CUPPING WITH RANDOM SETS

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ABSTRACT. We prove that a set is K-trivial if and only if it is not weakly ML-cuppable. Further, we show that a set below zero jump is K-trivial if and only if it is not ML-cuppable. These results settle a question of Kučera, who introduced both cuppability notions.

1. INTRODUCTION

The study of algorithmic randomness has uncovered a remarkable relationship between sets that are highly random and sets that have no random content at all. In this paper, we strengthen this relationship by looking at the computational power of a set joined with an incomplete random set. The sets with no random content are known as the K-trivials. We will prove that the K-trivial sets are precisely those that cannot be joined above $\emptyset'$ with an incomplete random. We will also show that all sets below $\emptyset'$ that are not K-trivial can be joined to $\emptyset'$ with a low random.

The question as to whether the K-trivial sets could be characterized in this manner was originally posed by Kučera [12]. This question appears in Miller and Nies's paper, *Randomness and computability: open questions*, as well as Downey and Hirschfeldt's recent monograph, *Algorithmic Randomness and Complexity* [2, 10].

We will assume familiarity with the basic theory of algorithmic randomness. For an introduction to this topic, see the recent books by Downey and Hirschfeldt and by Nies [2, 13]. We adopt the notational conventions used by Downey and Hirschfeldt. Recall that a set A is *K-trivial* if for some constant c, for all n we have that $K(A \upharpoonright n) \leq K(n) + c$ (where $K(n)$ is defined to be $K(1^n)$). Hence, a K-trivial set is indistinguishable from a computable set in terms of K complexity. The existence of non-computable K-trivial sets was first established by Solovay in an unpublished manuscript [2]. Later, Zambella constructed a non-computable K-trivial c.e. set [18].

A set R is *Martin-Löf random* if there is a constant c such that for all n, $K(R \upharpoonright n) > n - c$. The definition of a Martin-Löf random set can be

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relativized to any oracle A. We call a set A *low for Martin-Löf randomness* if every Martin-Löf random set is also Martin-Löf random relative to A.

The following theorem, a combination of results by Nies and by Hirschfeldt, Nies and Stephan, shows the relationship between Martin-Löf randomness and K-triviality [5, 11].

Theorem 1 (Nies; Hirschfeldt, Nies and Stephan)**.** *The following are equivalent:*

(i) *A is K-trivial.*
(ii) *A is low for Martin-Löf randomness.*
(iii) *There exists $X \geq_T A$ such that X is Martin-Löf random relative to A.*

Posner and Robinson proved that any non-computable set that is Turing below $\emptyset'$ can be cupped to $\emptyset'$ with a 1-generic set [14]. This is a fundamental result in computability theory. In 2004, Kučera asked which sets below $\emptyset'$ can be cupped to $\emptyset'$ with an incomplete Martin-Löf random [12]. In other words, does the Posner-Robinson theorem hold if we replace Baire category with Lebesgue measure, and if not, for which sets does it fail? This question motivated the following definition.

Definition 2. A set A is *weakly ML-cuppable* if there is an incomplete Martin-Löf random set X such that $A \oplus X \geq_T \emptyset'$. If one can choose $X <_T \emptyset'$, then A is *ML-cuppable.*

Kučera suggested, quite correctly, that these notions might characterize non-K-triviality. Nies gave a partial answer to Kučera's question by showing that there exists a non-computable K-trivial c.e. set that is not weakly ML-cuppable [12].

2. Proving Non-ML-cuppability

Theorem 3. *No K-trivial set is weakly ML-cuppable.*

The proof of Theorem 3 builds on work of Franklin and Ng, and Bienvenu, Hölzl, Miller and Nies. Franklin and Ng characterized the incomplete Martin-Löf random sets in terms of tests consisting of differences of Σ^0_1 classes [4]. Recently, Bienvenu, Hölzl, Miller and Nies showed that the incomplete Martin-Löf random sets are exactly those Martin-Löf random sets for which a particular density property fails [1]. Let P be a measurable set and τ a finite string. We write $d(\tau, P)$ for $\mu([\tau] \cap P) \cdot 2^{|\tau|}$, where μ is the Lebesgue measure. This is the density of P above τ. We say that X has lower density zero in P if for all $\delta > 0$ there exists an n such that $d([X \upharpoonright n], P) < \delta$.

