Recent progress on radial Fourier multipliers
and some generalizations

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Recent Developments in Dispersive Partial Differential
Equations and Harmonic Analysis
AMS Special Session, JMM, Seattle, Washington
January 6th, 2016
Convergence of Fourier Integrals
Bochner-Riesz means

\[ R_t^\lambda f(x) = \frac{1}{(2\pi)^d} \int \left( 1 - \frac{|\xi|^2}{t^2} \right)^\lambda \hat{f}(\xi) e^{ix \cdot \xi} d\xi \]

Question of convergence for \( f \in L^p, 1 \leq p \leq \infty \)

One would like to prove a (uniform) \( L^p \) boundedness of \( R_t^\lambda \) and the maximal operator \( \mathcal{R}^\lambda f(x) = \sup_{t>0} |R_t^\lambda f(x)|. \)
Convergence of Fourier Integrals
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\[ R^\lambda_t f(x) = \frac{1}{(2\pi)^d} \int \left( 1 - \frac{|\xi|^2}{t^2} \right)^\lambda \hat{f}(\xi) e^{ix \cdot \xi} d\xi \]

Question of convergence for \( f \in L^p, 1 \leq p \leq \infty \)

One would like to prove a (uniform) \( L^p \) boundedness of \( R^\lambda_t \) and the maximal operator \( \mathcal{R}^\lambda f(x) = \sup_{t>0} |R^\lambda_t f(x)| \).
Let $1 < p < 2$ and $d \geq 2$. Define $T_m$ by

$$\hat{T}_m f(\xi) = m(a(\xi)) \hat{f}(\xi),$$

where $m \in L^\infty([1/2, 2])$ and $a$ is a fixed smooth positive function on $\mathbb{R}^d \setminus 0$ which is homogeneous of degree 1.

- Model example: $a(\xi) = |\xi|$ and $m^\lambda(\rho) = (1 - \rho^2)^\lambda \chi(\rho)$
- We shall also consider $M_m$;

$$M_m f(x) := \sup_{t>0} |T_m(\cdot/t)f(x)|$$

- Under which conditions on $a$ and $m$ is $T_m$ or $M_m$ bounded on $L^p$?
### Quasiradial Fourier Multipliers

#### Necessary conditions

$T_m$ is bounded on $L^p$ \( \Rightarrow \)

- \( \| \mathcal{F}^{-1}[m \circ a] \|_{L^p(\mathbb{R}^d)} < \infty \)
- \( A(p) := \| (1 + | \cdot |)^{- (d-1)/2} \hat{m} \|_{L^p(\mathbb{R}, (1 + |r|)^{d-1} dr)} < \infty \)

The two quantities are comparable (Lee and Seeger 15)

#### Questions

- Is the converse statement true for some $p < \frac{2d}{d+1}$?
- Find minimal smoothness assumptions on $m$ for the $L^p$ boundedness of $T_m$. 

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**Motivation**

**Our Contribution**

**Related Work**

**The Problem That We Studied**

**Previous Work**

**Jongchon Kim**

Maximal operators generated by Fourier multipliers 4/20
Recent Results for Radial Fourier Multiplier: $a(\xi) = |\xi|$

**Theorem (Garrigós and Seeger 08)**

Let $d \geq 2$ and $1 < p < \frac{2d}{d+1}$. $T_m$ is bounded on $L^p_{rad}(\mathbb{R}^d)$ if and only if $A(p) < \infty$.

**Theorem (Heo, Nazarov and Seeger 11)**

$T_m$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $A(p) < \infty$, provided that $d \geq 4$ and $1 < p < \frac{2(d-1)}{d+1}$.

By duality, it follows that $\| T_m \|_{L^p' \to L^p'} \sim A(p)$ in the same $p$-range.
Local smoothing estimate for the wave equation

Their proof yields regularity results for the wave equation, giving an endpoint version of Sogge’s conjecture in the dual $p$-range.

**Theorem (Heo, Nazarov and Seeger 11)**

Let $I$ be a compact interval. Then

$$
\left( \int_I \left\| e^{it\sqrt{-\Delta}} f \right\|_q^q \, dt \right)^{1/q} \lesssim \|f\|_{L^q_{\delta(q)}},
$$

where $\delta(q) = (d - 1)\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{q}$ provided that $d \geq 4$ and $q > 2 + \frac{4}{d-3}$.

The admissible $q$-range (up to an $\epsilon$ loss in $\delta(q)$) was improved by Bourgain and Demeter 15.
Results for Associated Maximal Operators: \( a(\xi) = |\xi| \)

**Theorem (Garrigós and Seeger 09)**

Let \( d \geq 2 \) and \( 1 < p < \frac{2d}{d+1} \). \( M_m \) is bounded on \( L^p_{\text{rad}}(\mathbb{R}^d) \) if and only if

\[
\left\| (1 + |\cdot|)^{- (d-1)/2} \sup_{t \in [1,2]} \left| \widehat{m}(t\cdot) \right| \right\|_{L^p(\mathbb{R},(1+|r|)^{d-1} dr)} < \infty.
\]

- The statement fails if \( L^p_{\text{rad}} \) is replaced by \( L^p \): counterexample for maximal Bochner-Riesz operators due to Tao 98.
- Can we characterize \( L^{p'}_{\text{rad}} \) boundedness of \( M_m \)?
Maximal Operators Associated with Radial Fourier Multipliers: $a(\xi) = |\xi|$

**Theorem (JK 15)**

Let $d \geq 2$ and $1 < p < \frac{2d}{d+1}$. Then $M_m$ is bounded on $L^p_{rad}$ if and only if $A(p) < \infty$.

