MODELS PRESENTABLE FROM ALMOST EVERY REAL

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In (partial) answer to a question raised by Shinoda and Slaman [2]:

Theorem. *Let $\mathcal{M}$ be a countable model. If almost every real computes a presentation of $\mathcal{M}$, then $\mathcal{M}$ has a presentation which does not compute a DNR function (hence the presentation does not compute a 1-random).*

Proof. We may assume that $\mathcal{M}$ is a graph [1]. Let $\mathcal{M} = \langle V, E\rangle$ where $E \subseteq V^2$ and $V = \{v_0, v_1, v_2, \dots\}$.

It will be useful to specify a precise form for the presentations of $\mathcal{M}$. Call a presentation *total* if it has domain ω (instead of a subset of ω, as is generally allowed). We may restrict our attention to total presentations, because every presentation of $\mathcal{M}$ computes a total presentation. For any function $f\colon \omega \to V$, define $f^*\colon \omega \to \{0,1\}$ by

$$f^*(\langle a, b\rangle) = \begin{cases} 1 & \text{if } \langle f(a), f(b)\rangle \in E, \\ 0 & \text{otherwise.} \end{cases}$$

For a partial function $\psi\colon \omega \to V$, define the partial function ψ^* similarly, with the caveat that $\psi^*(\langle a, b\rangle)$ is only defined if $\psi(a)$ and $\psi(b)$ are both defined. If f is a bijection, then f^* is (a particular effective encoding of) a total presentation of $\mathcal{M}$. Furthermore, every total presentation of $\mathcal{M}$ is induced in this way.

We build a presentation of $\mathcal{M}$ in stages. For each $s \in \omega$, we determine a number $t_s \in \omega$ and an injection $f_s\colon \{0, \dots, t_s - 1\} \to V$. We require that $f_s \subseteq f_{s+1}$ and that t_s goes to infinity. Therefore, the function $f\colon \omega \to V$ defined by $f = \bigcup_{s\in\omega} f_s$ is well defined, injective and total. We also ensure that $v_s \in \operatorname{range}(f_{s+1})$, hence f is a bijection. Finally, for each $s \in \omega$, the construction will ensure that

$$Q_s\colon\ \varphi_s^{f^*} \text{ is not a DNR function.}$$

Therefore, f^* is the desired presentation of $\mathcal{M}$.

The Construction.

Stage 0: Let $t_s = 0$ and let f_s be the empty function.

Stage s+1: The main idea of the proof is to exploit the fact, which we now prove, that there is an effective procedure which enumerates the set of pairs $\langle t, \psi^*\rangle$ such that $t \in \omega$ and ψ^* corresponds to an injective partial function $\psi\colon \{0, \dots, t-1\} \to V$ extending f_s. Call the set of such pairs D.

Let $R = \operatorname{range}(f_s)$. For every $e \in \omega$ and every partial surjection $\tau\colon \omega \to R$, let

$$\mathcal{C}_{e,\tau} = \{A \in 2^\omega \mid \varphi_e^A = g^* \text{ for some bijection } g\colon \omega \to V \text{ such that } g \supset \tau\}.$$

Each $\mathcal{C}_{e,\tau}$ is a Σ^1_1 class, hence measurable by Lusin's theorem. Note that every real which computes a presentation of $\mathcal{M}$ must fall into at least one—indeed, infinitely many—of these classes. So we have a countable collection of measurable classes, the union of which has measure one. Therefore, $\mu(\mathcal{C}_{e,\tau}) > 0$ for some $e \in \omega$ and partial surjection $\tau\colon \omega \to R$. By the Lebesgue density theorem, there is a $\sigma \in 2^{<\omega}$ such that

$$\mu(\mathcal{C}_{e,\tau} \cap [\sigma]) > 2^{-|\sigma|-1}.$$

In other words, more than half of the oracles extending σ compute, via index e, presentations of $\mathcal{M}$ which are consistent with τ. This gives us a handle on $\mathcal{M}$; basically, we know that an extension of f_s^* occurs in $\mathcal{M}$ iff it occurs in φ_e^A for more than half of the oracles $A \in [\sigma]$. We now make this precise.

