splitting and operator splitting respectively for time. Compare the results at $t = 2, 8$ using 20 points and CFL number of 0.8. You can plot 1) the level curves of u ; 2) $u(x, 0, t)$; 3) $u(x, -0.4, t)$. For 2) and 3) plot the solutions versus the exact solution. Discuss your observations. What do you see if you refine the mesh?