POLYNOMIAL PARTITIONING AND THE
SZEMERÉDI-TROTTER THEOREM

1. Szemerédi-Trotter Theorem

Last time, Chandan discussed the proof of the polynomial partitioning lemma. We recall the statement.

**Theorem 1.1** (Guth-Katz [4]). Let \( X \subset \mathbb{R}^n \) be a finite set and \( D \geq 1 \). Then there is a non-zero polynomial \( P \) of degree \( \leq D \) so that each component of \( \mathbb{R}^n \setminus Z(P) \) contains \( \leq C_n |X|/D^n \) points of \( X \).

We also know, by a classical theorem from geometry, that the number of open connected components of \( \mathbb{R}^n \setminus Z(P) \) is \( O(D^n) \) if the degree of \( P \) is \( \leq D \). Solymosi-Tao gave a direct argument in Appendix of their paper [5] on higher dimensional versions of Szemerédi-Trotter theorem. We sketch their argument as follows. It suffices to show that for any large cube \( Q \) (say, centered at the origin), the number of components intersecting \( Q \) is \( O(D^n) \). The number of components intersecting the \( (n-1) \)-dimensional boundary of \( Q \) can be dealt by an induction on dimension. The number of components contained in \( Q \) is bounded by the number of critical points of \( P \), as each of them contains at least one critical point. Then they use Bézout’s theorem to estimate the number of critical points of \( P \), which is \( O(D^n) \).

In this note, we give an application of the polynomial partitioning lemma to Szemerédi-Trotter theorem based on the notes by Guth and a blog posting by Tao. The setting is as follows;

- \( \mathcal{L} \) : a set of \( L \) lines in \( \mathbb{R}^2 \).
- \( P_r(\mathcal{L}) \) : collection of \( r \)-rich points, i.e. points which lies in at least \( r \)-many lines in \( \mathcal{L} \).

The question is, how large can \( |P_r(\mathcal{L})| \) be? Consider the following examples. Take \( p \) points and then draw \( r \)-lines through each of them, then it is possible to get \( P_r(\mathcal{L}) = L/r \). Another example is the grid example; for e.g. 2-rich points, we use \( L/2 \) by \( L/2 \) grid, then we can obtain \( \sim L^2 \) many 2-rich points. Using a variant of this, one may obtain a configuration of lines so that \( |P_r(\mathcal{L})| \sim L^2/r^3 \). A fundamental result in incidence geometry is that these examples are extremal.
Theorem 1.2 (Szemerédi-Trotter).
\[ |P_r(\mathcal{L})| \lesssim \max(L/r, L^2/r^3). \]
Let \( S \) be set of \( S \) points in \( \mathbb{R}^2 \). Set
\[ I(S, \mathcal{L}) = |\{(p, l) \in S \times \mathcal{L} : p \in l\}|. \]

An equivalent statement of the Szemerédi-Trotter theorem is the following.

Theorem 1.3 (Szemerédi-Trotter).
\[ I(S, \mathcal{L}) \lesssim S^{2/3}L^{2/3} + S + L. \]

Theorem 1.3 implies Theorem 1.2 by setting \( S = P_r(\mathcal{L}) \). A proof of Theorem 1.3 using polynomial methods can be found in one of Guth’s course notes [1]. However, for my learning purposes, I wished to write a direct proof of Theorem 1.2 using similar techniques, following an outline from Guth’s Namboodiri lectures [2].

Proof of Theorem 1.2. If \( L \lesssim r^2 / r \), then the bound \( L/r \) is dominant. This case is easier, and follows from a simple counting argument.

Lemma 1.4. Assume that \( L \leq r^2 / 5 \). Then \( |P_r(\mathcal{L})| < 2L/r \).

Proof. Suppose not. Pick a subset \( P \subset P_r(\mathcal{L}) \) with \( 2L/r \) points. By assumption, \( P \) contains \( 2r/5 \) many \( r \)-rich points. We count the number of lines through points in \( P \). Observe that at least \( 3r/5 \) many lines through a point in \( P \) cannot pass any other points in \( P \). This gives the bound \( L \geq |P|3r/5 = 6L/5 \), which is contradiction. \( \square \)

It remains to treat the case \( r^2 \lesssim L \) and show that \( |P_r(\mathcal{L})| \lesssim L^2/r^3 \). We use the polynomial partitioning lemma to decompose \( P_r(\mathcal{L}) = P_Z \cup P_{\text{cell}} \), where \( P_Z = P_r(\mathcal{L}) \cap Z(P) \) for some polynomial \( P \) of degree \( \leq D \) to be chosen and \( P_{\text{cell}} = \cup_{i=1}^N P_{\text{cell},i} \), where \( P_{\text{cell},i} = P_r(\mathcal{L}) \cap O_i, |P_{\text{cell},i}| \lesssim |P_r(\mathcal{L})|/D^2 \) and \( N \lesssim D^2 \).