Assume that we are given $e \in \omega$, $\tau^{-1} \circ f_s\colon \{0, \dots, t_s - 1\} \to \omega$ and $\sigma \in 2^{<\omega}$ (which is only a finite amount of information). We want to give an effective enumeration of D. First, fix an oracle $A \in \mathcal{C}_{e,\tau}$ and a witness for its inclusion $g\colon \omega \to V$. In other words, g is a bijection extending τ and $\varphi_e^A = g^*$. Every pair $\langle t, \psi^* \rangle \in D$ corresponds to an injective function $\widetilde{\psi} = g^{-1} \circ \psi\colon \{0, \dots, t-1\} \to \omega$ which extends $\tau^{-1} \circ f_s$. Similarly, if $\widetilde{\psi}\colon \{0, \dots, t-1\} \to \omega$ is an injective function extending $\tau^{-1} \circ f_s$, then $\langle t, (g \circ \widetilde{\psi})^* \rangle \in D$. But we can enumerate the injective extensions $\widetilde{\psi}\colon \{0, \dots, t-1\} \to \omega$ of $\tau^{-1} \circ f_s$, hence by using the fact that

$$(g \circ \widetilde{\psi})^*(\langle a, b \rangle) = g^*(\langle \widetilde{\psi}(a), \widetilde{\psi}(b) \rangle) = \varphi_e^A(\langle \widetilde{\psi}(a), \widetilde{\psi}(b) \rangle),$$

we can enumerate D from A. Note that this enumeration procedure is uniform in A (although there is no telling what it will do for $A \notin \mathcal{C}_{e,\tau}$). Let D_A be the set enumerated using this procedure with an arbitrary oracle A. The point is that $\langle t, \psi^* \rangle \in D$ iff $\mu(\{A \in [\sigma] \mid \langle t, \psi^* \rangle \in D_A\}) > 2^{-|\sigma|-1}$, which is a Σ_1^0 condition. Therefore, D is computably enumerable.

The remainder of the construction is quite standard. Let $d \in \omega$ be the index of a function which enumerates D until it discovers a $\langle t, \psi^* \rangle \in D$ such that $\varphi_s^{\psi^*}(d)\downarrow$. (As usual, we can assume that φ_d knows its own index by the Recursion theorem.) If the search is successful and $\langle t, \psi^* \rangle$ is the first appropriate element of D which is found, then define $\varphi_d(d)$ so that $\varphi_d(d) \neq \varphi_s^{\psi^*}(d)$. Let $\psi\colon \{0, \dots, t-1\} \to V$ be an extension of f_s corresponding to ψ^*. Because f will be an extension of ψ, we have satisfied Q_s. On the other hand, if the search is not successful, then $\varphi_d(d)$ will diverge. But this is okay; it means that there is no way to extend f_s to a bijection $f\colon \omega \to V$ such that $\varphi_s^{f^*}(d)\downarrow$. Hence $\varphi_s^{f^*}$ cannot be total and Q_s is again satisfied. In this case, let $t = t_s$ and $\psi = f_s$.

All that remains is to ensure that $v_s \in \operatorname{range}(f_{s+1})$, but this is easy. If $v_s \in \operatorname{range}(\psi)$ then let $t_{s+1} = t$ and $f_{s+1} = \psi$. Otherwise, let $t_{s+1} = t + 1$ and let $f_{s+1}\colon \{0, \dots, t_{s+1} - 1\} \to V$ be the extension of ψ with $f_{s+1}(t) = v_s$.

This completes the construction and the proof. □

References

[1] Denis R. Hirschfeldt, Bakhadyr Khoussainov, Richard A. Shore, and Arkadii M. Slinko. Degree spectra and computable dimensions in algebraic structures. *Ann. Pure Appl. Logic*, 115(1-3):71–113, 2002.

[2] Juichi Shinoda and Theodore A. Slaman. Recursive in a generic real. *J. Symbolic Logic*, 65(1):164–172, 2000.

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