Theorem 4 (Bienvenu, Hölzl, Miller, Nies)**.** *If X is a Martin-Löf random set, then X is complete if and only if there exists a Π^0_1 class P such that $X \in P$ and X has lower density zero in P.*

The utility of Theorem 4 is that if a Martin-Löf random set is complete, then there is a single Π^0_1 class that witnesses this fact. The following corollary is a direct relativization of one direction of this theorem.

Corollary 5. *Fix A. If X is a set Martin-Löf random relative to A and $A \oplus X \geq_T A'$, then there exists a $\Pi^0_1(A)$ class P such that $X \in P$ and X has lower density zero in P.*

We need one more result in order to prove Theorem 3. Let S be a set of finite strings. We call S *bounded* if

$$\sum_{\sigma \in S} 2^{-|\sigma|} < \infty.$$

The following result is implicit in Miller, Kjos-Hanssen and Solomon [6], but it was first explicitly stated, in an even stronger form, by Simpson [16, Lemma 5.11].

Theorem 6. *Let A be a K-trivial set and let W_A be a bounded set of strings that is c.e. in A. Then there exists a bounded c.e. set of strings W such that $W_A \subseteq W$.*

It is not hard to see that Theorem 6 can be used to characterize the K-trivial sets.

Proof of Theorem 3. Let A be a K-trivial set. Let R be a Martin-Löf random set such that $A \oplus R \geq_T \emptyset'$. Nies showed that all K-trivial sets are low, and thus $A \oplus R \geq_T A'$ [11]. Further, because all K-trivial sets are low for Martin-Löf randomness, R is Martin-Löf random relative to A. Hence from Corollary 5, there exists a $\Pi^0_1(A)$ class P_A such that $R \in P_A$ and R has lower density zero in P_A.

Let W_A be an A-c.e. prefix-free set such that $P_A = \{X \in 2^\omega \colon (\forall \sigma \in W_A)\ \sigma \not\prec X\}$. Any prefix-free set has bounded weight. Hence by Theorem 6, there is a c.e. set W such that $W_A \subseteq W$ and W has bounded weight.

The fact that W has bounded weight means that W is a Solovay test. This implies that there are finitely many (perhaps zero) initial segments of R in W. No initial segment of R is in W_A, so we may remove them from W, preserving the fact that it is a bounded weight c.e. superset of W_A. Let $P = \{X \in 2^\omega \colon (\forall \sigma \in W)\ \sigma \not\prec X\}$. Observe that $R \in P$ and $P \subseteq P_A$. As R has lower density zero in P_A, it follows immediately that R has lower density zero in P. So by Theorem 4, R is complete. □

3. Constructing ML-cupping partners

We will now show that for any set A that is not K-trivial, there is an incomplete Martin-Löf random set R such that $A \oplus R \geq_T \emptyset'$. Further, we will show that if A is computable from $\emptyset'$, then such an R can be found that is low. This means that any set that is not K-trivial is weakly ML-cuppable, and any set below $\emptyset'$ that is not K-trivial is ML-cuppable.

The previous results in this direction have used Martin-Löf random sets that are also Martin-Löf random relative to A. The following was noted by Nies [12]. Assume that $A \leq_T \emptyset'$ is not K-trivial. Let Ω^A be Chaitin's Ω relativized to A (see [3]). It follows from Theorem 1 that $\Omega^A \not\geq_T A$, otherwise A would be K-trivial. Hence $\Omega^A \not\geq_T \emptyset'$. Relativizing the fact that $\Omega \equiv_T \emptyset'$ gives us $A \oplus \Omega^A \equiv_T A' \geq_T \emptyset'$, so A is weakly ML-cuppable. Further, if A is low, then $\Omega^A \leq_T A' \leq_T \emptyset'$, so A is ML-cuppable.

Our approach to finding a cupping partner for A is different. We will construct a Martin-Löf random set R that is *not* Martin-Löf random relative to A. We will control the places where R enters the levels of the universal Martin-Löf test relative to A and use this to compute a function from $A \oplus R$ that dominates the settling time function of $\emptyset'$.

The set R will be constructed using a $\emptyset'$ oracle. Control over R will be maintained by keeping R inside a sequence of Π^0_1 classes. During the construction, we need to keep R inside a Π^0_1 class while removing it from some $\Pi^0_1(A)$ class that contains only A-random sets (sets Martin-Löf random relative to A). To achieve this, we use the fact that A is not a K-trivial set, hence not low for Martin-Löf randomness.

