Endpoint bounds for a class of spectral multipliers on compact manifolds

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Spectral multipliers on compact manifolds

- $M$ : compact Riemannian manifold of dimension $d \geq 2$ without boundary.
- $\Delta$ : Laplace-Beltrami operator on $M$.
- Spectrum of $-\Delta$ is discrete and can be ordered as $0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots$.
- $\{ e_l \}$ : orthonormal basis for $L^2(M)$ such that $-\Delta e_l = \lambda_l^2 e_l$.
- For $f \in L^2(M)$,

$$f = \sum_l \langle f, e_l \rangle e_l = \sum_l E_l f.$$
For a given function $m$, we may define (with $A = \sqrt{-\Delta}$)

$$m(A)f = \sum_l m(\lambda_l)E_l f.$$ 

$m(A)$ is bounded on $L^2(M)$ if $m \in L^\infty$;

$$\|m(A)f\|_{L^2(M)}^2 = \sum_l |m(\lambda_l)|^2 \|E_l f\|_{L^2(M)}^2 \leq \|m\|_\infty^2 \sum_l \|E_l f\|_{L^2(M)}^2.$$ 

For which $m \in L^\infty$ may $m(A)$ be bounded on $L^p(M)$ for some $p < 2$?
Riesz means

- Let \( 1 \leq p \leq \frac{2(d+1)}{d+3} \). Stein-Tomas \( L^2 \) Fourier restriction theorem:
  \[
  \left\| \hat{f} \right\|_{L^2(S^{d-1})} \lesssim \left\| f \right\|_{L^p(\mathbb{R}^d)}.
  \]

- Sogge’s discrete restriction theorem:
  \[
  \left\| \chi_{[n,n+1]}(A)f \right\|_{L^2(M)} \lesssim (1 + n)^{\delta(p)} \left\| f \right\|_{L^p(M)}.
  \]

- Sogge obtained sharp estimates for Riesz means in the same \( p \)-range for general compact manifolds \( M \).

- More generally for \( A = \sqrt{P} \), where \( P \) is a positive second-order elliptic differential operator self-adjoint with respect to a smooth positive density \( dx \) on \( M \).
More general setting: Let $A$ be a first order elliptic pseudo-differential operator on $M$, positive and self-adjoint w.r.t. $dx$.

In what follows, we shall assume that for each $x \in M$, $\Sigma_x = \{\xi \in T^*_x(M) : a(x, \xi) = 1\}$ has everywhere non-vanishing Gaussian curvature. Here, $a(x, \xi)$ is the principal symbol of $A$.

Christ-Sogge and Seeger-Sogge: Sogge’s discrete restriction theorem continues to hold for such $A$. 
Motivation

Our Contribution

The Problem That We Studied

Previous Work

Weak-type endpoint bounds for Riesz means

- $m^\delta_t(A)$ fails to be uniformly bounded on $L^p$ if $\delta = \delta(p)$.

Theorem

Let $1 \leq p \leq \frac{2(d+1)}{d+3}$. Then

$$\sup_{t>0} \left\| m^\delta_t(A)f \right\|_{L^p,\infty(M)} \lesssim \|f\|_{L^p(M)}.$$

Due to Christ-Sogge, Seeger, and Tao.

Tao later proved in Euclidean case, strong type bounds at $p = p_0$ implies weak-type bounds for $1 \leq p < p_0$.

A decomposition for the Riesz multipliers ($t = 1$): $(1 - \lambda)^\delta_+ = \sum_j 2^{-j\delta} m^\delta_j(\lambda)$, where $m^\delta_j$ have essentially disjoint supports.
An endpoint result by Seeger

- Let $\alpha(p) = \delta(p) + 1/2 = d\left(\frac{1}{p} - \frac{1}{2}\right)$.
- More generally, Seeger obtained an endpoint $L^p \to L^{p,q}$ estimate of $m(A)$ for a function $m$ in localized $R^2_{\alpha(p),q}$ spaces, i.e. $\sup_{t>0} \|m(t\cdot)\psi\|_{R^2_{\alpha(p),q}} < \infty$.
- $R^2_{\alpha,q}$ has properties similar to, but is strictly contained in the Besov space $B^2_{\alpha,q}$ for $q > 1$.
- Orthogonality argument: there is a nice decomposition for a function in $R^2_{\alpha,q}$ which resembles the decomposition for the Riesz multipliers.
- Question: Can we replace $R^2_{\alpha,q}$ by $B^2_{\alpha,q}$?
Endpoint bounds in terms of Besov spaces

Theorem (JK, 16)

Let $1 < p < \frac{2(d+1)}{d+3}$ and $p \leq q \leq \infty$. Then

$$\| m(A)f \|_{L^{p,q}(M)} \lesssim \sup_{t > 0} \| m(t \cdot) \psi \|_{B^2_{\alpha(p),q}(\mathbb{R})} \| f \|_{L^p(M)}.$$  

- Previous results: Seeger-Sogge with $L^2_\alpha$ for any $\alpha > \alpha(p)$, Seeger with $R^2_{\alpha(p),q}$.
- Our proof is based on the work of Lee-Rogers-Seeger on radial Fourier multipliers.

