CONTRASTING PLAIN AND PREFIX-FREE KOLMOGOROV COMPLEXITY

JOSEPH S. MILLER

Abstract. Let $SCR_c = \{\sigma \in 2^n : K(\sigma) \geq n + K(n) - c\}$, where K denotes prefix-free Kolmogorov complexity. These are the strings with essentially maximal prefix-free complexity. We prove that SCR_c is not a Π^0_1 set for sufficiently large c. This implies Solovay's result that strings with maximal plain Kolmogorov complexity need not have maximal prefix-free Kolmogorov complexity, even up to a constant. We show that if $Q \subseteq SCR_c$ is an infinite Π^0_1 set, then Q is hyperimmune. Furthermore, assuming that $Q \in \Pi^0_1$ contains strings of every length, we derive a bound on the least element of $Q \setminus SCR_c$, matching the bound Solovay gave for $Q = KR_k = \{\sigma \in 2^n : C(\sigma) \geq n - k\}$.

We also give short derivations of Solovay's formulae relating plain and prefix-free complexity and An. A. Muchnik's result that these two complexity measures can disagree on the relative complexity of strings.

1. Introduction

This paper contributes to the investigation of the relationship between plain and prefix-free Kolmogorov complexity. These are denoted by C and K, respectively; their definitions will be reviewed in the next section. The most significant work on the interaction between these two notions of program size complexity is due to Solovay [10]. It has never been published, although it will be made available in the forthcoming text of Downey and Hirschfeldt [3]. Solovay expressed $C(\sigma)$ and $K(\sigma)$ in terms of each other, up to what is essentially a $O(\log^{(2)} |\sigma|)$ term.

Theorem 1.1 (Solovay [10])**.** *For all $\sigma \in 2^{<\omega}$:*
(i) $C(\sigma) = K(\sigma) - K^{(2)}(\sigma) + O(K^{(3)}(\sigma))$.
(ii) $K(\sigma) = C(\sigma) + C^{(2)}(\sigma) + O(C^{(3)}(\sigma))$.

Solovay also ruled out higher order analogs of these formulae.

Theorem 1.2 (Solovay [10])**.** *There are sequences $\{w_c\}_{c\in\omega}$ and $\{z_c\}_{c\in\omega}$ of strings such that $\lim_{c\to\infty} K(w_c) = \infty$ and $C(w_c) = C(z_c) + O(1)$ but $\lim_{c\to\infty} \dfrac{K(z_c) - K(w_c)}{\log^{(2)} |w_c|} = 1$.*

This demonstrates that it is not possible to express $K(\sigma)$ in terms of $C(\sigma)$ up to a $O(C^{(4)}(\sigma))$ term, because $C^{(4)}(\sigma) \leq 2\log^{(3)} |\sigma| + O(1)$ is asymptotically dominated by $\log^{(2)} |\sigma|$. Solovay's proof involves—and is motivated by a desire to understand—strings with (essentially) maximal complexity: the z_c from Theorem 1.2 have both maximal plain and prefix-free complexity and the w_c have maximal plain complexity, but clearly cannot have maximal prefix-free complexity.

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We continue the study of strings with maximal complexity. First, consider the plain complexity case. Define $KR_k = \{\sigma \in 2^{<\omega} : C(\sigma) \geq |\sigma| - k\}$. Note that KR_k is a Π^0_1 set for any $k \in \omega$. Elements of $KR = KR_0$ are called *Kolmogorov random*. Such strings exist in every length by a simple counting argument.

Proposition 1.3. $|2^n \smallsetminus KR_k| = |\{\sigma \in 2^n : C(\sigma) < n - k\}| \leq 2^{n-k} - 1$.

Proof. For every $\sigma \in 2^n \smallsetminus KR_k$, there is a τ such that $|\tau| < n - k$ and $V(\tau) = \sigma$.[1] The number of strings of length less than $n - k$ is $\sum_{m<n-k} 2^m = 2^{n-k} - 1$. □

It is clear that $C(\sigma) \leq |\sigma| + O(1)$ because V simulates the identity function, so Kolmogorov random strings indeed have (essentially) maximal plain Kolmogorov complexity.

The prefix-free case is similar, with a few complications. To get the upper bound on the complexity, consider the prefix-free machine M that acts as follows: $M(\tau\sigma)\downarrow= \sigma$ iff $U(\tau)\downarrow= |\sigma|$. By taking τ to be a minimal program for $|\sigma|$ we have $K(\sigma) \leq K_M(\sigma) + O(1) \leq |\sigma| + K(|\sigma|) + O(1)$. This upper bound is tight, although it is not as easy to prove as in the plain complexity case.

Proposition 1.4 (Chaitin [1])**.** $|\{\sigma \in 2^n : K(\sigma) < n + K(n) - c\}| \leq 2^{n-c+O(1)}$.

Now define $SCR_c = \{\sigma \in 2^n : K(\sigma) \geq n + K(n) - c\}$. Elements of SCR_c are called *strongly Chaitin random with constant* c. Chaitin's result implies that if c is sufficiently large, then SCR_c has strings of every length. Comparing the two notions, Solovay proved that strings with essentially maximal prefix-free complexity have essentially maximal plain complexity (although up to a different constant). In other words:

Proposition 1.5 (Solovay [10])**.** $(\forall c)(\exists k)\ SCR_c \subseteq KR_k$.

Proof. Fix c. Define a prefix-free machine M as follows: if $U(\sigma) = n$, $U(\tau) = m$ and $|\nu| = m - n$, then $M(\sigma\tau\nu) = V(\nu)$. Assume $\rho \notin KR_k$, for some k. Let $n = |\rho|$. Define $k' = n - C(\rho)$; so $k' > k$. Let ν be a minimal V-program for $|\rho|$. Let σ and τ be minimal U-programs for n and k', respectively. Then $M(\sigma\tau\nu) = \rho$, so

$$K(\rho) \leq K_M(\rho) + O(1) \leq |\sigma\tau\nu| + O(1) \leq$$
$$C(\rho) + K(n) + K(k') + O(1) \leq n - k' + K(n) + 2\log(k') + O(1) \leq$$
$$n + K(n) - k' + k'/2 + O(1) \leq n + K(n) - k/2 + O(1),$$

where the constant does not depend on ρ, n or k. Therefore, if k is sufficiently large (not depending on ρ or n), $\rho \notin SCR_c$. For this k, we have $SCR_c \subseteq KR_k$. □

As was mentioned above, Solovay's proof of Theorem 1.2 also shows that having maximal plain complexity does not guarantee maximal prefix-free complexity, even up to a constant. This is reproved below as Corollary 3.5.

