THE K-DEGREES, LOW FOR K DEGREES, AND WEAKLY LOW FOR K SETS

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Abstract. We call A *weakly low for K* if there is a c such that $K^A(\sigma) \geq K(\sigma) - c$ for infinitely many σ; in other words, there are infinitely many strings that A does not help compress. We prove that A is weakly low for K iff Chaitin's Ω is A-random. This has consequences in the K-degrees and the low for K (i.e., low for random) degrees. Furthermore, we prove that the initial segment prefix-free complexity of 2-random reals is infinitely often maximal. This had previously been proved for plain Kolmogorov complexity.

1. Introduction

If $A \in 2^\omega$ is 1-random, then there is a connection between the degree of randomness of A, the prefix-free (Kolmogorov) complexity of initial segments of A, and the (lack of) power of A as an oracle. We explore some aspects of this connection. See the next section for a brief introduction to effective randomness.

We say that $A \in 2^\omega$ is *weakly low for K* if $(\exists c)(\exists^\infty n)\ K(n) \leq K^A(n) + c$. Making use of $\leq^+$ to indicate a suppressed additive constant, we can write this as $(\exists^\infty n)\ K(n) \leq^+ K^A(n)$. Nies, Stephan and Terwijn [22] call A *low for Ω* if Chaitin's Ω is A-random. In Section 3, we show that A is weakly low for K iff it is low for Ω. This result is analogous to a celebrated result of Nies. Call $A \in 2^\omega$ *low for K* if $K(\sigma) \leq^+ K^A(\sigma)$ and *low for 1-random* if every 1-random is A-random. Nies [21] proved that these two notions—each stating that A is useless as an oracle in a specific context—are equivalent.

The equivalence of weakly low for K and low for Ω has a variety of consequences. In Section 4, we use it to prove that the initial segment prefix-free complexity of 2-random reals is infinitely often maximal. This had previously been proved for plain Kolmogorov complexity [22, 16].

Section 5 looks at consequences in the LR/LK-degrees. Nies partially relativized the notions of low for K and low for 1-random to introduce two ways of comparing the power of oracles in the context of effective randomness. He defined $X \leq_{LK} Y$ to mean that $K^Y(\sigma) \leq^+ K^X(\sigma)$, and $X \leq_{LR} Y$ to mean that every Y-random is X-random. These partial orders induce the *low for K degrees* and the *low for random degrees*, respectively. They turn out to be the same. It is clear that $X \leq_{LK} Y$ implies $X \leq_{LR} Y$; Kjos-Hanssen, Miller and Solomon [12] proved the converse.

2000 *Mathematics Subject Classification.* Primary 68Q30; Secondary, 03D30, 03D80.

Key words and phrases. Martin-Löf randomness, prefix-free Kolmogorov complexity.

The author was supported by the National Science Foundation under grants DMS-0945187 and DMS-0946325, the latter being part of a Focused Research Group in Algorithmic Randomness. The author also thanks André Nies and Liang Yu, without whom this paper would be greatly diminished, or more likely, would not exist at all.

Note that this extends the result of Nies, since X is low for K iff $X \leq_{LK} \emptyset$ and X is low for 1-random iff $X \leq_{LR} \emptyset$.

We prove that if $X \leq_{LR} Y$ and Y is low for Ω, then $X \leq_T Y'$. Thus, if Y is low for Ω, it has countably many predecessors in the LR-degrees; the converse is open. It also follows that if X and Y are 2-random relative to each other, then they form a minimal pair in the LR-degrees.

In Section 6, we consider the K-degrees. Downey, Hirschfeldt and LaForte [7, 8] defined $X \leq_K Y$ to mean that $K(X \upharpoonright n) \leq^+ K(Y \upharpoonright n)$. In other words, Y has higher initial segment prefix-free complexity than X, up to a constant. The induced partial order is called the *K-degrees*. If higher complexity implies more randomness, then one can interpret $X \leq_K Y$ as saying that Y is more random than X.