The following lemma is an immediate corollary of a theorem of Kučera, who proved that for any Martin-Löf random set X and any Π^0_1 class P of positive measure, there is a tail of X in P [7]. This lemma was also used by Kučera and Terwijn in a construction of a noncomputable c.e. low for random set [9].

Lemma 7. *Let P be any Π^0_1 class of positive measure and let Q be any $\Pi^0_1(A)$ class that only contains A-random sets. If $P \subseteq Q$, then A is K-trivial.*

Proof. If X is a Martin-Löf random set, then some tail of X is in P. This implies that some tail of X is in Q, and hence that X is A-random. □

Our objective is to build sequences that are inside some Π^0_1 classes and outside others. We will now establish the conventions for Π^0_1 classes that we will use in the rest of this paper. For any Π^0_1 class P, there exists a c.e. set W such that $P = \{X \in 2^\omega \colon (\forall n)(X \upharpoonright n \notin W)\}$. Hence we can define an approximation to P by letting $P[s] = \{X \in 2^\omega \colon (\forall n)(X \upharpoonright n \notin W[s])\}$, where $W[s]$ is the set obtained by enumerating W for s steps. Clearly, for all s, $P[s+1] \subseteq P[s]$ and $P = \bigcap_s P[s]$. We will make use of the fact that $W[s]$ is always a finite set, and hence $P[s]$ is the set of sequences that avoid some finite set of strings. Note that this relativizes, and so we can talk about $P(A)[s]$ where $P(A)$ is a $\Pi^0_1(A)$ class.

Because we will build our cupping partners by finite extension, we need to know whether or not a finite string τ can be extended to a sequence within a Π^0_1 class P (i.e., whether or not $[\tau] \cap P = \emptyset$). We will abuse notation slightly and write $\tau \in P$ if $[\tau] \cap P \neq \emptyset$. We will write $\tau \notin P$ if $[\tau] \cap P = \emptyset$.

In general we will use $Q(A)$ to denote $\Pi^0_1(A)$ classes that only contain A-randoms. We will use $Q_n(A)$ to denote the complement of the nth level of the universal Martin-Löf test relative to A. Of course we will use these conventions together. For example, $\rho \notin Q_n(A)[s]$ means that if we run the A-approximation to the complement of the nth level of the universal Martin-Löf test relative to A for s steps, then any infinite sequence extending ρ will be removed.

Lemma 8. *Let P be a Π^0_1 class, $\tau \in 2^{<\omega}$, and A a set that is not K-trivial. Let Q be any $\Pi^0_1(A)$ class that only contains A-random sets. If $d(\tau, P) \geq \delta$, then there exists $\rho \succ \tau$ such that:*

(i) *$\rho \notin Q$.*
(ii) *$d(\rho, P) \geq \delta/2$.*
(iii) *For all $\sigma \prec \rho$, there is an $s \in \omega$ with $\sigma \in Q[s]$ and $\rho \notin Q[s]$.*

Proof. Define the following Π^0_1 class:

$$\hat{P} = \{X \in 2^\omega : X \succeq \tau \wedge (\forall \sigma)(\tau \preceq \sigma \prec X \rightarrow d(\sigma, P) \geq \delta/2)\}.$$

If there is an $X \in \hat{P} \backslash Q$, then we can take $\rho \prec X$ to be the shortest extension of τ that is not in Q. So assume that $\hat{P} \subseteq Q$. If $\hat{P}$ has positive measure, we can apply the previous lemma to obtain a contradiction to the fact that A is not K-trivial.

Let S be a prefix-free set of strings that defines the complement of $\hat{P}$ above τ. If $\sigma \in S$, then the measure of the complement of P above σ is at least $2^{-|\sigma|}(1 - \delta/2)$. Hence,

$$\mu S \cdot (1 - \delta/2) \leq \mu([\tau] \setminus P) \leq (1 - \delta) \cdot 2^{-|\tau|}.$$

This implies that the measure of S is strictly less than $2^{-|\tau|}$, so $\hat{P}$ has positive measure. □

Theorem 9. *Let $A <_T \emptyset'$ be a set that is not K-trivial. Let $D \leq_T \emptyset'$ be a non-computable set. There exists a Martin-Löf random set R such that:*