The following section reviews the fundamentals of Kolmogorov complexity. In Section 3 we introduce our basic proof method and show that if $Q \subseteq SCR_c$ is a Π^0_1 set, then it cannot contain strings of every length. This gives a short derivation of Corollary 3.5: Solovay's result from the previous paragraph. It also implies that SCR_c is not a Π^0_1 set when c is large, answering the question that originally motivated this work. Section 4 gives two proofs of a theorem of An. A. Muchnik

[1] V denotes a universal (plain) machine and U a universal prefix-free machine. This and other notation is fixed in Section 2.

[9]; he showed that for every d, there are strings σ and τ such that $C(\sigma)-C(\tau) \geq d$ and $K(\tau) - K(\sigma) \geq d$. In other words, the two complexity measures disagree, by as much as we like, about which of σ and τ is more complex. The first derivation uses Corollary 3.5. The other uses facts beyond the scope of this paper, but is included for comparison. The proof method introduced in Section 3 is further exploited in Sections 5 and 6. In the former, we show that a Π^0_1 set $Q \subseteq SCR_c$ must be hyperimmune. We also prove, using another result of Solovay, that there is an infinite Π^0_1 set $Q \subseteq SCR_c$. In Section 6, we consider a Π^0_1 set Q that contains strings of every length. We already know from Theorem 3.3 that $Q \nsubseteq SCR_c$, for any c. In Section 6 we bound the length of the shortest member of $Q \setminus SCR_c$. The bound was suggested by Theorem 1.2, and in fact, provides an alternate proof of that theorem. A derivation of Theorem 1.1 is given in Section 7. It takes several elements from Solovay's derivation but lets the symmetry of information and a result of Gács do some of the work.

2. Preliminaries

We briefly review the basics of Kolmogorov complexity and some of the other notions used in this paper. For an proper introduction to Kolmogorov complexity see Li and Vitanyi [8] or the upcoming monograph of Downey and Hirschfeldt [3].

Let $f(n)$ and $g(n)$ be real valued functions defined on ω. We write $f(n) \sim g(n)$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 1$. We say that $f(n)$ is $O(g(n))$ if there are $m, M \in \omega$ such that $|f(n)| \leq M|g(n)|$ for all $n > m$. In a formula, such as $h(n) = q(n) + O(g(n))$, we use $O(g(n))$ to stand in for an indeterminate $O(g(n))$ function. Our most frequent use will be of the form $h(n) \leq q(n)+O(1)$ or $h(n) \geq q(n)+O(1)$, where $O(1)$ can be taken to be a (positive or negative) constant.

We refer to partial computable functions $M\colon 2^{<\omega} \to 2^{<\omega}$ as *machines* and call their arguments *programs*. The *Kolmogorov complexity* of $\sigma \in 2^{<\omega}$ with respect to a machine M is $C_M(\sigma) = \min\{|\tau|\colon M(\tau) = \sigma\}$. A machine V is *universal* if for any other machine M, there is a $\tau \in 2^{<\omega}$ such that $(\forall\sigma)\ U(\tau\sigma) = M(\sigma)$. In this case, $(\forall\sigma)\ C_V(\sigma) \leq C_M(\sigma) + O(1)$, where the constant depends on M. Fix a universal machine V and call $C(\sigma) = C_V(\sigma)$ the *plain (Kolmogorov) complexity* of σ.

For $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \prec \tau$ to mean that σ is a proper prefix of τ. A set of strings $D \subseteq 2^{<\omega}$ is *prefix-free* if $(\forall\sigma, \tau \in D)\ \sigma \not\prec \tau$. A machine is *prefix-free* if its domain is a prefix-free set. If M is prefix-free, we will write K_M instead of C_M for the Kolmogorov complexity with respect to M. There is a prefix-free machine U *universal* with respect to prefix-free machines. As before, $(\forall\sigma \in 2^{<\omega})\ K_U(\sigma) \leq K_M(\sigma) + O(1)$ for any prefix-free machine M. We write $K(\sigma)$ for $K_U(\sigma)$ and call it the *prefix-free (Kolmogorov) complexity* of σ.

It is convenient to fix a bijection between $2^{<\omega}$ and ω. In particular, we identify n with σ whenever the binary expansion of $n+1$ is 1σ. This gives meaning to $K(n)$ when $n \in \omega$. It is easy to see that $K(\sigma) \leq 2|\sigma| + O(1)$, which for natural numbers translates to $K(n) \leq 2\log(n)+O(1)$. Iterating, we get $K^{(3)}(\sigma) \leq 2\log^{(2)}|\sigma|+O(1)$. In the introduction we observed that $K(\sigma) \leq |\sigma| + K(|\sigma|) + O(1)$, so in fact we have $K(\sigma) \leq |\sigma| + 2\log|\sigma| + O(1)$. We will freely use such inequalities.

If M is a prefix-free machine, then $\sum_{\sigma\in 2^{<\omega}} 2^{-K_M(\sigma)} \leq 1$; this is a form of *Kraft's inequality*. Its converse is the *Kraft–Chaitin theorem*. Say we are given a computably enumerable set $\{\langle d_n, \sigma_n\rangle\}_{n\in\omega}$, where $\sigma_n \in 2^{<\omega}$ and $d_n \in \omega$. If

$\sum_{n\in\omega} 2^{-d_n} \leq 1$, then we can (effectively) build a prefix-free machine M such that $K_M(\sigma_n) \leq d_n$ for all n. This is the main tool used to build prefix-free machines.

To define conditional prefix-free complexity we consider prefix-free machines with input: partial computable functions $\widehat{M}\colon 2^{<\omega} \times 2^{<\omega} \to 2^{<\omega}$ such that for every $\rho \in 2^{<\omega}$ the domain of $\widehat{M}(\cdot, \rho)$ is prefix-free. Let $\widehat{U}$ be universal among such machines and define *conditional prefix-free complexity* by $K(\sigma \mid \rho) = \min\{|\tau|\colon \widehat{U}(\tau, \rho) = \sigma\}$. Fix a pairing function, i.e., an effective bijection $\langle\cdot,\cdot\rangle\colon \omega\times\omega \to \omega$. Define $K(\sigma, \tau) = K(\langle\sigma, \tau\rangle)$. Let σ^* denote the first minimal length programs for σ to converge. The *symmetry of algorithmic information*, due to Levin (see Gács [5]) and Chaitin [1], states that $K(\sigma, \tau) = K(\sigma) + K(\tau \mid \sigma^*) + O(1)$.