**Theorem (JK 15)**

Let $d \geq 4$ and $1 < p < \frac{2(d-1)}{d+1}$. Then $M_m$ is bounded on $L^p$ if and only if $A(p) < \infty$.

- $\|M_m\|_{L^p' \to L^p'} \sim \|T_m\|_{L^p' \to L^p'} \sim \|T_m\|_{L^p \to L^p} \sim A(p)$.
- The proof relies on an inequality due to Heo, Nazarov and Seeger.
Theorem (JK 15)

Let \( d \geq 4 \) and \( 1 < p < \frac{2(d-1)}{d+1} \). Assume that the "cosphere" \( \Sigma = \{ \xi \in \mathbb{R}^d : a(\xi) = 1 \} \) has everywhere non-vanishing Gaussian curvature. Then

\[
\| M_m \|_{L^{p'} \to L^{p'}} \simeq \| T_m \|_{L^p \to L^p} \simeq A(p).
\]
This observation was utilized by Lee, Rogers and Seeger\cite{LeeRogersSeeger} in order to obtain endpoint $L^{p'}$ bounds for $M_m$ in terms of Besov spaces.

Here, it is crucial that $p' > 2$. 
Fix a cutoff function $\eta$ such that $m(\cdot/t) = \eta(\cdot)m(\cdot/t)$ for all $t \in [1, 2]$. By Fourier inversion formula,

$$T_{m(\cdot/t)}f_t(x) = \int \eta(a(\xi))m(a(\xi)/t)\hat{f}_t(\xi)e^{ix\cdot\xi}d\xi$$

$$= \int\int\int \eta(a(\xi))e^{i(x-y)\cdot\xi+ira(\xi)}d\xi t\hat{m}(tr)f_t(y)drdy.$$

Set $K_r(x) = \int \eta(a(\xi))e^{ix\cdot\xi+ira(\xi)}d\xi$. Then we may write

$$\int_1^2 T_{m(\cdot/t)}f_t(x)dt = \int\int K_r(x - y)g(r, y)drdy,$$

where $g(r, y) = \int_1^2 t\hat{m}(tr)f_t(y)dt$. 
Main Inequality

Under the same assumptions on $d, p, a$, we have

**Theorem (JK 15)**

\[
\left\| \int \int K_r(x - y) g(r, y) \, dr \, dy \right\|_p^p \\
\lesssim \int \int \left| g(r, y) \right| \left( 1 + |r| \right)^{d-1} \, dr \, dy.
\]

- Variant of an inequality due to Heo, Nazarov and Seeger
- Applying this with $g(r, y) = \int_1^2 t \hat{m}(tr) f_t(y) \, dt$ finishes the estimate for $M_m$
- Local smoothing estimates for $e^{ita(D)}$ (Lee and Seeger 13)
Multipliers in Besov Spaces

Let $d \geq 2$, $1 < p < \frac{2(d+1)}{d+3}$, $p \leq q \leq \infty$ and $\alpha(p) = d \left( \frac{1}{p} - \frac{1}{2} \right)$.

**Theorem (JK 15)**

Assume that the cosphere $\Sigma$ has everywhere non-vanishing Gaussian curvature. Then

$$\| T_m \|_{L^p \rightarrow L^p, q} \leq \| M_m \|_{L^{p'}, q' \rightarrow L^{p'}} \lesssim \| m \|_{B^{2 \alpha(p), q}}$$

- The case $a(\xi) = |\xi|$ is due to Lee, Rogers and Seeger 14
- For general $m$, $\| T_m \|_{L^p \rightarrow L^{p, q}} \lesssim \sup_{t>0} \| m(t\cdot) \phi \|_{B^{2 \alpha(p), q}}$
  c.f. Hörmander type condition: $\sup_{t>0} \| m(t\cdot) \phi \|_{L^{2 \alpha(p)+\epsilon}} < \infty$
- This yields endpoint bounds for (generalized) Bochner-Riesz multipliers and associated maximal operators in the above $p$-range
Let $M$ be a smooth compact Riemannian manifold of dimension $d \geq 2$ without boundary. Let $1 < p < \frac{2(d+1)}{d+3}$ and $p \leq q \leq \infty$.