We show first that for $m$ compactly supported in $[1/2, 2]$,

$$\sup_{t>0} \| m(A/t)f \|_{L^p,q(M)} \lesssim \| m \|_{B^2_{\alpha(p), q(\mathbb{R})}} \| f \|_{L^p(M)}.$$ 

For general $m$, we combine frequency localized pieces by the atomic decomposition using Peetre’s square function.

$m = \sum_{j \geq 0} m_j$ for $\hat{m}_j$ supported in $\{ r : |r| \sim 2^j \}$ for $j \geq 1$.

Then $\| m \|_{B^2_{\alpha,q}}$ is comparable to $\left( \sum_{j \geq 0} [2^{j\alpha} \| m_j \|_2]^q \right)^{1/q}$ and $\| m \|_{L^2_{\alpha}} = \| m \|_{B^2_{\alpha,2}}$. 
Strategy

- \( m(A/t)f = \sum_j m_j(A/t)f = \sum_j \int t\hat{m}_j(tr)e^{irA}fdr \), where \( e^{irA}f \) solves \((i\partial_r + A)u = 0\) and \( u(0) = f \).

- For \( 2^j < \epsilon t \) (small \( |r| \)), use the parametrix constructed by Lax and Hörmander.

- For \( 2^j \geq \epsilon t \) (large \( |r| \)), use the discrete restriction theorem.
Strategy: $2^j \geq \epsilon t$

- (Seeger-Sogge) For $2^j \geq \epsilon t$, we have
  \[ \| m_j(A/t)f \|_{L^2(M)} \lesssim 2^{j/2} \| m_j \|_2 t^{\delta(p)} \| f \|_{L^p(M)}. \]

- From the result, we obtain
  \[ \left\| \sum_{2^j \geq \epsilon t} m_j(A/t)f \right\|_{L^2(M)} \lesssim \| m \|_{B^2_{\alpha(p),\infty}} \| f \|_{L^p(M)}, \]
  since
  \[ \sum_{2^j \geq \epsilon t} 2^{j/2} \| m_j \|_2 = \sum_{2^j \geq \epsilon t} 2^{-j\delta(p)} 2^{j\alpha(p)} \| m_j \|_2 \lesssim \| m \|_{B^2_{\alpha(p),\infty}} t^{-\delta(p)}. \]
Strategy: $2^j < \epsilon t$

- Decompose $m_j = m_j \ast \tilde{\eta}_j = [m_j \eta] \ast \tilde{\eta}_j + [m_j(1 - \eta)] \ast \tilde{\eta}_j$, where $\eta$ is supported on $[1/2C, 2C]$ and 1 on $[1/C, C]$ for some large $C \geq 4$.

- As $m_j$ is essentially supported on $\{\lambda : |\lambda| \in [1/4, 4]\}$, $m_j = (m_j \eta) \ast \tilde{\eta}_j$ up to a negligible error.

- Assume that $b_j$ are functions satisfying the followings;
  
  1. $\|b_j\|_2 \leq C$ for all $j$.
  2. $\text{supp} \hat{b}_j \subset \{r : |r| \in [2^{-3}, 2^{j+3}]\}$.
  3. For $n, N \geq 0$, we have

\[
|b_j^{(n)}(\lambda)| \leq C_{n,N}2^{-jN}(1 + 2^j|\lambda|)^{-N}, \quad \text{if } |\lambda| \notin [1/16, 16].
\]

- Example of $b_j$: $\|m_j\|_2^{-1}(m_j \eta) \ast \tilde{\eta}_j$
**Strategy:** \(2^j < \epsilon t\)

- Assume that \(f_j\) are functions supported in a compact subset \(\Omega_0\) of a coordinate patch \(\Omega \subset M\).
- Let \(1 \leq p < \frac{2(d+1)}{(d+3)}\). Then we prove that

\[
\left\| \sum_{1 < 2^j < \epsilon t} 2^{jd/2} b_j(A/t) f_j \right\|_{L^p(M)} \lesssim \left( \sum_j 2^j \| f_j \|_{L^p(\Omega)}^p \right)^{1/p}.
\]

