A NOTE ON THE IMAGINARY POWER OF THE LAPLACIAN

JONGCHON KIM

1. INTRODUCTION

This is a part of what I learned from the reading course with Andreas Seeger in Fall 2012. This note might work as a companion to the paper [1] which covers more general operators.

Definition 1.1 (Multiplier operators). Let $m$ be a tempered distribution and $f \in S(\mathbb{R}^n)$, the Schwarz space. Define the operator $T_m$ on $S(\mathbb{R}^n)$ by

$$
\hat{T}_m f(\xi) = m(\xi) \hat{f}(\xi).
$$

We say $m \in M_p$ if $T_m$ is bounded on $L^p$ and define $||m||_{M_p} = ||T_m||_{L^p \to L^p}$.

Example 1.1. The Hilbert transform is a multiplier operator with $m(\xi) = i \text{sgn}(\xi)$.

Theorem 1.1. Here are some useful facts about multiplier operators.

(i) $T_m f = K \ast f$ for $\hat{K} = m$
(ii) $||m||_{M_2} = ||m||_{L^\infty}$
(iii) $||m||_{M_p} = ||m||_{M_{p'}}$ for $1 \leq p \leq \infty$.

Example 1.2 (Powers of the Laplacian). Define the operator $(-\Delta)^\beta$ by

$$
\hat{(-\Delta)^\beta} f(\xi) = |\xi|^\beta \hat{f}(\xi).
$$

When $-n < \beta < 0$, the Hardy-Littlewood-Sobolev inequality gives $L^p \to L^q$ boundedness for the operator for $1/q = 1/p + \beta n$. We are interested in the case when $\beta$ is purely imaginary.

Definition 1.2. Let $m_\tau(\xi) = |\xi|^{i\tau}$ for $\tau \in \mathbb{R} \setminus \{0\}$ and $T_\tau = T_{m_\tau}$.

First note that $|m_\tau(\xi)| = 1$, so $T_\tau$ is bounded on $L^2$. Moreover, Hörmander-Mikhlin multiplier theorem shows that $T_\tau$ is of weak type $(1,1)$ with

$$
|\{x : |T_\tau f(x)| > \alpha\}| \leq C_\tau (1 + |\tau|)^{\frac{\alpha}{2} + \epsilon} ||f||_{L^1}
$$

for any $\epsilon > 0$.

By interpolation, we get $||m_\tau||_{M_p} \leq C_\tau (1 + |\tau|)^{\frac{n}{p} - \frac{1}{2} + \epsilon}$, for $1 < p < \infty$. Our goal is to get rid of the $\epsilon$ and get a sharp norm estimate as $|\tau| \to \infty$. We show

Theorem 1.2. $||T_\tau||_{H^{1/2} \to L^1} \lesssim |\tau|^{\frac{\beta}{2}}$ for all large $|\tau|$.

Theorem 1.3. $||T_\tau||_{L^1 \to L^{1,\infty}} \lesssim |\tau|^{\frac{\beta}{2}}$ for all large $|\tau|$.

In Section 3 we also show that Theorem 1.2 and Theorem 1.3 are sharp.

2. D YADIC DECOMPOSITION OF THE MULTIPLIER

Let $\varphi \in C_0^\infty$ such that $\text{supp} \varphi \subset B_2 \setminus B_{1/2}$ and $\sum_{j=-\infty}^{\infty} \varphi(\frac{\xi}{2^j}) = 1$ if $\xi \neq 0$. Define $K^\tau(x)$ by $K^\tau(\xi) = \varphi(\xi) m_\tau(\xi)$. Note that $m_\tau$ is “almost” homogeneous of degree 0, in the sense that $m_\tau(r\xi) = c_r m_\tau(\xi)$ for all $r > 0$ with $|c_r| = 1$. Since $c_r$
does not make any difference in the norm estimate, let us assume that \( c_r = 1 \) for simplicity. Let \( K_j^n(x) = 2^{nj}K^r(2^jx) \). Then we have,

\[
m_r(\xi) = \sum_j \varphi\left(\frac{\xi}{2^j}\right)m_r(\xi) = \sum_j \hat{K}^r\left(\frac{\xi}{2^j}\right) = \sum_j \tilde{K}^r_j(\xi)
\]

Thus, \( Tf(x) = \sum_j K_j^n * f(x) \).