We prove that if X is 1-random, then prefix-free complexity relative to X can be expressed in terms of the prefix-free complexity of initial segments of X. In particular, $K^X(\sigma) =^+ \min_{s\in\omega} K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle$. This implies that if X is 1-random and $X \leq_K Y$, then $Y \leq_{LK} X$. Note that this result is not new; it follows from the corresponding result for the LR-degrees [19] and the equivalence between $\leq_{LR}$ and $\leq_{LK}$. As a corollary to the work of Section 5, the cones above 2-random reals in the K-degrees are countable. This is not true for all 1-random reals.

2. Preliminaries

We assume that the reader is familiar with basic computability (recursion) theory, as would be found in Part I of Soare [25]. We give a quick introduction to effective randomness, touching on the definitions and results needed in this paper. For a more thorough introduction, see Li and Vitányi [14], Nies [20], or the upcoming monograph of Downey and Hirschfeldt [6].

By *strings* we refer to elements of $2^{<\omega}$. We identify strings with natural numbers using an effective bijection; for concreteness, identify $\sigma \in 2^{<\omega}$ with $n \in \omega$ if 1σ is the binary expansion of $n+1$. We call elements of 2^ω *reals* and abuse notation by conflating $X \in 2^\omega$ with the element of $[0,1]$ that has binary expansion $0.X$. The non-uniqueness of binary expansion will not be an issue below.

For $\sigma \in 2^{<\omega}$, let $[\sigma] = \{X \in 2^\omega \colon \sigma \prec X\}$, i.e., the set of reals extending σ. If $S \subseteq 2^\omega$ is a c.e. set, then $\bigcup_{\sigma\in S}[\sigma]$ is called a Σ^0_1 class. The complement of a Σ^0_1 class is called a Π^0_1 class. A *Martin-Löf test* is a uniform sequence $\{V_n\}_{n\in\omega}$ of Σ^0_1 classes such that $\mu(V_n) \leq 2^{-n}$, where μ is the standard Lebesgue measure on 2^ω. A real $X \in 2^\omega$ is said to *pass* a Martin-Löf test $\{V_n\}_{n\in\omega}$ if $X \notin \bigcap_{n\in\omega} V_n$. We say that $X \in 2^\omega$ is *1-random* (or *Martin-Löf random*) if it passes all Martin-Löf tests. There is a *universal* Martin-Löf test, i.e., a single test $\{U_n\}_{n\in\omega}$ that is passed only by the 1-random reals.

Define *A-randomness* by relativizing Martin-Löf's definition to an oracle $A \in 2^\omega$. An essential tool in understanding relativized randomness is Van Lambalgen's theorem [27]: $X \oplus Y$ is 1-random iff X is 1-random and Y is X-random. Note that by applying Van Lambalgen's theorem twice, we can show that if X and Y are both 1-random, then X is Y-random iff Y is X-random. We call X *n-random* if it is $\emptyset^{(n-1)}$-random.

There is an important connection between the randomness of real numbers and the complexity, or information content, of finite binary strings. A set $D \subseteq 2^{<\omega}$ is *prefix-free* if no element of D is a proper prefix of another element. A *prefix-free machine* is a partial computable function $M \colon 2^{<\omega} \to 2^{<\omega}$ with prefix-free domain.

If M is a prefix-free machine, then let

$$K_M(\sigma) = \min\{|\tau| \colon M(\tau) = \sigma\},$$

so $K_M(\sigma)$ is the length of the shortest M-description of σ. There is a *universal prefix-free machine* $U \colon 2^{<\omega} \to 2^{<\omega}$ that is optimal for prefix-free machines: if M is any such machine, then $K_U(\sigma) \leq^+ K_M(\sigma)$. We write $K(\sigma)$ for $K_U(\sigma)$ and call it the *prefix-free* (*Kolmogorov*) *complexity* of $\sigma \in 2^{<\omega}$. Plain Kolmogorov complexity C is defined in the same way as prefix-free complexity except without restricting the domains of machines. It is well known that the 1-random reals can be characterized in terms of the prefix-free complexity of their initial segments. Schnorr proved that $X \in 2^\omega$ is 1-random iff $K(X \upharpoonright n) \geq^+ n$.