We assume that the reader is familiar with the basics of computability theory, in particular, with partial computable functions and computably enumerable (c.e.) sets. Recall that a set is Π^0_1 if its complement is c.e. Finally, we will use the *recursion theorem*. Informally, it allows us to define a prefix-free machine M in terms of a prefix τ by which U simulates M, in other words, such that $(\forall\sigma)\ U(\tau\sigma) = M(\sigma)$.

3. The Basic Proof

This section introduces the proof method that will return in Sections 5 and 6. We begin with a key technical lemma.

Lemma 3.1. *Assume that $Q \subseteq 2^{<\omega}$ is a Π^0_1 set. For each $c \in \omega$, there is a $k \in \omega$ such that if*

(i) *$Q \cap 2^n \subseteq SCR_c$,*
(ii) *$|Q_s \cap 2^n| \leq b2^{n-k-c-1}$ for $b > 0$, and*
(iii) *$K_{s-1}(n) \neq K(n)$, then*

$$|Q \cap 2^n| \leq (b-1)2^{n-k-c-1}. \tag{1}$$

Proof. The assumption that $Q\cap 2^n \subseteq SCR_c$ allows us to force strings of length n out of Q by ensuring that they are not in SCR_c. This is the main idea of the proof. We will define a prefix-free machine M. By the recursion theorem, we may assume that we know in advance the prefix ρ by which U simulates M. Set $k = |\rho|$. We define M in stages as follows: if $U(\tau)\downarrow= n$ at stage s, then M finds the first $2^{n-k-c-1}$ strings of length n that are still in Q_s and (using the Kraft–Chaitin theorem) gives these strings descriptions of length $n + |\tau| - k - c - 1$. If there are not enough strings in $Q_s \cap 2^n$, then M makes due with the strings that are available. To see that M will not run out of room in its domain (i.e., that the application of the Kraft–Chaitin theorem is valid), observe that the total cost of all M-descriptions will be at most

$$\sum_{\tau\in\operatorname{dom} U} 2^{n-k-c-1}2^{-(n+|\tau|-k-c-1)} = \sum_{\tau\in\operatorname{dom} U} 2^{-|\tau|} \leq 1.$$

This completes the definition of M.

Now assume that $|Q_s \cap 2^n| \leq b2^{n-k-c-1}$ for $b > 0$ and $Q \cap 2^n \subseteq SCR_c$. Assume also that $K_{s-1}(n)$ is wrong. Let $t \geq s$ be the first stage at which $K_t(n) = K(n)$. So there is a string τ of length $K(n)$ for which $U(\tau)\downarrow= n$ at stage t. This means that M gives $2^{n-k-c-1}$ strings in $Q_t\cap 2^n$ (or as many as exist, if there are fewer) descriptions of length $n+K(n)-k-c-1$. These strings have resulting U-descriptions of length $n+K(n)-c-1$, so they are not in SCR_c and hence not in Q. If $|Q_t\cap 2^n| \leq 2^{n-k-c-1}$, then M guarantees that every string in $Q_t \cap 2^n$ is not in Q. In other words, $Q \cap 2^n$

is empty, so (1) is satisfied. On the other hand, if $|Q_t \cap 2^n| > 2^{n-k-c-1}$, then M guarantees that at least $2^{n-k-c-1}$ strings in $Q_t \cap 2^n \subseteq Q_s \cap 2^n$ are not in Q, from which (1) again follows. □

Remark 3.2. The k from the previous lemma is a function of c and (the index of) Q. Fixing Q, we can quantify the dependence on c as follows. We will define a prefix-free machine N. By the recursion theorem, we may define N in terms of its coding constant h. If $U(\sigma)\downarrow= c$, then let $k = h + |\sigma|$ and define $N(\sigma\tau) = M_{c,k}(\tau)$, where $M_{c,k}$ is the machine defined in the proof above for constants c and k. Note that U simulates $M_{c,k}$ with a prefix of length k, so its definition is consistent with the lemma. We are particularly interested in the case where σ is a minimal length program for c, so that $k = h + |\sigma| = h + K(c)$.

We can now restate the lemma to make its dependence on c explicit. Assume that $Q \cap 2^n \subseteq SCR_c$. If $|Q_s \cap 2^n| \leq b2^{n-h-k(c)-c-1}$ for $b > 0$, and $K_{s-1}(n) \neq K(n)$, then $|Q \cap 2^n| \leq (b-1)2^{n-h-K(c)-c-1}$. This uniform version of Lemma 3.1 will be useful in Section 6.

We turn to the main result of this section.

Theorem 3.3. *If $Q \subseteq 2^{<\omega}$ is a Π^0_1 set such that $(\forall n)\ Q \cap 2^n \neq \emptyset$, then $Q \nsubseteq SCR_c$.*

Proof. Assume, for a contradiction, that $Q \subseteq SCR_c$. Let k be the constant given by Lemma 3.1. Let b be the least natural number such that $(\exists^\infty n)\ |Q \cap 2^n| \leq b2^{n-k-c-1}$. Note that b cannot be zero, because $|Q \cap 2^n| > 0$ for every $n \in \omega$. Define a partial computable function ψ as follows. If $|Q_s \cap 2^n| \leq b2^{n-k-c-1}$ at some stage s, then let $\psi(n) = K_{s-1}(n)$. By Lemma 3.1, if $\psi(n)$ is defined and $\psi(n) \neq K(n)$, then $|Q \cap 2^n| \leq (b-1)2^{n-k-c-1}$. Therefore, this can happen only finitely often. Furthermore, the choice of b guarantees that ψ has an infinite domain.

It is now straightforward to derive a contradiction. Take v such that if $n \geq v$ and $\psi(n)\downarrow$, then $\psi(n) = K(n)$. We define a prefix-free machine N, assuming that N knows its coding constant h. Wait for an $n \geq v$ such that $\psi(n)\downarrow\geq h+1$. When such an n is found, let $N(\lambda) = n$. But then U gives n a description of length $h \leq \psi(n) - 1 = K(n) - 1$, which is impossible. □

It appears that even the simplest case of Theorem 3.3 was unknown.