**Conjecture**

\[
\sup_{t>0} \left\| m\left(\sqrt{-\Delta}/t\right)f \right\|_{L^p,q(M)} \lesssim \|m\|_{B^2_\alpha(p),q} \|f\|_{L^p(M)}
\]

- The above statement with $B^2_\alpha(p),q$ replaced by $L^2_\alpha(p)+\epsilon$ is due to Seeger and Sogge 89.
- This conjecture was posed by Seeger 91, who obtained a slightly weaker endpoint results.
Open Problems

- What happens in the case when the cosphere $\Sigma$ has some points of vanishing curvature?
- Improve the $p$-ranges in the results; $p < \frac{2d}{d+1}$ is the conjectured range.
For Further Reading

- Y. Heo, F. Nazarov, A. Seeger
  *Radial Fourier multipliers in high dimensions*

- S. Lee, K. M. Rogers, A. Seeger
  *Square functions and maximal operators associated with radial Fourier multipliers*
  Advances in Analysis: The Legacy of Elias M. Stein, 2014

- J. Bourgain, C. Demeter
  *The proof of the $l^2$ Decoupling Conjecture*

- J. Kim
  *Endpoint bounds for quasiradial Fourier multipliers*
Restricted Weak-type Inequality

For each \((n, z) \in \mathbb{N} \times \mathbb{Z}^d\), assume that \(b_{n,z}\) is a function bounded by 1 and supported on 1-neighbourhood of \(z\).

Reduction of the main inequality to a restricted weak-type inequality

\[
\text{meas}\{x : \left| \sum_{j \geq 1} 2^{j(d-1)/2} \sum_{(n,z) \in \mathcal{E}_j} K_n \ast b_{n,z} \right| > \lambda \} \lesssim \lambda^{-p} \sum_{j \geq 1} 2^{j(d-1)} \# \mathcal{E}_j,
\]

for all large \(\lambda\), where \(C\) is independent of the choice of \(b_{n,z}\) and \(\mathcal{E}_j \subset ([2^j, 2^{j+1}) \cap \mathbb{N}) \times \mathbb{Z}^d\).
Weak Spatial Localization

The proof uses ideas from Heo, Nazarov, and Seeger 11.

For simplicity, assume that $a(\xi) = |\xi|$.

Weak spatial localization: $K_n * b_{n, z}$ is “essentially" supported on an $O(1)$-nbd of $z + nS^{d-1}$.

Thus, if $Q \subset [2^j, 2^{j+1}) \times \mathbb{R}^d$ is a cube of side length $l(Q) \geq 1$ centered at $(r, y)$, then $\sum_{(n,z) \in E_j \cap Q} K_n * b_{n, z}$ is essentially supported in $A_Q$: an $O(l(Q))$-nbd of $y + rS^{d-1}$, whose measure is $O(2^{j(d-1)}l(Q))$. 
Density Decomposition

- Perform a variant of Calderón-Zygmund decomposition; Decompose $\mathcal{E}_j$ into $\mathcal{E}_j(\lambda) \cup \bigcup_{Q \in Q_j} \mathcal{E}_j \cap Q$, where $Q_j$ is a collection of disjoint dyadic subcubes $Q$ of $[2^j, 2^{j+1}) \times \mathbb{R}^d$, over which $\mathcal{E}_j$ has high density; $\mathcal{E}_j(\lambda) \cap Q > \lambda^p l(Q)$.

- For the high density part, we can avoid estimates on the exceptional set $A = \bigcup_{j \geq 1} \bigcup_{Q \in Q_j} A_Q$ since

$$|A| \lesssim \lambda^{-p} \sum_{j \geq 1} 2^{j(d-1)} \# \mathcal{E}_j.$$ 

Thus, we may control

$$\text{meas}\{ x : | \sum_{j \geq 1} 2^{j(d-1)/2} \sum_{Q \in Q_j} \sum_{(n,z) \in \mathcal{E}_j \cap Q} K_n * b_{n,z} | > \lambda \}$$

by $|A|$ plus an $L^1$ estimate off $A$. 
$L^2$ Estimate for the Low Density Part

Let $G_j = \sum_{(n,z) \in \mathcal{E}_j(\lambda)} K_n \ast b_{n,z}$. 

- $\mathcal{E}_j(\lambda)$ is the low density part; $\mathcal{E}_j(\lambda) \cap Q \leq \lambda^{p} l(Q)$ for any dyadic subcube of $[2^j, 2^{j+1}) \times \mathbb{R}^d$. By Chebyshev, it suffices to show 

$$\left\| \sum_{j \geq 1} 2^{j(d-1)/2} G_j \right\|_2^2 \lesssim \lambda^{2p/(d-1)} \log_2 \lambda \sum_{j \geq 1} 2^{j(d-1)} \# \mathcal{E}_j$$

since $-2 + \frac{2p}{d-1} < -p$ if $p < \frac{2(d-1)}{d+1}$.

- The $L^2$ estimate is completed by 

$$\left\| G_j \right\|_2^2 \lesssim \lambda^{2p/(d-1)} \# \mathcal{E}_j$$

$$|\langle G_j, G_k \rangle| \lesssim \lambda^{p} 2^{j(d-1)/2} 2^{-k(d-2)} \# \mathcal{E}_j \text{ for } k < j - 9$$