Since U has prefix-free domain, $\sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} \leq \sum_{\tau \in \operatorname{dom} U} 2^{-|\tau|} \leq 1$; this is called *Kraft's inequality*.[1] It has a useful effective converse. A *Kraft–Chaitin set* is a c.e. set $W \subseteq \omega \times 2^{<\omega}$ such that $\sum_{\langle d,\sigma\rangle \in W} 2^{-d} \leq 1$. Given such a set, the Kraft–Chaitin theorem says that there is a prefix-free machine M such that $\langle d, \sigma\rangle \in W$ implies that $K_M(\sigma) \leq d$. Thus, $K(\sigma) \leq^+ d$ for all $\langle d,\sigma\rangle \in W$. Closely related to the Kraft–Chaitin theorem is the fact that K is an optimal information content measure. A function $\widehat{K} \colon \omega \to \mathbb{R} \cup \{\infty\}$ is an *information content measure* if $\sum_{n\in\omega} 2^{-\widehat{K}(n)} \leq 1$ and $\{\langle k, n\rangle \colon \widehat{K}(n) \leq k\}$ is computable enumerable. Not only is K an information content measure (when viewed as a function of ω), but it is not hard to see that if $\widehat{K}$ is another information content measure, then $W = \{\langle k+1, n\rangle \colon \widehat{K}(n) < k\}$ is a Kraft–Chaitin set, hence $K(n) \leq^+ \widehat{K}(n)$.

We write U^A and K^A for the relativizations of the universal prefix-free machine and prefix-free complexity, respectively, to an oracle $A \in 2^\omega$. The results mentioned above remain true in their relativized forms. In particular, $X \in 2^\omega$ is A-random iff $K^A(X \upharpoonright n) \geq^+ n$. The following result relates K^A to unrelativized prefix-free complexity when $A \in 2^\omega$ is 1-random (see also Lemma 6.1).

Ample Excess Lemma (Miller and Yu [19]). *Let $A \in 2^\omega$ be 1-random.*

(1) $\sum_{n\in\omega} 2^{n-K(A\upharpoonright n)} < \infty$.
(2) $K^A(n) \leq^+ K(A \upharpoonright n) - n$.

Note that (1) implies (2) by applying the Kraft–Chaitin theorem relativized to A.

Chaitin proved that $\Omega = \sum_{\tau\in\operatorname{dom} U} 2^{-|\tau|}$ is 1-random. It is easy to see that Ω is a *c.e. real*, meaning that there is a computable, nondecreasing sequence of rational numbers $\{\Omega_s\}_{s\in\omega}$ such that $\Omega = \lim \Omega_s$. It follows from Calude, Hertling, Khoussainov and Wang [4] and Kučera and Slaman [13] that every 1-random c.e. real is Ω for the right choice of universal machine. Chaitin showed that $\Omega \equiv_T \emptyset'$ (this also follows from Arslanov's completeness criterion). It is easy to see, given what we know, that the 2-random reals are exactly the 1-random, low for Ω reals.

Proposition 2.1 (Nies, Stephan and Terwijn [22]). *Let $A \in 2^\omega$ be 1-random. Then A is 2-random iff it is low for Ω.*

Proof. By definition, A is 2-random iff A is 1-random relative to $\emptyset'$. Since $\emptyset' \equiv_T \Omega$, this is equivalent to A being Ω-random. By Van Lambalgen's theorem, A is Ω-random iff Ω is A-random, in other words, exactly when A is low for Ω. □

[1] Strictly speaking, Kraft considered finite prefix-free codes.

3. Weakly low for K is the same as low for Ω

We show that being weakly low for K is equivalent to being low for Ω. An interesting alternate proof of the harder direction, that low for Ω implies weakly low for K (Theorem 3.3), has recently been found by Laurent Bienvenu. A *Solovay function* is a computable $f\colon \omega \to \omega$ such that $K(n) \leq^+ f(n)$ (which is equivalent to $\sum_{n\in\omega} 2^{-f(n)}$ converging) and $(\exists^\infty n)\ f(n) \leq^+ K(n)$. Bienvenu and Downey [3] proved that a computable function f is a Solovay function iff $\sum_{n\in\omega} 2^{-f(n)}$ is finite and 1-random. To see how this implies Theorem 3.3, let f be a computable function such that $\sum_{n\in\omega} 2^{-f(n)} = \Omega$. Then f is a Solovay function, so $K(n) \leq^+ f(n)$. If A is not weakly low for K, then $\lim_{n\to\infty} K(n) - K^A(n) = \infty$, hence $\lim_{n\to\infty} f(n) - K^A(n) = \infty$. Therefore, f is not a Solovay function relative to A. Relativizing the result of Bienvenu and Downey, Ω is not A-random, i.e., A is not low for Ω.