Corollary 3.4. *If $c \in \omega$ is sufficiently large, then SCR_c is not Π^0_1.*

Proof. Take $c \in \omega$ to be large enough to ensure that $(\forall n)\ SCR_c \cap 2^n \neq \emptyset$. If SCR_c were Π^0_1, then the previous theorem would imply that $SCR_c \nsubseteq SCR_c$. □

The theorem also implies a version of Solovay's separation of the strong Chaitin random and Kolmogorov random strings. As we observed above, KR is Π^0_1 and $(\forall n)\ KR \cap 2^n \neq \emptyset$. Applying the theorem:

Corollary 3.5. $(\forall c)$ *$KR \nsubseteq SCR_c$.*

4. Muchnik's Theorem

In this section, we give a short derivation of a theorem of An. A. Muchnik, who proved that plain and prefix-free Kolmogorov complexity need not agree on the relative complexity of strings, even up to a finite error. For comparison, let us recall how Muchnik derived this result. The *overgraph* of C is the c.e. set $\{(\sigma, n)\colon C(\sigma) < n\}$. The overgraph of K can be defined similarly. Kummer [7] proved that the overgraph of C is tt-complete among the c.e. sets.[2] On the other hand, An. A. Muchnik [9] proved that the overgraph of K may or may not be tt-complete, depending on the choice of the universal prefix-free machine U. Muchnik exploited this difference to prove Theorem 4.1. We instead derive his result from Corollary 3.5. It is interesting to note that both proofs come out of work contrasting the computability theoretic properties of plain and prefix-free complexity.

Theorem 4.1 (An. A. Muchnik [9]). *For every $d \in \omega$, there are $\sigma, \tau \in 2^{<\omega}$ such that $C(\sigma) - C(\tau) \geq d$ and $K(\tau) - K(\sigma) \geq d$.*

Proof. Fix a constant $k \in \omega$ such that $(\forall n)\ SCR_k \cap 2^n \neq \emptyset$. For $c \in \omega$, take $\sigma_c \in KR \setminus SCR_c$. Choose $\tau_c \in SCR_k$ such that $|\tau_c| = |\sigma_c| - \lfloor c/2 \rfloor$. Then $C(\sigma_c) - C(\tau_c) \geq |\sigma_c| - |\tau_c| - O(1) = c/2 + O(1)$. Thus $\lim_{c\to\infty} C(\sigma_c) - C(\tau_c) = \infty$.

On the other hand,

$$\begin{aligned} K(\tau_c) - K(\sigma_c) &> (|\tau_c| + K(|\tau_c|) - k) - (|\sigma_c| + K(|\sigma_c|) - c) \\ &= c/2 + K(|\tau_c|) - K(|\sigma_c|) - O(1). \end{aligned}$$

But $|\sigma_c| = |\tau_c| + \lfloor c/2 \rfloor$ implies that $K(|\sigma_c|) \leq K(|\tau_c|) + K(c) + O(1) \leq K(|\tau_c|) + O(\log c)$. Thus,

$$K(\tau_c) - K(\sigma_c) > c/2 + K(|\tau_c|) - K(|\sigma_c|) - O(1) \geq c/2 - O(\log c).$$

and $\lim_{c\to\infty} K(\tau_c) - K(\sigma_c) = \infty$.

Therefore, if $c \in \omega$ is sufficiently large, σ_c and τ_c satisfy our requirements. □

We can easily add the restriction that $|\sigma| = |\tau|$.

Proposition 4.2. *For every $d \in \omega$, there are $\sigma, \tau \in 2^{<\omega}$ such that $C(\sigma) - C(\tau) \geq d$ and $K(\tau) - K(\sigma) \geq d$ and $|\sigma| = |\tau|$.*

Proof. For $c \in \omega$, consider σ_c and τ_c as constructed in the proof above. Let $\widetilde{\tau}_c = \tau_c 0^{\lfloor c/2 \rfloor}$. Thus $|\widetilde{\tau}_c| = |\sigma_c|$. Note that $|C(\tau_c) - C(\widetilde{\tau}_c)| \leq C(c) + O(1) \leq O(\log c)$. Therefore,

$$\lim_{c\to\infty} C(\sigma_c) - C(\widetilde{\tau}_c) \geq \lim_{c\to\infty} c/2 - O(\log c) = \infty.$$

By the same reasoning, $\lim_{c\to\infty} K(\widetilde{\tau}_c) - K(\sigma_c) = \infty$. Therefore, if $c \in \omega$ is sufficiently large, σ_c and $\widetilde{\tau}_c$ satisfy our requirements. □

A similar proof of Muchnik's result can be given using strings with very low complexity. We say that $X \in 2^\omega$ is K-*trivial* if $(\forall n)\ K(X \upharpoonright n) \leq K(n) + O(1)$. Solovay [10] proved that there is a noncomputable K-trivial; see Downey, Hirschfeldt, Nies and Stephan [4] for a simple construction. On the other hand, Chaitin [2] proved that if $(\forall n)\ C(X \upharpoonright n) \leq C(n) + O(1)$, then X is computable. From this disparity we can derive Muchnik's theorem.

Second proof of Theorem 4.1. Let $X \in 2^\omega$ be a noncomputable K-trivial. Fix c. By Chaitin result, there is an n such that $C(X \upharpoonright n) \geq C(n) + c$. Let $\sigma_c = X \upharpoonright n$. Take $\rho \in KR$ with $|\rho| = 2^{c/2}$. Let $\tau_c = \rho 0^n$. Then

$$\begin{aligned} C(\sigma_c) - C(\tau_c) &\geq (C(n) + c) - (K(\rho) + C(n) + O(1)) \\ &\geq c - |\rho| - 2\log|\rho| - O(1) \geq c/2 - 2\log c - O(1), \end{aligned}$$

[2]Actually, Kummer proved that KR is tt-complete, from which the stated result follows.

where the constants do not depend on c. Thus $\lim_{c\to\infty} C(\sigma_c) - C(\tau_c) = \infty$. On the other hand,

$$\begin{aligned} K(\tau_c) - K(\sigma_c) &\geq (K(\rho) + K(n) - O(1)) - K(X \upharpoonright n) \\ &\geq C(\rho) + K(n) - K(n) - O(1) = c/2 - O(1). \end{aligned}$$

Again the constants are independent of c and so $\lim_{c\to\infty} K(\tau_c) - K(\sigma_c) = \infty$. Therefore, if $c \in \omega$ is sufficiently large, σ_c and τ_c satisfy our requirements. □

Again, we could add the restriction that $|\sigma| = |\tau|$.