(i) *$R \not\geq_T D$.*
(ii) *R is low.*
(iii) *$A \oplus R \equiv_T \emptyset'$.*

Proof. We will construct R using a $\emptyset'$ oracle. We will build a sequence $\{(\tau_s, P_s)\}_{s\in\omega}$, where τ_s is a finite string and P_s is a Π^0_1 class. For all s, we will ensure that $\tau_{s+1} \succeq \tau_s$ and $P_{s+1} \subseteq P_s$. We will take $R = \bigcup_s \tau_s$ and further ensure that $R \in \bigcap_s P_s$. Recall our convention that for all n, $Q_n(A)$ is the complement of the nth level of the universal Martin-Löf test relative to A.

The idea behind this proof is to construct an R that is not Martin-Löf random relative to A. We can then use the stage that R leaves $Q_{l(s)}(A)$ (l will be a function computable in $A \oplus R$) to compute a function that dominates the settling time function of $\emptyset'$. Let m be the settling time function for $\emptyset'$. The requirement that $R \not\geq_T D$ will be achieved at odd stages of the construction and the requirement that R is low will be achieved at even stages in the construction. In particular, we will ensure that:

(i) If s is odd and $e = (s-1)/2$, then $(\forall X)(X \in [\tau_s] \cap P_s \to \Gamma_e^X \neq D)$.
(ii) If $s > 0$ is even and $e = s/2$, then either $((\forall X)(X \in [\tau_s] \cap P_s \to \Gamma_e^X(e)\uparrow))$ or $((\forall X)(X \in [\tau_s] \cap P_s \to \Gamma_e^X(e)\downarrow))$.

Define $f : \omega \to \mathbb{Q}$ by $f(s) = 2^{-4s-1}$. The function f will be a lower bound for $d(\tau_s, P_s)$. At stage 0, let $\tau_0 = \lambda$ and let P_0 be the complement of the first level of the universal Martin-Löf test.

At stage $s+1$, assume that $d(\tau_s, P_s) \geq f(s)$. This clearly holds for the case that $s = 0$. Define $l(s) = |\tau_s| + 4s + 2$. Observe that

$$\begin{aligned} d(\tau_s, P_s \cap Q_{l(s)}(A)) &\geq d(\tau_s, P_s) - 2^{|\tau_s|}\mu(\overline{Q_{l(s)}(A)}) \\ &\geq f(s) - 2^{-4s-2} \\ &= f(s)/2. \end{aligned}$$

Hence if we let $\hat{P}_s = P_s \cap (Q_{l(s)}(A)[m(s)])$, then the above inequality establishes that $d(\tau_s, \hat{P}_s) \geq f(s)/2$. Note that $\hat{P}_s$ is a Π^0_1 class because $Q_{l(s)}(A)[m(s)]$ is the set of sequences that do not extend some finite set of strings. Let $t \in \omega$ be the least number such that there exists $\rho \succ \tau_s$ with the following properties:

(i) $\rho \notin Q_{l(s)}(A)[t]$.
(ii) $d(\rho, \hat{P}_s) \geq f(s)/4$.

Such a t and ρ exist by Lemma 8. Let τ_{s+1} be the least ρ for which the above holds for this particular t.

Now we will define P_{s+1} such that $d(\tau_{s+1}, P_{s+1}) \geq f(s)/16$. The definition of P_{s+1} depends on the requirement being met. If $s+1$ is odd, then let $e = s/2$. Using a $\emptyset'$ oracle, we can find a number n such that the measure of the set $\{X \in 2^\omega : \Gamma_e^X \succ D \restriction n\}$ is less than $2^{-|\tau_{s+1}|} \cdot f(s)/8$. The existence of such an n follows from Sacks's theorem that the measure of the Turing cone above any non-computable set is zero [15]. The class P_{s+1} is defined to be the intersection of $\hat{P}_s$, and the class of sets that do not compute an extension $D \restriction n$ via Γ_e.

If $s+1$ is even, then let $e = (s+1)/2$. The measure of the Π^0_1 class $[\tau_s + 1] \cap \hat{P}_s \cap \{X : \Gamma_e^X(e)\uparrow\}$ is a right-c.e. real. Given a rational q, $\emptyset'$ can determine whether the measure of this class drops strictly below q or whether it is greater than or equal to q. Hence, we can use $\emptyset'$ to determine whether $d(\tau_{s+1}, \{X \in \hat{P}_s : \Gamma_e^X(e)\uparrow\}) \geq f(s)/8$.