Theorem 3.1. *If A is weakly low for K, then A is low for Ω.*

Proof. We show the contrapositive. First, we define a two families of c.e. sets $\{W_\sigma\}_{\sigma\in 2^{<\omega}}$ and $\{D_\sigma\}_{\sigma\in 2^{<\omega}}$. Fix $\sigma \in 2^{<\omega}$. Search for the least stage $s \in \omega$ such that $\sigma \prec \Omega_s$, in other words, such that σ appears to be a prefix of Ω. If no such stage is found, then W_σ and D_σ will be empty. Now, for any $\tau \in 2^{<\omega}$ such that $U(\tau)\!\downarrow$ after stage s, enumerate $\langle |\tau|, U(\tau)\rangle$ into D_σ. Also enumerate $\langle |\tau|, U(\tau)\rangle$ into W_σ as long as it preserves the condition that $\sum_{\langle d,n\rangle\in W_\sigma} 2^{-d} \leq 2^{-|\sigma|}$. Note that if $K_s(n) \neq K(n)$, then $\langle K(n), n\rangle \in D_\sigma$.

We claim that if $\sigma \prec \Omega$, then $W_\sigma = D_\sigma$. It follows from our definition that $\sum_{\langle d,n\rangle\in D_\sigma} 2^{-d} \leq \Omega - \Omega_s$. Observe that if $\sigma \prec \Omega$, then $\Omega - \Omega_s \leq 2^{-|\sigma|}$. In this case, $\sum_{\langle d,n\rangle\in D_\sigma} 2^{-d} \leq 2^{-|\sigma|}$, so $W_\sigma = D_\sigma$. The idea is that we have used an approximation of Ω to efficiently approximate all but finitely many values of $K(n)$.

Consider the A-c.e. set $W = \{\langle d + |\tau| - |\sigma|, n\rangle \colon U^A(\tau) = \sigma \text{ and } \langle d,n\rangle \in W_\sigma\}$. By the construction of $\{W_\sigma\}_{\sigma\in 2^{<\omega}}$ and Kraft's inequality,

$$\sum_{\langle e,n\rangle\in W} 2^{-e} = \sum_{U^A(\tau)\downarrow=\sigma}\ \sum_{\langle d,n\rangle\in W_\sigma} 2^{-d-|\tau|+|\sigma|} \leq \sum_{U^A(\tau)\downarrow} 2^{-|\tau|} \leq 1.$$

This proves that W is a Kraft–Chaitin set relative to A. Therefore, there is a constant $k \in \omega$ such that if $\langle e,n\rangle \in W$, then $K^A(n) \leq e + k$.

Now, assume that Ω is not A-random. For any $c \in \omega$, there are $\tau, \sigma \in 2^\omega$ such that $U^A(\tau) = \sigma$, $|\sigma| - |\tau| \geq c$ and $\sigma \prec \Omega$. Let $s \in \omega$ be the least stage such that $\sigma \prec \Omega_s$ (which must exist because Ω is not a dyadic rational). There is an $N \in \omega$ such that if $n \geq N$, then $K_s(n) \neq K(n)$ (by the usual conventions on stages, $N = s + 1$ is sufficient). For all $n \geq N$, we have $\langle K(n), n\rangle \in W_\sigma$, hence $\langle K(n) + |\tau| - |\sigma|, n\rangle \in W$. But this means that $K^A(n) \leq K(n) + |\tau| - |\sigma| + k \leq K(n) + k - c$, for all but finitely many n. But c was arbitrary, so A is not weakly low for K. □

For the other direction we use the fact that Ω is essentially interchangeable with any other 1-random c.e. real. This follows from work on *Solovay reducibility.* Write $X \leq_S Y$ to mean that there is a $c \in \omega$ and a partial computable function $f\colon \mathbb{Q} \to \mathbb{Q}$ such that if $q < Y$ is rational, then $0 \leq X - f(q) \leq c(Y - q)$ [26]. In other words, good approximations of Y from the left give us good approximations of X from the left. Kučera and Slaman [13] showed that if X is a 1-random c.e. real, then $X \equiv_S \Omega$.