5. Infinite Π^0_1 Subsets of SCR_c are Hyperimmune

Our next goal is to modify the proof of Theorem 3.3 to show that every infinite Π^0_1 subset of SCR_c is hyperimmune. Before we do that, it is worth proving that infinite Π^0_1 subsets of SCR_c do exist.

Lemma 5.1 (Solovay [10])**.** *There is a computable function $g\colon \omega \to \omega$ and a $t \in \omega$ such that:*

(i) $(\forall m)\ g(m) \geq K(m)$.
(ii) $(\exists^\infty m)\ g(m) \leq K(m) + t$.

Proof. Let

$$h(m) = \begin{cases} K_{s+1}(n), & \text{if } m = \langle n, s\rangle \text{ and } K_{s+1}(n) \neq K_s(n) \\ m, & \text{otherwise.} \end{cases}$$

It is clear that $h\colon \omega \to \omega$ is computable. Note that $\sum_{m\in\omega} 2^{-h(m)} \leq \sum_{m\in\omega} 2^{-m} + \sum_{n\in\omega}\sum_{d\geq K(n)} 2^{-d} = 2 + 2\sum_{n\in\omega} 2^{-K(n)} < \infty$. Therefore, it follows from the Kraft–Chaitin theorem that $(\forall m)\ K(m) \leq h(m) + c$, for some c. Hence $g(m) = h(m) + c$ satisfies condition (i). We claim that g also satisfies condition (ii). For $n \in \omega$, let s be the least stage such that $K_{s+1}(n) = K(n)$. Then $g(\langle n, s\rangle) = h(\langle n, s\rangle) + c = K_{s+1}(n) + c = K(n) + c \leq K(\langle n, s\rangle) + O(1)$, where the constant does not depend on n. Because such an $m = \langle n, s\rangle$ exists for all n, condition (ii) is proved. □

Proposition 5.2. *For a sufficiently large $c \in \omega$, there is an infinite Π^0_1 subset of SCR_c.*

Proof. Choose k large enough that $|SCR_k \cap 2^n| > 0$ for all n. Take $g\colon \omega \to \omega$ and t from the lemma above. Let $c \geq t + k$ and define

$$P = \{\sigma \in 2^{<\omega} \colon K(\sigma) \geq |\sigma| + g(|\sigma|) - (t + k)\}.$$

It is clear that $P \subseteq 2^{<\omega}$ is a Π^0_1 set. Note that $\sigma \in P \implies K(\sigma) \geq |\sigma| + g(|\sigma|) - (t + k) \geq |\sigma| + K(|\sigma|) - c \implies \sigma \in SCR_c$, so $P \subseteq SCR_c$.

We must prove that P is infinite. There are infinitely many n such that $g(n) \leq K(n) + t$. Choose such an n and take $\sigma \in SCR_k \cap 2^n$, which is nonempty by the choice of k. But then $K(\sigma) \geq |\sigma| + K(|\sigma|) - k \geq |\sigma| + g(|\sigma|) - (t+k)$, so $\sigma \in P$. □

Now that we have verified that SCR_c contains infinite Π^0_1 sets, we are ready to prove that such sets must be fairly sparse. Recall that a set $A \subset \omega$ is *not hyperimmune* iff there is a computable function $f\colon \omega \to \omega$ such that $(\forall n)\ |A \cap [0, f(n)]| \geq n$. Having identified binary strings with natural numbers, we can transfer this definition to subsets of $2^{<\omega}$. A standard technique provides a useful equivalent formulation.

Lemma 5.3. *$Q \subseteq 2^{<\omega}$ is not hyperimmune iff there is a computable function $g\colon \omega \to \omega$ such that, for all $m \in \omega$, there is an $n \in [g(m), g(m+1))$ such that $Q \cap 2^n \neq \emptyset$.*

Proof. Take a computable $f\colon \omega \to \omega$ such that $(\forall n)\ |Q \cap [0, f(n)]| \geq n$. We define g recursively. Let $g(0) = 0$. Assume that we have defined $g(m)$. If $p = f(2^{g(m)})$, then $Q \cap [2^{g(m)} - 1, p]$ is nonempty. Let $g(m+1) = \lfloor \log_2(p+1) \rfloor + 1$. Since $p < 2^{g(m+1)} - 1$, we have $Q \cap [2^{g(m)} - 1, 2^{g(m+1)} - 1) \neq \emptyset$. In other words, there is an $n \in [g(m), g(m+1))$ such that $Q \cap 2^n \neq \emptyset$. This is because strings of length n are identified with natural numbers in the interval $[2^n - 1, 2^{n+1} - 1)$. Thus g satisfies the lemma. □

Theorem 5.4. *If $Q \subseteq SCR_c$ is a infinite Π^0_1 set, then Q is hyperimmune.*

Proof. Let k be the constant given by Lemma 3.1. Assume, for a contradiction, that Q is not hyperimmune. By the lemma, there is a computable function $g\colon \omega \to \omega$ such that, for all m, there is an n in the interval $G_m = [g(m), g(m+1))$ for which $Q \cap 2^n$ is nonempty.

Let $b \in \omega$ be the least natural number such that, for infinitely many m, b is the least number such that, for all $n \in G_m$,

$$|Q \cap 2^n| \leq b 2^{n-k-c-1}.$$

If $b = 0$, then for infinitely many m, there is no element in Q with length in G_m. This contradicts the definition of g, hence $b > 0$.

Define a partial computable function ψ as follows: if for some m and s,

$$|Q_s \cap 2^n| \leq b 2^{n-k-c-1},$$

for every $n \in G_m$, then let $\psi(n) = K_{s-1}(n)$ for all such n. The choice of b guarantees that ψ is defined on infinitely many intervals G_m. By Lemma 3.1, if $\psi(n)$ is defined and $\psi(n) \neq K(n)$, then $|Q \cap 2^n| \leq (b-1)2^{n-k-c-1}$. Therefore, the choice of b ensures that there are only finitely many m such that ψ is defined on G_m but never equals K. Take $v \in \omega$ such that if $m > v$ and ψ is defined on G_m, then there is an $n \in G_m$ for which $\psi(n) = K(n)$.

