1. Evaluate each sum or show that it diverges.

(a) [8 points] $\sum_{n=1}^{\infty} \frac{2^n + (-2)^n}{3^n}$ $= \sum_{n=1}^{\infty} \frac{2}{3}\cdot\left(\frac{2}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{-2}{3}\left(-\frac{2}{3}\right)^{n-1}$

geometric!

need to reindex

$$= \frac{\frac{2}{3}}{1-\frac{2}{3}} + \frac{-\frac{2}{3}}{1+\frac{2}{3}}$$

$$= \frac{\frac{2}{3}+\frac{4}{9}-\frac{2}{3}+\frac{4}{9}}{1-\frac{4}{9}} = \frac{\frac{8}{9}}{\frac{5}{9}} = \boxed{\frac{8}{5}}$$

(b) [8 points] $\sum_{n=1}^{\infty} \sqrt[n+1]{n+1} - \sqrt[n]{n}$

tellscoping!

$$= (2)^{1/2} - 1$$
$$+ (3)^{1/3} - (2)^{1/2}$$
$$+ (4)^{1/4} - (3)^{1/3}$$
$$\vdots$$
$$+ \lim_{K\to\infty} (K+1)^{\frac{1}{K+1}}$$

$$= -1 + \lim_{N\to\infty} \sqrt[N]{N}$$

$$= -1 + 1 = \boxed{0}$$

$$\lim_{N\to\infty} e^{\ln(N^{1/N})} = \lim_{N\to\infty} e^{\frac{\ln N}{N}} = \lim_{N\to\infty} e^{\frac{1}{N}}$$

$$= 1$$

$\tan^2 x + 1 = \sec^2 x$

2. Evaluate the following integrals.

(a) [10 points] $\int \sec^3 x \tan^3 x \, dx$ $= \int \sec^3 x \tan x (\sec^2 x - 1) \, dx$

$= \int \sec^4 x \sec x \tan x - \sec^2 x \sec x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x$

$= \int u^4 - u^2 \, du$

$= u^5/5 - u^3/3 + C$

$$\boxed{= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

(b) [10 points] $\int \ln^2 x \, dx$

$u = \ln^2 x$ $\quad dv = 1\,dx$

$du = 2\ln x \cdot \frac{1}{x}\,dx$ $\quad v = x$

$= x\ln^2 x - \int 2\ln x \, dx$

$$\boxed{= x\ln^2 x - 2x\ln x + 2x + C}$$

$\int 2\ln x \, dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x + C$

$u = \ln x$ $\quad dv = 2\,dx$

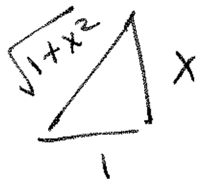
$du = \frac{1}{x}dx$ $\quad v = 2x$

(c) [12 points] $\int \frac{dx}{(x^2+1)^2}$

Trig sub!

$\sin^2 + \cos^2 = 1$

$\tan^2 + 1 = \sec^2$



$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$\int \frac{\sec^2\theta \, d\theta}{(\tan^2\theta+1)^2} = \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$$= \int \cos^2\theta \, d\theta = \int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$$= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C$$

$$= \frac{\tan^{-1}x}{2} + \frac{\left(\frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right)}{2} + C$$

$$\boxed{= \frac{\tan^{-1}x}{2} + \frac{x}{2(1+x^2)} + C}$$

3. [14 points] Find the partial fractions decomposition of the following rational functions. *You do not have to integrate.*

(a) $\dfrac{x^2+3x-1}{x^4-x^3}$ $=$ $\boxed{\dfrac{-3}{x} + \dfrac{-2}{x^2} + \dfrac{1}{x^3} + \dfrac{3}{x-1}}$

$$\frac{x^2+3x-1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}$$

$$\Rightarrow \quad Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3 = x^2+3x-1$$