If so, set $P_{s+1} = \{X \in \hat{P}_s : \Gamma_e^X(e)\uparrow\}$. Otherwise we have that $d(\tau_{s+1}, \{X \in \hat{P}_s : \Gamma_e^X(e)\downarrow\}) \geq f(s)/8$. Determine a number n such that $d(\tau_{s+1}, \{X \in \hat{P}_s : \Gamma_e^X(e)\downarrow[n]\}) \geq f(s)/16$, and then set $P_{s+1} = \{X \in \hat{P}_s : \Gamma_e^X(e)\downarrow[n]\}$.

Observe that $d(\tau_{s+1}, P_{s+1}) \geq f(s)/16 = f(s+1)$; hence our construction assumption holds for the following stage. This ends the construction.

Verification. The construction ensures that $R \not\geq_T D$ and $R' \leq_T \emptyset'$. We will show that $R \oplus A \equiv_T \emptyset'$. Given R and A, it is possible to compute the sequence $\{\tau_s\}_{s\in\omega}$ and the function $l(s)$. Firstly, $l(s)$ can be computed from

$|\tau_s|$. Secondly, given $l(s)$, we can compute the least stage t_s such that for some $\tau \prec R$, $\tau \notin Q_{l(s)}(A)[t_s]$. The string τ_{s+1} is the least such τ for this t_s. For all s, because $R \in Q_{l(s)}(A)[m(s)]$, t_s is greater than $m(s)$, and thus $R \oplus A \geq_T \emptyset'$. □

Corollary 10. *If A is a set below $\emptyset'$, then A is ML-cuppable if and only if A is not K-trivial.*

If we remove the requirements that A and D are below $\emptyset'$, then the construction is computable in $A \oplus D \oplus \emptyset'$ and we get a Martin-Löf random R such that $R \not\geq_T D$ and $A \oplus \emptyset' \leq_T A \oplus R \leq_T A \oplus D \oplus \emptyset'$. Letting $D = \emptyset'$, we obtain the following corollary.

Corollary 11. *If $A \subseteq \omega$, then A is weakly ML-cuppable if and only if A is not K-trivial.*

Using a slightly different construction, we can construct, for any non-K-trivial A and non-computable D, a Martin-Löf random $R \not\geq_T D$ such that $A \oplus R \equiv_T A \oplus D \oplus \emptyset'$. In order to achieve this, we need a new technique to encode D into $A \oplus R$ while ensuring that $R \not\geq_T D$. First we need to improve on Lemma 8.

Lemma 12. *Let P be a Π^0_1 class, $\tau \in 2^{<\omega}$, and A a set that is not K-trivial. Let Q be any $\Pi^0_1(A)$ class that only contains A-random sets. If $d(\tau, P \cap Q) \geq \delta$, then there exists an $A \oplus \emptyset'$ computable prefix-free sequence of strings $\langle \rho_i \colon i \in \omega \rangle$ such that for all i,*

(i) *$\rho_i \notin Q$.*
(ii) *$d(\rho_i, P) \geq \delta/2$.*
(iii) *For all $\sigma \prec \rho_i$, there is an $s \in \omega$ with $\sigma \in Q[s]$ and $\rho_i \notin Q[s]$.*

Proof. It is sufficient to show that $A \oplus \emptyset'$ can compute a sequence of strings $\langle \rho_i \colon i \in \omega \rangle$, each with the properties (i), (ii) and (iii), such that for all $i, j \in \omega$, if $i < j$, then $\rho_i \not\prec \rho_j$. From such a sequence, an infinite prefix-free set can be formed by removing any string that is a proper initial segment of an earlier string in the sequence.

We define $\langle \rho_i \colon i \in \omega \rangle$ by induction. Let ρ_0 be the first string found with properties (i), (ii) and (iii). Such a string exists by Lemma 8. Once ρ_s has been defined, let $P_s = P \setminus \bigcup_{i \leq s} [\rho_i]$. Observe that $P_s \cap Q = P \cap Q$, so we can again apply Lemma 8 to find another extension of τ with the desired properties. Let this extension be ρ_{s+1}. □

Theorem 13. *If A is a set that is not K-trivial and D is a set that is non-computable, then there exists a Martin-Löf random set R such that:*

(i) *$A \oplus R \equiv_T A \oplus D \oplus \emptyset'$.*
(ii) *$R \not\geq_T D$.*
(iii) *$R' \leq_T A \oplus R$.*

Proof. If $D \leq_T A \oplus \emptyset'$, then the result follows from the proof of Theorem 9, so we will assume that $D \not\leq_T A \oplus \emptyset'$.