To derive a contradiction, we define a prefix-free machine N, assuming that N knows its coding constant h. Wait for an $m > v$ such that ψ converges on G_m and $\sum_{n \in G_m} 2^{-\psi(n)+h+1} \leq 1$. To see that such an m exists, note that $\psi(n) \geq K(n)$ when it is defined. So,

$$\sum_{G_m \subset \operatorname{dom} \psi} \sum_{n \in G_m} 2^{-\psi(n)+h+1} \leq \sum_{n \in \omega} 2^{-K(n)+h+1} \leq 2^{h+1}.$$

Because the outer series converges, its terms must tend to zero. So an appropriate m is eventually found. When it is, N gives each $n \in G_m$ a description of length $\psi(n) - h - 1$, which is possible by the Kraft–Chaitin theorem. But this means that U gives some $n \in G_m$ a description of length $\psi(n) - 1 = K(n) - 1$, which is a contradiction.

Therefore, Q is hyperimmune. □

6. Bounding the Shortest Element of $KR \setminus SCR_c$

Consider a Π^0_1 set Q that has at least one string of each length. For each $c \in \omega$, Theorem 3.3 says that there is a $w_c \in Q \setminus SCR_c$. Our goal is to refine the argument given there to bound the length of w_c. The nonuniformity of our basic proof makes this calculation somewhat interesting.

Theorem 6.1. *Let $Q \subseteq 2^{<\omega}$ be a Π^0_1 set such that $(\forall n)\ Q \cap 2^n \neq \emptyset$. There is a family of strings $\{w_c\}_{c\in\omega} \subseteq Q$ such that:*

(i) $w_c \notin SCR_c$.
(ii) $|w_c| \leq 2^{2^{c+O(\log c)}}$.

Proof. Let h be the constant from Remark 3.2. For $c, n \in \omega$, let $b_c(n)$ be the least number such that $|Q \cap 2^n| \leq b_c(n)2^{n-h-K(c)-c-1}$. Clearly, $0 < b_c(n) \leq 2^{h+K(c)+c+1}$. By the remark, if

- $Q \cap 2^n \subseteq SCR_c$, and
- $|Q_s \cap 2^n| \leq b_c(n)2^{n-h-K(c)-c-1}$,

then $K_s(n) = K(n)$. Let $b_{c,s}(n)$ be the stage s approximation of $b_c(n)$.

The idea of the proof is to build a prefix-free machine $N : 2^{<\omega} \to 2^{<\omega}$ that, when given c, looks for an $n \in \omega$ and a stage s such that $b_{c,s}(n) = b_c(n)$ and $K_s(n)$ is large enough that N can force $K(n) < K_s(n)$. This would imply that $Q \cap 2^n \nsubseteq SCR_c$. The obvious problem is that we cannot compute $b_c(n)$. This was circumvented in Section 3 by restricting attention to the least value of $b_c(n)$ that occurs infinitely often and the $v \in \omega$ beyond which no lower value of $b_c(n)$ occurs. Unfortunately, this is not conducive to finding a small n. Our new method will be to feed N the value b that we are interested in and the number of m's—below a specific upper bound—for which $b_c(m) < b$. We will show that if b is chosen correctly, then N can find an n and s for which $b_{c,s}(n) = b_c(n) = b$ and $K_s(n)$ is large enough that N can further compress n.

We turn to the details. By the recursion theorem, we may assume that N knows its coding constant k. Consider strings $\sigma, \tau, \nu \in 2^{<\omega}$ for which there exist natural numbers c, b and d such that

- $U(\sigma) = c$;
- $|\tau| = h + |\sigma| + c + 1$;
- τ is the binary expansion of $b-1$, padded with zeros on the left, if necessary;
- $|\nu| = (b-1)(k + h + 2|\sigma| + c + 2)$; and
- ν is the padded binary expansion of d.

Note that the set of all strings $\sigma\tau\nu$ for which $\sigma, \tau, \nu \in 2^{<\omega}$ satisfy these criteria is a prefix-free set. The intention is that d will code the number of $m's$ below the bound $D = 2^{2^{h+|\sigma|+c+1}(k+h+2|\sigma|+c+2)}$ for which $b_c(m) < b$. Given $\sigma\tau\nu$ as input, N looks for a stage t such that there are d numbers $m < D$ such that $b_{c,t}(m) < b$. Then N searches for a stage $s \geq t$ and a number $n < D$ such that $b_{c,s}(n) = b$ and $K_s(n) > k + |\sigma\tau\nu| = b(k + h + 2|\sigma| + c + 2) - 1$. Finally, let $N(\sigma\tau\nu) = n$, ensuring that $K(n) < K_s(n)$. As observed above, if $b_c(n) = b_{c,s}(n) = b$, then $Q \cap 2^n \nsubseteq SCR_c$.

Fix $c \in \omega$ and let σ be a minimal program for c. So, for all programs we consider below, the upper bound in the description of N is

$$D = 2^{2^{h+K(c)+c+1}(k+h+2K(c)+c+2)}.$$

Note that $D = 2^{2^{c+O(\log c)}}$ because h and k are independent from c. Assume, for a contradiction, that $Q \cap 2^n \subseteq SCR_c$ for all $n < D$. Let $S_b = \{n < D \colon b_c(n) = b\}$, for $0 \leq b \leq 2^{h+K(c)+c+1}$. Note that $S_0 = \emptyset$ and $\sum_{i \leq 2^{h+K(c)+c+1}} |S_i| = D$. We prove by induction on $b \leq 2^{h+K(c)+c+1}$ that:

(i) If $n \in S_b$, then $K(n) \leq b(k + h + 2K(c) + c + 2) - 1$.
(ii) $\sum_{i \leq b} |S_i| \leq 2^{b(k+h+2K(c)+c+2)} - 1$.