3. Estimates on \( K^r(x) \)

In this section, we get several estimates which will be used later. Notice that \( K^r(x) = \frac{1}{(2\pi)^n} \int e^{i(\tau^{\log|\xi|} + x \cdot \xi)} \varphi(x) \, dx \). So the following theorem applies.

**Theorem 3.1.** Let \( I(\lambda) = \int e^{i\lambda \Phi(\xi)} \varphi(\xi) \, d\xi \), where \( \varphi \in C_0^{\infty} \) and \( \Phi \) is smooth on the supp \( \varphi \).

(i) *(Non-stationary phase)* Suppose \( \nabla \Phi(\xi) \neq 0 \) for \( \xi \in \text{supp} \varphi \). Then \( I(\lambda) = O_N(\lambda^{-N}) \) as \( \lambda \to \infty \).

(ii) *(Stationary phase)* Suppose there is a non-degenerate stationary point \( \xi_0 \in \text{supp} \varphi \), i.e. \( \nabla \Phi(\xi_0) = 0 \) and \( \det \left( \frac{\partial^2 \Phi}{\partial \xi_i \partial \xi_j}(\xi_0) \right) \neq 0 \). Then \( I(\lambda) = C\lambda^{-\frac{N}{2}} + O(\lambda^{-\frac{N}{2} - 1}) \).

Let \( \phi(\xi) = \tau \log|\xi| + x \cdot \xi \). Then \( \nabla \phi(\xi) = \frac{\xi}{|\xi|^2} + x \) and there’s one non-degenerate stationary point \( \xi_0 = -\frac{x}{|x|^2} \) with \( |\xi_0| = \frac{|x|}{|x|^2} \). Therefore, if \( |x| > 2|\tau| \) or \( |x| < \frac{1}{2}|\tau| \), then \( \nabla \phi(\xi) \neq 0 \) for \( \xi \in \text{supp} \varphi \). If \( \frac{1}{2}|\tau| < |x| < 2|\tau| \), then \( \xi_0 \in \text{supp} \varphi \). Applying the theorem with \( \lambda = |x| \) and \( \lambda = |\tau| \), we get

**Theorem 3.2.** We have the following estimates

(i) \( K^r(x) = O_N(|x|^{-N}) \) when \( |x| > 2|\tau| \)

(ii) \( K^r(x) = O_N(|\tau|^{-N}) \) when \( |x| < \frac{1}{2}|\tau| \)

(iii) \( K^r(x) \approx C|\tau|^{-\frac{N}{2}} \) when \( \frac{1}{2}|\tau| < |x| < 2|\tau| \)

(iv) \( \|K^r\|_{L^p} \approx |\tau|^{n(N - \frac{1}{2})} \)

where the implied constants does not depend on \( \tau \) and \( x \).

Notice that (iv) shows that \( ||m_r||_{L^p} \gtrsim |\tau|^{n(N - \frac{1}{2})} \), so this implies that Theorem 1.2 and Theorem 1.3 are optimal.

Using the above estimates, we get the following:

**Corollary 3.3.** \( \int_{|x| > R} |K^r(x - y)| \, dx \leq \frac{C}{R^r} \) for all \( y \) with \( |y| < \frac{R}{2} \) if \( R > 4|\tau| \).

**Corollary 3.4.** \( \int |K^r(x + y) - K^r(x)| \, dx \lesssim |y| |\tau|^{\frac{N}{2}} \).

**Proof.** First notice that \( \partial_y K^r(x) \) satisfies the same estimate as \( K^r(x) \). Therefore, using the above theorem, we see that \( ||\nabla K^r||_{L^1} \approx |\tau|^{\frac{N}{2}} \). Define \( g(t) = K^r(x + ty) \), then \( K^r(x + y) - K^r(x) = \int_0^1 g'(t) \, dt = \int_0^1 y \cdot \nabla K(x + ty) \, dt \). Fubini’s theorem gives the estimate. \( \square \)
4. Proof of Theorem 1.2

Let \( a \) be an atom supported on a ball \( B \). Here we may assume that the ball is centered at the origin since \( T_m \) is translation invariant. Moreover, we may assume that \( B = B_1 \) because \( T_m \) is invariant under the scaling which follows from \( m \tau \) being homogeneous of degree 0.