When he introduced the reducibility, Solovay [26] proved that $X \leq_S Y$ implies $X \leq_K Y$. Relativizing the proof to an oracle A, we see that $X \leq_S Y$ implies that $K^A(X \upharpoonright n) \leq^+ K^A(Y \upharpoonright n)$, from which it follows that A-randomness is closed upward in the Solovay degrees. Together with the result of Kučera and Slaman, if X is a 1-random c.e. real, then X is random relative to A iff A is low for Ω.

We also need a simple lemma.

Lemma 3.2. *Let A be an oracle. If V is any $\Sigma^0_1[A]$ class, then there is a $\Sigma^0_1[A]$ class $\widehat{V}$ such that:*

(1) $\mu(\widehat{V}) \leq 3\mu(V)$.
(2) *If X is an endpoint of an open interval in V, then $X \in \widehat{V}$.*

Furthermore, an index for $\widehat{V}$ can be found uniformly from an index from V and is independent of A.

Proof. Let $\widehat{V} = \bigcup\{(a-\varepsilon, a+2\varepsilon)\colon [a, a+\varepsilon] \subseteq V\}$. It is easy to check that $\widehat{V}$ has the required properties. □

Theorem 3.3. *If A is low for Ω, then A is weakly low for K.*

Proof. Assume that A is not weakly low for K. Let $S \subseteq 2^\omega$ be a Σ^0_1 class such that $\mu(S) \leq 1/2$ and $2^\omega \setminus S$ contains only 1-random reals. For example, we could take $S = U_1$, where $\{U_n\}_{n\in\omega}$ is a universal Martin-Löf test. Let $X = \inf(2^\omega \setminus S)$. Note that X is a 1-random c.e. real; from the discussion above, if we prove that X is not A-random, then A is not low for Ω.

For each n, we define a $\Sigma^0_1[A]$ class V_n such that $\mu(V_n) \leq 2^{-n-2}$. It will not be the case that $X \in V_n$; in fact, we will have $V_n \subseteq S$. On the other hand, it *will* always be true that X is an endpoint of an open interval in V_n. We claim that this is sufficient. By the previous lemma, we can form a computable sequence $\{\widehat{V_n}\}_{n\in\omega}$ of $\Sigma^0_1[A]$ classes such that $X \in \bigcap_{n\in\omega} \widehat{V_n}$ and $\mu(\widehat{V_n}) \leq 3\mu(V_n) \leq 3 \cdot 2^{-n-2} < 2^{-n}$. Therefore, X is covered by a Martin-Löf test relative to A, so X is not A-random.

We turn to the definition of $\{V_n\}_{n\in\omega}$. Assume that $S = \bigcup_{s\in\omega}[\sigma_s]$, where $\{\sigma_s\}_{s\in\omega}$ is a prefix-free computable sequence of strings. Fix $n \in \omega$. If $m = |\sigma_s|$, put $[\sigma_s]$ into V_n as long as σ_s is among the first $2^{m-K^A(m)-n-2}$ strings of length m in $\{\sigma_s\}_{s\in\omega}$. In other words, V_n is built from the same sequence that defines S but with the restriction that strings of length m can contribute at most $2^{-K^A(m)-n-2}$ to its measure. Note that the stage-wise approximations to $2^{m-K^A(m)-n-2}$ approach it from below, so V_n is $\Sigma^0_1[A]$. Also note that

$$\mu(V_n) \leq \sum_{m\in\omega} 2^{-K^A(m)-n-2} = 2^{-n-2} \sum_{m\in\omega} 2^{-K^A(m)} \leq 2^{-n-2},$$

where the last step uses Kraft's inequality.