Both (i) and (ii) are trivially true for $b = 0$. Now take $b > 0$ and assume that both conditions hold for all $i < b$. Note that there are at less than $2^{b(k+h+2K(c)+c+2)}$ strings with descriptions of length at most $b(k+h+2K(c)+c+2)-1$. Hence if (ii) fails for b, then there is some $n \in \bigcup_{i \leq b} S_i$ such that $K(n) > b(k+h+2K(c)+c+2)-1$. But this n cannot be in $\bigcup_{i<b} S_i$ by the inductive assumption, so it must be in S_b. Therefore, (i) would fail for b. So, it is enough to prove that (i) holds for b. Let τ be the padded binary expansion of $b-1$ of length $h + K(c) + c + 1$ (recall that $|\sigma| = K(c)$). Let $d = \sum_{i<b} |S_i| \leq 2^{(b-1)(k+h+2K(c)+c+2)} - 1$ and take ν to be the padded binary expansion of d of length $(b-1)(k + h + 2K(c) + c + 2)$. On input $\sigma\tau\nu$, N first finds a t such that $b_{c,t}(m) < b$ for all $m \in \bigcup_{i<b} S_i$. Then N searches for a stage $s \geq t$ and a number $n < D$ such that $b_{c,s}(n) = b$ and $K_s(n) > b(k+h+2K(c)+c+2)-1$. If (i) fails, then such an n is found, in which case $b_c(n) = b$ and so $Q \cap 2^n \not\subseteq SCR_c$. Since this is not the case, (i) holds for b.

Taking $b = 2^{h+K(c)+c+1}$, (ii) states that $\sum_{i \leq b} |S_i| \leq D - 1$, which is a contradiction. Therefore, there is an $n < D$ such that $Q \cap 2^n \not\subseteq SCR_c$. Take $w_c \in (Q \cap 2^n) \setminus SCR_c$ □

Theorem 6.1 is a thinly disguised version of Theorem 1.2. Solovay expressed his result in a somewhat different form and specifically for $Q = KR_d$, for sufficiently large $d \in \omega$. To derive his version we use Theorem 6.1 and Lemma 7.2, which is proved below.

Theorem 1.2 (Solovay [10]). *There are sequences $\{w_c\}_{c\in\omega}$ and $\{z_c\}_{c\in\omega}$ of strings such that $\lim_{c\to\infty} K(w_c) = \infty$ and $C(w_c) = C(z_c)+O(1)$ but $\lim_{c\to\infty} \dfrac{K(z_c) - K(w_c)}{\log^{(2)} |w_c|} = 1$.*

Proof. Let $\{w_c\}_{c\in\omega}$ be the sequence constructed in Theorem 6.1 for $Q = KR$. It is obvious that $\lim_{c\to\infty} K(w_c) = \infty$ because a string can only appear in the sequence finitely often. From $w_c \notin SCR_c$, we have $|w_c| + K(|w_c|) - K(w_c) > c$. But $|w_c| \leq 2^{2^{c+O(\log c)}}$, so $\log^{(2)} |w_c| \leq c + O(\log c)$. This proves that

$$\liminf_{c\to\infty} \frac{|w_c| + K(|w_c|) - K(w_c)}{\log^{(2)} |w_c|} \geq 1.$$

For the reverse inequality, note that $C(w_c) = |w_c| + O(1)$ because $w_c \in KR$. Together with Lemma 7.2 this implies

$$|w_c| \leq K(w_c) + K(|w_c| \mid w_c^*) - K(|w_c|) + K^{(2)}(|w_c|) + O(1).$$

But $K(|w_c| \mid w_c^*) = O(1)$, so $|w_c| + K(|w_c|) - K(w_c) \leq K^{(2)}(|w_c|) + O(1) \leq \log^{(2)} |w_c| + O(\log^{(3)} |w_c|)$. Therefore, $\limsup_{c\to\infty} \frac{|w_c|+K(|w_c|)-K(w_c)}{\log^{(2)} |w_c|} \leq 1$. So,

$$\lim_{c\to\infty} \frac{|w_c| + K(|w_c|) - K(w_c)}{\log^{(2)} |w_c|} = 1.$$

Now take k large enough that SCR_k contains strings of every length. Take a sequence $\{z_c\}_{c\in\omega} \subseteq SCR_k$ such that $|z_c| = |w_c|$ for all c. Then $K(z_c) = |z_c| + K(|z_c|) + O(1) = |w_c| + K(|w_c|) + O(1)$, which implies that

$$\lim_{c\to\infty} \frac{K(z_c) - K(w_c)}{\log^{(2)} |w_c|} = 1,$$

as required. Finally, by Proposition 1.5, $C(z_c) = |z_c| + O(1) = |w_c| + O(1) = C(w_c) + O(1)$. □

7. A Derivation of Solovay's Formulae

To end the paper, we present a new proof of Theorem 1.1, Solovay's formulae relating C and K. Our derivation relies on a lemma of Gács that allows us to express plain complexity as a conditional prefix-free complexity, thus opening the way for us to apply the symmetry of information.

We begin with a proof of Gács' result. The proof uses the fact that if $c \leq K(c) + O(1)$, then c is bounded above. This is immediate because $K(c) \leq 2\log c + O(1)$. More generally, note that if $f(n) \leq K(|f(n)|) + O(g(n))$, then $f(n)/2 \leq f(n) - 2\log(|f(n)|) + O(1) \leq f(n) - K(|f(n)|) + O(1) \leq O(g(n))$, so $f(n) \leq O(g(n))$. This observation, which also appears in Solovay [10], is used in the proof of Lemma 7.3.

Lemma 7.1 (Gács [6])**.** *$C(\sigma) = K(\sigma \mid C(\sigma)) + O(1)$.*

Proof. Recall that the universal (plain) machine is denoted by $V : 2^{<\omega} \to 2^{<\omega}$. We define a two variable machine $M : 2^{<\omega} \times 2^{<\omega} \to 2^{<\omega}$ that is prefix-free in the first coordinate as follows: if $n = |\sigma|$, let $M(\sigma, n) = V(\sigma)$. Otherwise, $M(\sigma, n)$ should diverge. If σ^+ is a minimal length V-program for σ, then $M(\sigma^+, C(\sigma)) = \sigma$. This proves that $K(\sigma \mid C(\sigma)) \leq C(\sigma) + O(1)$.