Construction. Define $f : \omega \to \mathbb{Q}$ by $f(s) = 2^{-5s-1}$. At stage 0, let $\tau_0 = \lambda$. Let P_0 be the complement of the first level of the universal Martin-Löf test. At stage $s+1$, let $l(s) = |\tau_s| + 5s + 2$. Define $\hat{P}_s = P_s \cap (Q_{l(s)}(A)[m(s)])$. Again note that $\hat{P}_s$ is a Π^0_1 class. Provided that $d(\tau_s, P_s) \geq f(s)$, then by a simple comparison of set sizes, we have that $d(\tau_s, \hat{P}_s \cap Q_{l(s)}(A)) \geq f(s)/2$.

If $s+1$ is odd, then let $e = (s-1)/2$. Apply Lemma 12 to obtain a prefix-free sequence of strings $\langle \rho_i : i \in \omega \rangle$ by taking P, Q, τ and δ to be $\hat{P}_s$, $Q_{l(s)}(A)$, τ_s and $f(s)/2$ respectively. We claim that there is a k such that

$$d(\rho_{\langle D(e),k \rangle}, \{X \in \hat{P}_s : \Gamma^X_e(k) = D(k)\}) \leq \frac{1}{2}\, d(\rho_{\langle D(e),k \rangle}, \hat{P}_s).$$

This claim holds because otherwise D could be computed from $A \oplus \emptyset'$ by majority voting within $\hat{P}_s$. Specifically, for any k, determine the string $\rho_{\langle D(e),k \rangle}$; then using $\emptyset'$ determine for which value $i \in \{0,1\}$, $\mu\{X \in \hat{P}_s \cap [\rho_{\langle D(e),k \rangle}] : \Gamma^X_e(k) = i\} > \frac{1}{2}\mu(\hat{P}_s \cap [\rho_{\langle D(e),k \rangle}])$. By assumption, $D(k) = i$.

From $A \oplus D \oplus \emptyset'$ we can find a k such that

$$d(\rho_{\langle D(e),k \rangle}, \{X \in \hat{P}_s : \Gamma^X_e(k) = D(k)\}) < \frac{9}{16}\, d(\rho_{\langle D(e),k \rangle}, \hat{P}_s). \tag{3.1}$$

Let $\tau_{s+1} = \rho_{\langle D(e),k \rangle}$ for this k. This string τ_{s+1} allows us to encode the value of $D(e)$ and at the same time meet the requirement that $\Gamma^R_e \neq D$ by a judicious choice of P_{s+1}. Define P_{s+1} as follows. Let $F = \{X \in 2^\omega : \Gamma^X_e(k) \downarrow = 1\}$ and $G = \{X \in 2^\omega : \Gamma^X_e(k) \downarrow = 0\}$ be Σ^0_1 classes. If any of the following conditions apply, then define P_{s+1} as per the first condition that is found to hold by a $\emptyset'$ search:

(i) If $d(\tau_{s+1}, \hat{P}_s \cap (F \cup G)) < \frac{7}{8}\, d(\tau_{s+1}, \hat{P}_s)$, then let $P_{s+1} = \hat{P}_s \setminus (F \cup G)$.

(ii) If $d(\tau_{s+1}, \hat{P}_s \cap F) > \frac{9}{16}\, d(\tau_{s+1}, \hat{P}_s)$, then $D(k) = 0$ by (3.1), and so define $P_{s+1} = \hat{P}_s \setminus G$.

(iii) If $d(\tau_{s+1}, \hat{P}_s \cap G) > \frac{9}{16}\, d(\tau_{s+1}, \hat{P}_s)$, then $D(k) = 1$ by (3.1), and so define $P_{s+1} = \hat{P}_s \setminus F$.