$$Ax^3 - Ax^2 + Bx^2 - Bx + Cx - C + Dx^3 = x^2+3x-1$$

x^3: $A + D = 0$ $\qquad \boxed{D = 3}$

x^2: $-A + B = 1$ $\qquad -A - 2 = 1$ $\qquad \boxed{A = -3}$

x: $-B + C = 3$ $\qquad \Rightarrow 1 - B = 3$ $\qquad \boxed{B = -2}$

1: $\boxed{-C = -1}$

$\boxed{C = 1}$

$-A-$

$\cancel{\frac{5}{x} + \frac{4}{x^2} + \frac{1}{x^3} + \frac{5}{x-1}}$

(b) $\dfrac{x^3 - 2x + 1}{(x^2 + 2x + 2)^2}$

$$\boxed{= \frac{x-2}{x^2+2x+2} + \frac{5}{(x^2+2x+2)^2}}$$

$$= \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}$$

$$\Leftrightarrow (Ax+B)(x^2+2x+2) + Cx+D = x^3-2x+1$$

$$Ax^3 + 2Ax^2 + 2Ax + Bx^2 + 2Bx + 2B + Cx + D = x^3 - 2x + 1$$

x^3 $\quad \boxed{A = 1}$

x^2 $\quad 2A + B = 0 \quad \Rightarrow \boxed{B = -2}$

x $\quad 2A + 2B + C = -2$

$\quad 2 - 4 + C = -2 \quad \Rightarrow \boxed{C = 0}$

1 $\quad 2B + D = 1$

$\quad -4 + D = 1 \quad \boxed{D = 5}$

4. [10 points] Evaluate $\int_{0\,1}^{\infty} \frac{dx}{\sqrt{x}(1+x)}$ (or show that it diverges).

$u = \sqrt{x}$

$du = \frac{-1}{2\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

$$\int_{u(1)=1}^{u(\infty)=\infty} \frac{2\,du}{(1+u^2)}$$

$$= \lim_{b\to\infty} \int_1^b \frac{2\,du}{1+u^2}$$

$$= \lim_{b\to\infty} 2\arctan u \Big|_1^b$$

$$= \lim_{b\to\infty} 2\arctan b - 2\arctan 1$$

$$= 2\cdot\pi/2 - 2\cdot\pi/4$$

$$= \pi - \pi/2 = \boxed{\pi/2}$$

5. [8 points] Evaluate $\lim_{n\to\infty} \left(\ln\left(e+\frac{1}{n}\right)\right)^n$.

$\ln(e) = 1$

$\to$ indeterminate

$$\lim_{n\to\infty} e^{\ln\left[\left(\ln\left(e+\frac{1}{n}\right)\right)^n\right]}$$

$$= \lim_{n\to\infty} e^{n \ln\left(\ln\left(e+\frac{1}{n}\right)\right)}$$

$$= \lim_{n\to\infty} e^{\frac{\ln\left(\ln\left(e+\frac{1}{n}\right)\right)}{1/n}} \to \frac{\infty}{\infty} \quad \text{L'Hôpital}$$

$$= \lim_{n\to\infty} e^{\frac{\frac{1}{\ln(e+\frac{1}{n})} \cdot \frac{1}{e+\frac{1}{n}} \cdot \frac{-1}{n^2}}{-\frac{1}{n^2}}} = e^{\frac{1}{\ln(e)} \cdot \frac{1}{e}} \boxed{= e^{\frac{1}{e}}}$$

6. [20 points] Which of the following series converge and which diverge? *Give reasons for your answers.*

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n n!}{(2n)!}$ do ratio test

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n^n\, n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)^n (\cancel{n+1}) \cancel{n!}\ \cancel{(2n)!}}{2(\cancel{2n+2})(2n+1)\cancel{(2n)!}\ n^n\ \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{2(2n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(n+1)}{(2n+1)} \cdot \left(1+\frac{1}{n}\right)^n \qquad \boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

$$= \lim_{n\to\infty} \frac{1}{2}\cdot\frac{1}{2}\cdot e = \frac{e}{4} < 1$$

Converges by ratio test

(2n+2)! correctly

(b) $\sum_{n=0}^{\infty} \sqrt{\frac{n^2+5}{n^5+2}}$ do limit comparison +2

$$\sqrt{\frac{n^2}{n^5}} = \sqrt{\frac{1}{n^3}} = \frac{1}{n^{3/2}} \qquad +3$$

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n^2+5}{n^5+2}}}{\frac{1}{(n^3)^{1/2}}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{1} \cdot \sqrt{\frac{n^2+5}{n^5+2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^5+5n^3}{n^5+2}} = 1 \qquad +3$$

so by LCT since $\underbrace{\sum_{\infty} \frac{1}{n^{3/2}} \text{ conv}}$

so does $\sum_{n=0} \sqrt{\frac{n^2+5}{n^5+2}}$ +2