If $C(\sigma) < K(\sigma \mid C(\sigma))$, then we are done. Assume not. Recall that $\widehat{U} : 2^{<\omega} \times 2^{<\omega} \to 2^{<\omega}$ is universal among machines prefix-free in the first coordinate. Let ρ be a minimal length program such that $\widehat{U}(\rho, C(\sigma)) = \sigma$. Let $c = C(\sigma) - K(\sigma \mid C(\sigma)) = C(\sigma) - |\rho|$. Given c and ρ we can compute $C(\sigma)$ and then use $\widehat{U}$ to compute σ. Therefore, $C(\sigma) \leq K(c) + |\rho| + O(1) = K(c) + K(\sigma \mid C(\sigma)) + O(1)$. But this means that $c \leq K(c) + O(1)$, so c is bounded above. Therefore, $C(\sigma) \leq K(\sigma \mid C(\sigma)) + O(1)$. □

Lemma 7.2. *$C(\sigma) \leq K(\sigma) + K(C(\sigma) \mid \sigma^*) - KC(\sigma) + K^{(2)}C(\sigma) + O(1)$.*

Proof. Note that $K(\sigma \mid C(\sigma)) \leq K(C(\sigma)^* \mid C(\sigma)) + K(\sigma \mid C(\sigma)^*) + O(1)$. This is because if we know how to compute $C(\sigma)^*$ from $C(\sigma)$ and we know how to compute σ from $C(\sigma)^*$, then we can compute σ from $C(\sigma)$. It is elementary that $K(\tau^* \mid \tau) = K(K(\tau) \mid \tau) + O(1)$, from which it follows that $K(C(\sigma)^* \mid C(\sigma)) = K(KC(\sigma) \mid C(\sigma)) + O(1) \leq K^{(2)}C(\sigma) + O(1)$. Putting these inequalities together, and invoking Lemma 7.1,

$$C(\sigma) \leq K(\sigma \mid C(\sigma)^*) + K^{(2)}C(\sigma) + O(1). \tag{2}$$

The symmetry of algorithmic information implies that $K(\sigma \mid C(\sigma)^*) = K(\sigma, C(\sigma)) - KC(\sigma) + O(1) = K(\sigma) + K(C(\sigma) \mid \sigma^*) - KC(\sigma) + O(1)$. Substituting this into equation (2) completes the derivation. □

Lemma 7.3 (Solovay [10])**.** *$C(\sigma) = K(\sigma) - KC(\sigma) + O(K^{(2)}C(\sigma))$.*

Proof. First note that $K(\sigma) \leq C(\sigma) + KC(\sigma) + O(1)$, which implies that $C(\sigma) \geq K(\sigma) - KC(\sigma) + O(K^{(2)}C(\sigma))$.

For the other direction, note that $K(C(\sigma) \,|\, \sigma^*) \leq K(C(\sigma) \,|\, K(\sigma)) + O(1) \leq K(|C(\sigma) - K(\sigma)|) + O(1) \leq K(|C(\sigma) + KC(\sigma) - K(\sigma)|) + O(K^{(2)}C(\sigma))$. Putting this together with Lemma 7.2 and rearranging, we get

$$C(\sigma) + KC(\sigma) - K(\sigma) \leq K(|C(\sigma) + KC(\sigma) - K(\sigma)|) + O(K^{(2)}C(\sigma)).$$

As noted at the beginning of this section, this completes the proof. □

The remainder of the derivation is very similar to what appears in Solovay's unpublished manuscript. We need the fact that $K(n + m) = K(n) + O(K(m))$. This is true because $K(n + m) \leq K(n) + K(m) + O(1)$ and $K(n) \leq K(n + m) + K(m) + O(1)$. Also note that if $h(n) = O(q(n))$, then $K(h(n)) = O(\log q(n))$. Hence if $g(n) = f(n) + O(K(f(n)))$, then $K(g(n)) = K(f(n)) + O(\log K(f(n)))$. Therefore, $K(g(n)) \sim K(f(n))$.

Theorem 1.1 (Solovay [10])**.** *For all $\sigma \in 2^{<\omega}$:*

(i) $C(\sigma) = K(\sigma) - K^{(2)}(\sigma) + O(K^{(3)}(\sigma))$.
(ii) $K(\sigma) = C(\sigma) + C^{(2)}(\sigma) + O(C^{(3)}(\sigma))$.

Proof. (i) Apply K to both sides of the previous lemma to get

$$KC(\sigma) = K^{(2)}(\sigma) + O(K^{(2)}C(\sigma)). \tag{3}$$

Now apply K to both sides of $K^{(2)}(\sigma) = KC(\sigma) + O(K^{(2)}C(\sigma))$ to get

$$K^{(3)}(\sigma) \sim K^{(2)}C(\sigma). \tag{4}$$

Finally, substitute equations (3) and (4) into Lemma 7.3.

(ii) Apply part (i) to $C(\sigma)$ to get

$$C^{(2)}(\sigma) = KC(\sigma) + O(K^{(2)}C(\sigma)). \tag{5}$$

Using equations (3) and (4) we get $C^{(2)}(\sigma) = K^{(2)}(\sigma) + O(K^{(3)}(\sigma))$. Applying this formula to $C(\sigma)$ produces $C^{(3)}(\sigma) = K^{(2)}C(\sigma) + O(K^{(3)}C(\sigma))$, so

$$C^{(3)}(\sigma) \sim K^{(2)}C(\sigma). \tag{6}$$

Substituting equations (5) and (6) into Lemma 7.3 completes the proof. □

References

[1] Gregory J. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.
[2] Gregory J. Chaitin. Information-theoretic characterizations of recursive infinite strings. *Theoret. Comput. Sci.*, 2(1):45–48, 1976.
[3] R. Downey and D. Hirschfeldt. *Algorithmic randomness and complexity.* Springer-Verlag, Berlin. To appear.
[4] Rod G. Downey, Denis R. Hirschfeldt, André Nies, and Frank Stephan. Trivial reals. In *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 103–131, Singapore, 2003. Singapore Univ. Press.
[5] Péter Gács. The symmetry of algorithmic information. *Dokl. Akad. Nauk SSSR*, 218:1265–1267, 1974.
[6] Péter Gács. Exact expressions for some randomness tests. *Z. Math. Logik Grundlag. Math.*, 26(5):385–394, 1980.
[7] Martin Kummer. On the complexity of random strings (extended abstract). In *STACS 96 (Grenoble, 1996)*, volume 1046 of *Lecture Notes in Comput. Sci.*, pages 25–36. Springer, Berlin, 1996.

[8] M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Texts and Monographs in Computer Science. Springer-Verlag, New York, 1993.

[9] Andrej A. Muchnik and Semen Ye. Positselsky. Kolmogorov entropy in the context of computability theory. *Theoret. Comput. Sci.*, 271(1-2):15–35, 2002.

[10] Robert M. Solovay. Draft of paper (or series of papers) on Chaitin's work. Unpublished notes, 215 pages, May 1975.

Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009
E-mail address: `joseph.miller@math.uconn.edu`