First consider the “cellular” case, i.e. the case \( |P_{\text{cell}}| \geq |P_r(\mathcal{L})|/2 \). Note that, in this case, the number of cells \( N \) must be comparable to \( D^2 \). Let \( \mathcal{L}_i \) be the lines entering a cell \( O_i \). We claim that there is a cell \( O_i \) such that
\begin{itemize}
  \item \( |P_{\text{cell},i}| \sim |P_r|/D^2 \),
  \item \( |\mathcal{L}_i| \lesssim L/D \).
\end{itemize}

We assume the claim for the moment. Then
\[ |P_r(\mathcal{L})| \sim D^2|P_{\text{cell},i}| = D^2|P_r(\mathcal{L}_i)|. \]
Take $D \sim L/r^2$ so that $|\mathcal{L}_i| \leq r^2/5$. Then we may apply Lemma 1.4 to get $|P_r(\mathcal{L}_i)| \lesssim L/(Dr) \sim r$. This gives the $L^2/r^3$ bound for $|P_r(\mathcal{L})|$.

To verify the claim, observe that there should be $\sim D^2$ many cells $O_i$ such that $|P_{cell,i}| \sim |P_r|/D^2$. Denote by $O$ the collection of such cells. Next, consider incidences between lines and $O$, i.e.

$$I(\mathcal{L}, O) = |\{(l, O) : l \in \mathcal{L} \text{ enters a cell } O \in O\}|.$$

Let $\mathcal{L}_Z$ be the lines contained in $Z(P)$ and $\mathcal{L}_{cell} = \mathcal{L} \setminus \mathcal{L}_Z$. By Bézout’s theorem, $|l \cap Z(P)| \leq D$ if $l \in \mathcal{L}_{cell}$. Therefore, such line can enter at most $D+1$ cells and thus, $I(\mathcal{L}, O) \lesssim LD$. Let $N$ to be the minimum of $|\mathcal{L}_i|$ associated with cells $O_i$ contained in $O$. This gives $ND^2 \lesssim I(\mathcal{L}, O)$, and thus $N \lesssim L/D$. This verifies the claim.

Next, consider the “algebraic” case, i.e. $|P_Z| \geq |P_r(\mathcal{L})|/2$. We also decompose $P_Z$ as

- Type 1: $x$ is a singular point; $P(x) = \nabla P(x) = 0$.
- Type 2: $x$ is a non-singular point.

Let $Z_S(P)$ be the collection of singular points in $Z(P)$. We claim that

$$(1) \quad |l \cap \text{Type 1}| \leq |l \cap Z_S(P)| \leq D.$$  

Since (1) is clear for $l \in \mathcal{L}_{cell}$, we assume that $l \in \mathcal{L}_Z$. Without loss of generality, we may assume that $P$ does not have any square factor. Then, it is impossible for $l$ to be contained in both $Z(\partial_x P)$ and $Z(\partial_y P)$. Using this, another application of Bézout’s theorem gives (1). By (1) and the fact that each point lies on at least $r$ lines,

$$r|\text{Type 1}| \leq I(\text{Type 1}, \mathcal{L}) \leq DL.$$  

This gives the $L^2/r^3$ bound for Type 1.

On the other hand, each Type 2 point lies in at most one line contained in $Z(P)$. Thus, $I(\text{Type 2}, \mathcal{L}_Z) \leq |\text{Type 2}|$. By a similar argument for Type 1 points, we get

$$r|\text{Type 2}| \leq I(\text{Type 2}, \mathcal{L}) \leq I(\text{Type 2}, \mathcal{L}_{cell}) + I(\text{Type 2}, \mathcal{L}_Z) \leq DL + |\text{Type 2}|.$$  

This gives the $L^2/r^3$ bound for Type 2.  