First, we exploit the \( L^2 \) boundedness of \( T_m \) to get

\[
\int_{|x|<4|\tau|} |T_m a(x)| dx \leq C||T_m a||_{L^2} |\tau|^{\frac{3}{2}} \leq C||a||_{L^2} |\tau|^{\frac{3}{2}} \lesssim |\tau|^{\frac{3}{2}}.
\]

Therefore, it is enough to get the estimate

\[
(1) \quad \int_{|x|>4|\tau|} |T_m a(x)| dx \leq \sum_j \int_{|x|>4|\tau|} |K_j^\tau * a(x)| dx \lesssim |\tau|^{\frac{3}{2}}.
\]

Change of the variable and Corollary 3.3 we get the following estimate:

\[
(2) \quad \int_{|x|>4|\tau|} |K_j^\tau * a(x)| dx \leq \int \int_{|x|>4|\tau|} 2^{nj} |K^\tau (2^j (x-y))| dx |a(y)| dy
\]

\[
\leq \int_{|y|\leq 1} \int_{|x|>2^{j+2}|\tau|} |K^\tau (x-2^j y)| dx |a(y)| dy \lesssim \frac{|a||L_1|}{2^j |\tau|} \lesssim \frac{1}{2^j}.
\]

This estimate is good for \( j \geq 0 \), but not for \( j < 0 \). To get an estimate which is good for \( j < 0 \), we exploit the mean zero property of \( a \) and Corollary 3.4

\[
(3) \quad \int_{|x|>4|\tau|} |K_j^\tau * a(x)| dx \leq \int \int_{|x|>4|\tau|} 2^{nj} |K^\tau (2^j (x-y)) - K^\tau (2^j x)| dx |a(y)| dy
\]

\[
\lesssim 2^j |a||L_1| |\tau|^{\frac{3}{2}} \lesssim 2^j |\tau|^{\frac{3}{2}}.
\]

Combining (2) and (3), we get (1).

5. Proof of Theorem 1.3

Let \( f \in L^1 \) and \( \alpha > 0 \). We use the Calderón-Zygmund decomposition \( f = g + b \) with height \( |\tau|^{\frac{3}{2}} \alpha \), which have the following properties:

(i) \( |g(x)| \leq c|\tau|^{\frac{3}{2}} \alpha \)

(ii) \( b = \sum_k b_k, \text{ supp } b_k \subset Q_k, \text{ and } \int b_k = 0. \)

(iii) \( \int_{Q_k} |b_k| \leq c|\tau|^{\frac{3}{2}} \alpha |Q_k| \)

(iv) \( \sum_k |Q_k| \leq \frac{c}{|\tau|^{\frac{3}{2}} \alpha} ||f||_{L^1}. \)

Using this with Chebyshev’s inequality, we get

\[
(4) \quad |\{|T_{\tau} g(x)| > \alpha/2\}| \leq \frac{4}{\alpha^2} \int |g|^2 \lesssim \frac{|\tau|^{\frac{3}{2}}}{\alpha} \int |g| \lesssim \frac{|\tau|^{\frac{3}{2}}}{\alpha} ||f||_{L^1}. \]

Let \( \Omega^* = \cup_k Q_k^* \), where \( Q_k^* \) is the cube which has the same center as \( Q_k \) but of side length \( 2c|\tau| \) times that of \( Q_k \) for some \( c > 0 \). Then \( |\Omega^*| \lesssim |\tau|^n \sum_k |Q_k| \lesssim \frac{|\tau|^{\frac{3}{2}}}{\alpha} ||f||_{L^1}. \) Therefore, it remains to show that

\[
|\{|T_{\tau} b(x)| > \alpha/2\} \cap (\Omega^*)^c| \lesssim \frac{|\tau|^{\frac{3}{2}}}{\alpha} ||b||_{L^1}. \]
which follows from

\[ \int_{(Q_k^c)^*} |T_{\tau}b_k| \leq C|\tau|^{\frac{n}{2}} \int_{Q_k} |b_k|. \]  

As in the proof of Theorem 1.2, we may assume that \( Q_k = [-1, 1]^n \). The remaining part is also essentially the same as the proof of Theorem 1.2

References