Next we prove that there is a $v \in \omega$ such that if $|\sigma_s| \geq v$, then $[\sigma_s] \subseteq V_n$. Let $J(m) = |\{s \in \omega\colon |\sigma_s| = m\}|$. We claim that $I(m) = m - \log(J(m))$ is an information content measure. Clearly, I is computable from above. Note that $2^{-I(m)} = J(m)2^{-m}$ is exactly the contribution to the measure of S made by the strings in $\{\sigma_s\}_{s\in\omega}$ of length m. Since $\{\sigma_s\}_{s\in\omega}$ is prefix-free, $\sum_{m\in\omega} 2^{-I(m)} = \mu(S) \leq 1/2$. This shows that I is an information content measure, so there is c such that

$(\forall m)\ K(m) \leq I(m) + c$. Because A is not weakly low for K, there is a v large enough that $K^A(m) \leq K(m) - c - n - 2$, for all $m \geq v$. For such m,

$$2^{m-K^A(m)-n-2} \geq 2^{m-K(m)+c} \geq 2^{m-I(m)} = 2^{\log(J(m))} = J(m).$$

Therefore, $[\sigma_s]$ is put into V_n as long as $|\sigma_s| \geq v$.

We can now show that X is an endpoint of an open interval in V_n. This is because X is not a binary rational and thus not an endpoint of $[\sigma_s]$, for any s. Since there are only finitely many strings in $\{\sigma_s\}_{s\in\omega}$ of length less than v, there is an ε small enough such that $(X - \varepsilon, X)$ is disjoint from all corresponding intervals. But $(X - \varepsilon, X) \subseteq S$, so $(X - \varepsilon, X) \subseteq V_n$. This completes the proof. □

It has been shown that every nonempty Π^0_1 class has a low for Ω member [9, 23], giving us a weakly low for K basis theorem.

4. All 2-random reals maximize K infinitely often

While it is impossible for every initial segment of a real to have maximal complexity (with respect to either C or K), almost every real infinitely often achieves maximal initial segment complexity up to a constant. Call $A \in 2^\omega$ *infinitely often (i.o.) K maximizing* if $(\exists^\infty n)\ K(A \upharpoonright n) \geq^+ n + K(n)$. Similarly, A is *i.o. C maximizing* if $(\exists^\infty n)\ C(A \upharpoonright n) \geq^+ n$.[2] The right side of each inequality represents the maximal possible complexity for a string of length n.

Solovay [26] proved that i.o. K maximizing implies i.o. C maximizing. In fact, he proved that strings with (essentially) maximal K complexity must have (essentially) maximal C complexity. Solovay also proved that almost all reals are i.o. K maximizing. Yu, Ding and Downey [28] analyzed his argument to prove that 3-randomness is sufficient to imply that a real is i.o. K maximizing. In the other direction, Martin-Löf [15] showed that every i.o. C maximizing real is 1-random, while Schnorr [24] refuted the converse. Nies, Stephan and Terwijn [22] showed that i.o. C maximizing implies 2-randomness. They also showed the converse, as did Miller [16], classifying the i.o. C maximizing reals. Putting these facts together,

$$\text{3-random} \Longrightarrow \text{i.o. } K \text{ maximizing} \Longrightarrow \text{i.o. } C \text{ maximizing} \Longleftrightarrow \text{2-random}.$$

We resolve the status of i.o. K maximizing, answering a question in [17].

Theorem 4.1. *A is 2-random iff it is infinitely often K maximizing.*

Proof. Assume that A is 2-random. Then A is low for Ω by Proposition 2.1, hence weakly low for K by Theorem 3.3. By the ample excess lemma, $K^A(n) \leq^+ K(A \upharpoonright n) - n$. Rearranging, we have $K(A \upharpoonright n) \geq^+ n + K^A(n)$. Because A is weakly low for K, there are infinitely many n such that $K^A(n) \geq^+ K(n)$. Note that $K(A \upharpoonright n) \geq^+ n + K(n)$ for these n, so A is infinitely often K maximizing. □

It is interesting to note that Solovay [26] proved that strings with maximal C-complexity need not have maximal K-complexity, so the equivalence of i.o. C and K maximizing is not true on the level of strings.

[2] Starting with Yu, Ding and Downey [28], these notions have been called *strong Chaitin random* and *Kolmogorov random*, respectively. Since they were neither introduced by Kolmogorov nor Chaitin, and since Chaitin has used "strong Chaitin randomness" to denote one of his characterizations of 1-randomness, it seems reasonable to look for alternative names. One may even question the need for names, since both notions are equivalent to 2-randomness. For these reasons, we have adopted the descriptive—if artless—terms used in this paper.