I learned the consideration of singular/non-singular points from [6]. The proof of Theorem 1.3 from [1] is simpler than the proof of Theorem 1.2 given here. In particular, for the “algebraic” case, it does not require the consideration of singular/non-singular points but it uses a clever induction argument, using the fact that $|\mathcal{L}_Z| \leq D$. However, I don’t know how to adapt such an inductive argument for a “direct” proof of Theorem 1.2.
2. Incidence theorems in 3-dimension

What can we say about \( r \)-rich points in 3-dimension? Can we improve the bound \( \max(L/r, L^2/r^3) \)? The answer is no unless we impose some conditions on \( \mathcal{L} \). For instance, when all lines are contained in a plane, one cannot do better. If the configuration of \( \mathcal{L} \) is “truly” 3-dimensional in some sense, then the estimate might be improved. We briefly introduce some results in this direction.

**Theorem 2.1.** If \( \mathcal{L} \) contains at most \( \lesssim L^{1/2} \) lines in any plane and \( r \geq 3 \), then \( |P_r(\mathcal{L})| \lesssim L^{3/2}/r^2 \).

This result is due to Guth-Katz (Elekes-Kaplan-Sharir for \( r = 3 \)). What is an analogue for 2-rich points? It turns out that there is a barrier to achieve the \( L^{3/2} \) bound. For instance consider the surface \( z = xy \) which contains many lines. Indeed, each point \((a, b, ab)\) on the surface is an intersection of two lines parameterized by \( t \rightarrow (a, t, ta) \) and \( t \rightarrow (t, b, tb) \). Thus, it is possible that we may get \( \sim L^2 \) 2-rich points with \( L \) lines contained in the surface \( z = xy \). However, if one excludes such degenerate cases, then an improvement is still possible.

**Theorem 2.2** (Guth-Katz [4]). If \( \mathcal{L} \) contains at most \( \lesssim L^{1/2} \) lines in any algebraic surface of degree \( \leq 2 \), then \( |P_2(\mathcal{L})| \lesssim L^{3/2} \).

A three dimensional version of Theorem 1.3 can be found at [1].

One may use a simpler induction argument if one is willing to allow an \( \epsilon \) loss in the exponents as was done by Solymosi-Tao. In his restriction paper [3], Guth included a proof of the following weaker version of Theorem 2.2 using such an induction argument.

**Theorem 2.3.** For \( \epsilon > 0 \), there is \( D = D(\epsilon) > 1 \) such that the following holds. Assume that \( \mathcal{L} \) contains at most \( B \) lines in any algebraic surface of degree \( \leq D \). Then there is a constant \( C(\epsilon, B) \) such that

\[
P_2(\mathcal{L}) \leq C(\epsilon, B)L^{3/2+\epsilon}.
\]

In fact, one can take \( C(\epsilon, S) = 2(D(\epsilon) + \sqrt{B}) \).

**Proof.** The result is true for small \( L \). So we assume that the statement is valid for \( \leq L - 1 \) lines.

Cellular case; By the same argument as in the proof of Theorem 1.2 we have for some \( i \)

\[
|P_2(\mathcal{L})| \sim D^3|P_{\text{cell},i}| = D^3|P_2(\mathcal{L}_i)|,
\]

where \( \mathcal{L}_i \) is the part of \( \mathcal{L} \) contained in the \( i \)-th cell.
where $|\mathcal{L}_i| \lesssim L/D^2$ with degree $D$ to be chosen. By the induction hypothesis,

$$|P_2(\mathcal{L})| \leq CD^3C(\epsilon,B)(L/D^2)^{3/2+\epsilon} = CD^{-2\epsilon}C(\epsilon,B)L^{3/2+\epsilon}.$$

Thus, we chose $D$ so large that $CD^{-2\epsilon} \leq 1$ and then the induction closes.

Algebraic case; We decompose $P_Z$ as

- Type 1: there is $l \in \mathcal{L}_{\text{cell}}$ containing $x$.
- Type 2: Not type 1.

As each of line in $\mathcal{L}_{\text{cell}}$ intersects $\leq D$ points in $Z(P)$, this gives that there are at most $LD$ Type 1 points. Type 2 points are contained in $P_2(\mathcal{L}_Z)$. We have a trivial estimate $|P_2(\mathcal{L}_Z)| \leq \left( \frac{|\mathcal{L}_Z|}{2} \right) \leq B^2$ by assumption. In sum,

$$|P_2(\mathcal{L})| \leq 2|P_Z| \leq 2(LD + B^2) \leq 2(D + B^{1/2})L^{3/2} \leq C(\epsilon,B)L^{3/2}.$$

$\square$

References