(iv) If $d(\tau_{s+1}, \hat{P}_s \cap F) > \frac{1}{4}\, d(\tau_{s+1}, \hat{P}_s)$ and $d(\tau_{s+1}, \hat{P}_s \cap G) > \frac{1}{4}\, d(\tau_{s+1}, \hat{P}_s)$, then if $D(k) = 0$, let $P_{s+1} = \hat{P}_s \cap F[n]$, where n is the least number such that $d(\tau_{s+1}, \hat{P}_s \cap F[n]) \geq d(\tau_{s+1}, \hat{P}_s)/8$. If $D(k) = 1$, then do the same with F replaced by G.

Note that one of these conditions must hold. If (i) and (ii) both fail, then

$$d(\tau_{s+1}, \hat{P}_s \cap G) \geq \left(\frac{7}{8} - \frac{9}{16}\right)\, d(\tau_{s+1}, \hat{P}_s) > \frac{1}{4}\, d(\tau_{s+1}, \hat{P}_s),$$

so the second conjunct in (iv) holds. If (i) and (iii) both fail, we get the first conjunct in the same way. Also note that however we have defined P_{s+1}, we have

$$d(\tau_{s+1}, P_{s+1}) \geq d(\tau_{s+1}, \hat{P}_s)/8 \geq d(\tau_s, \hat{P}_s)/16 \geq d(\tau_s, P_s)/32 \geq f(s+1).$$

If $s+1$ is even, then act as in the construction of Theorem 9.

Verification. The odd stages in the construction ensure that $R \not\geq_T D$. At stage $s+1 = 2e+1$, whichever condition (i)–(iv) is used to define P_{s+1} ensures that there is some k such that no element of P_{s+1} correctly computes $D(k)$ via Γ_e. It is still the case that $A \oplus R \geq_T \emptyset'$ by the same argument given in the proof of Theorem 9. We will show that using $A \oplus R$ we can determine indices for the Π^0_1 classes used in the construction. Note that once we have determined τ_{s+1}, then $\emptyset'$ can determine which condition (i)–(iv) is used to define P_{s+1}. We only need D in this step if (iv) is used. In this case, P_{s+1} is defined to be either $\hat{P}_s \cap F[n]$ or $\hat{P}_s \cap G[n]$. But this means that $\Gamma^R_e(k)\downarrow$, and F is used in the construction if and only if $\Gamma^R_e(k)\downarrow= 1$. The n is computable from $\emptyset'$. This allows us to compute an index for P_{s+1}.

From the index for P_s, along with A and R, we can determine the index of the string $\rho_{\langle i,k\rangle}$ that is equal to τ_{s+1} when $s+1$ is odd. Hence we know that $D(e) = i$ for $e = (s-1)/2$. Finally, the even stages of the construction ensure that R' is computable from the construction. Hence $R' \leq_T A \oplus D \oplus \emptyset'$, which we have just shown is Turing equivalent to $A \oplus R$. □

Theorem 13 lets us characterize K-triviality in terms of its degree-theoretic interaction with Martin-Löf random sequences *without* mentioning $\emptyset'$.

Corollary 14. *If $A \subseteq \omega$, then A is not K-trivial if and only if for all $D >_T \emptyset$, there is a Martin-Löf random R such that $R \not\geq_T D$ and $A \oplus R \geq_T D$.*

Proof. If A is not K-trivial and $D >_T \emptyset$, then Theorem 13 gives us the necessary Martin-Löf random. On the other hand, if A is K-trivial, then let $D = \emptyset'$ and use the fact that A is not weakly ML-cuppable. □

Slaman and Steel extended the Posner-Robinson theorem to show that any non-computable set A that is strictly Turing below $\emptyset'$ can be cupped to $\emptyset'$ with a 1-generic set X such that A and X form a minimal pair [17]. The analogous result for A not K-trivial and X Martin-Löf random does not hold. Any Martin-Löf random computes a diagonally non-computable function. Kučera showed that if A and B both compute diagonally non-computable functions and are both below $\emptyset'$, then A and B do not form a minimal pair [8]. Hence no Martin-Löf random set below $\emptyset'$ forms a minimal pair with any set A below $\emptyset'$ that computes a diagonally non-computable function.

The problem with adding minimal pair requirements to the construction used in the proof of Theorem 9 is that $\emptyset'$ cannot enumerate the non-computable, A-computable sets. However, A'' can, and hence we can obtain the following.

Corollary 15. *If A is a set that is not K-trivial and $X \geq_T A''$, then there exists an incomplete Martin-Löf random set R such that $A \oplus R \equiv_T X$ and A and R form a minimal pair.*

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