- Applying the result with \(b_j = \| m_j \|_2^{-1} (m_j \eta) * \tilde{\eta}_j\) and \(f_j = 2^{-jd/2} \| m_j \|_2 f\) yields

\[
\left\| \sum_{1 < 2^j < \epsilon t} m_j(A/t) f \right\|_{L^p(M)} \lesssim \| m \|_{B_2^{\alpha(p),p}} \| f \|_{L^p(\Omega)}.
\]

- Interpolation gives \(L^p \to L^{p,q}\) estimates.
Comparison with quasiradial Fourier multipliers

- Let \( x \in \Omega \). \( b_j(A/t)f(x) \) can be written in local coordinates as, up to a negligible error,

\[
\int \int t\hat{m}_j(tr) \int e^{i\varphi(x,y,\xi)+ira(y,\xi)} q(r, x, y, \xi) \eta(\xi/t) d\xi dr f(y) dy,
\]

where \( \varphi(x, y, \xi) \) is a small perturbation of \( \langle x - y, \xi \rangle \) and \( q \) is a symbol of order 0.

- \( m(a(D)) : \mathcal{F}[m(a(D))f](\xi) = m(a(\xi))\mathcal{F}f(\xi) \), where \( a \) is a smooth positive function which is homogeneous of degree 1. Model case: \( a(\xi) = |\xi| \).

- \( m(a(D)/t)f(x) \) can be written as

\[
\int \int t\hat{m}(tr) \int e^{i\langle x-y,\xi \rangle + ira(\xi)} \eta(\xi/t) d\xi dr f(y) dy.
\]
Corollary (JK, 15)

Let $1 < p < \frac{2(d+1)}{d+3}$ and $p \leq q \leq \infty$. Assume that

$\Sigma = \{\xi : a(\xi) = 1\}$ has everywhere non-vanishing Gaussian curvature. Then

$$\|m(a(D))f\|_{L^{p,q}(\mathbb{R}^d)} \lesssim \sup_{t > 0} \|m(t \cdot)\psi\|_{B^{2,\alpha(p),q}_{\alpha(p),q}(\mathbb{R})} \|f\|_{L^p(\mathbb{R}^d)}.$$  

- Transference theorem of Mityagin.
- Proof is based on the work of Lee-Roger-Seeger on the radial case $a(\xi) = |\xi|$, where they utilize some ideas from the work by Heo-Nazarov-Seeger.
Ingredients of the proof for compact manifolds

- Proof is based on the work on quasiradial Fourier multipliers, but it requires finer estimates.
- Main difficulty: quasi-orthogonality estimates which control the interaction between operators $b_j(A/t)$ and $b_k(A/t)$.
- Use the second dyadic decomposition (e.g. Christ-Sogge, Seeger-Sogge-Stein, Tao) and adapt a construction of an exceptional set used by Lee-Seeger.
Assume that $m$ is compactly supported in $(0, \infty)$ and consider the maximal operator $M_m f(x) = \sup_{t > 0} |m(a(D)/t)f(x)|$.

**Theorem (JK 15)**

Let $1 < p < \frac{2(d+1)}{d+3}$ and $p \leq q \leq \infty$. With the same curvature assumption on $\Sigma$, 

$$\|M_m f\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|m\|_{B^2_{\alpha(p), q}(\mathbb{R})} \|f\|_{L^{p'}, q'}(\mathbb{R}^d).$$

- Yields the maximal, dual version of the weak-type endpoint bounds for (generalized) Bochner-Riesz means.
- For a smaller $p$-range, a necessary and sufficient condition for $L^{p'}, q' \rightarrow L^{p'}$ boundedness can be obtained.
Maximal operators generated by spectral multipliers

- Let $\mathcal{M}_mf(x) = \sup_{t>0} |m(A/t)f(x)|$ for $m$ compactly supported in $(0, \infty)$. Does the following hold?

$$\|\mathcal{M}_mf\|_{L^p'(M)} \lesssim \|m\|_{B^2_{\alpha(p)},q}(\mathbb{R}) \|f\|_{L^{p',q}(M)}.$$ 

- Recall the $L^p \rightarrow L^2$ estimate for $2^j \geq \epsilon t$;

$$\|m_j(A/t)f\|_{L^2(M)} \lesssim 2^{j/2} \|m_j\|_2 t^{\delta(p)} \|f\|_{L^p(M)}.$$ 

- In view of Euclidean argument, it is likely that the maximal estimate would follow once the vector valued version of the above lemma is established;

$$\left\| \int_1^2 m_j(A/st)f_s ds \right\|_{L^2(M)} \lesssim 2^{j/2} \|m_j\|_2 t^{\delta(p)} \left\| \int_1^2 |f_s| ds \right\|_{L^p(M)},$$

whose Euclidean counterpart is not hard to prove.
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