5. Applications to the LR/LK-degrees

The work of Section 3 has consequences in the LR/LK-degrees.

Theorem 5.1. *If $X \leq_{LR} Y$ and Y is low for Ω, then $X \leq_T Y'$.*

Proof. We have $X \leq_{LK} Y$, so

$$K^Y(X \upharpoonright n) \leq^+ K^X(X \upharpoonright n) \leq^+ K^X(n) \leq^+ K(n), \tag{1}$$

for all $n \in \omega$. Because Y is low for Ω, it is weakly low for K, so there is a $c \in \omega$ such that $S = \{n \in \omega \colon K(n) \leq K^Y(n) + c\}$ is infinite. Together with (1), there is a $d \in \omega$ such that if $n \in S$, then $K^Y(X \upharpoonright n) \leq K^Y(n) + d$. By relativizing Chaitin's counting theorem [5, Lemma I3], there is an $e \in \omega$ such that

$$|\{\sigma \in 2^n \colon K^Y(\sigma) \leq K^Y(n) + d\}| \leq e,$$

for all $n \in \omega$. Let $T \subseteq 2^{<\omega}$ be the tree defined by

$$\sigma \in T \text{ iff } (\forall n < |\sigma|)[\, n \in S \rightarrow K^Y(\sigma \upharpoonright n) \leq K^Y(n) + d\,].$$

Note that $|[T]| \leq e$ and $X \in [T]$, where $[T]$ denotes the set of infinite paths through T. Also note that $S \leq_T Y'$ and so $T \leq_T Y'$. But every isolated infinite path through a tree is computable from the tree, hence $X \leq_T Y'$. □

Corollary 5.2. *If Y is low for Ω, then it has countably many predecessors in the LR-degrees.*

It is possible that the converse holds.

Open Question. If Y is not low for Ω, must it have continuum many predecessors in the LR-degrees?

Not all reals have countably many LR-predecessors. Barmpalias, Lewis and Soskova [2] proved that if $Y \in 2^\omega$ is non-GL$_2$ (i.e., $Y'' \not\leq_T (Y \oplus \emptyset')'$), then it has continuum many predecessors in the LR-degrees. Furthermore, Barmpalias [1] has answered the question positively for Δ^0_2 reals. Note that the LR-predecessors of a real form a Borel set, so if Y has uncountably many LR-predecessors, then it has continuum many. In fact, Y has continuum many *LR-degrees* below it, because each LR-degree is countable (if $X \equiv_{LK} Y$, then $X' \equiv_T Y'$ [21]).

The next proof uses some basic facts about 2-randomness. By definition, $X \in 2^\omega$ is 2-random iff X is $\emptyset'$-random. This is equivalent to X being Ω-random, since $\Omega \equiv_T \emptyset'$. Hence by Van Lambalgen's theorem, $X \oplus \Omega$ is 1-random. Second, Kautz [11] proved that every 2-random X is GL$_1$; in other words, $X' \leq_T X \oplus \emptyset'$. Finally, we say that Y is 2-random relative to X if it is X'-random. Note that almost every pair of reals are two random relative to each other.

Corollary 5.3. *If $X, Y \in 2^\omega$ are 2-random relative to each other, then they form a minimal pair in the LR-degrees.*

Proof. Since X is 2-random relative to Y, it must be 2-random, so $X' \equiv_T X \oplus \emptyset' \equiv_T X \oplus \Omega$ and $X \oplus \Omega$ is 1-random. We know that Y is X'-random, hence $X \oplus \Omega$-random. Applying Van Lambalgen's theorem, $X \oplus \Omega$ is Y-random. So if we assume that $A \leq_{LR} Y$, then $X \oplus \Omega$ is A-random.

Now assume that $A \leq_{LR} X$. By Theorem 5.1 and the fact that X is low for Ω, we have $A \leq_T X' \equiv_T X \oplus \Omega$. So we have proved that A is computed by an A-random real. This means that A is a *base for 1-randomness*, which Hirschfeldt, Nies and Stephan proved to be equivalent to A being low for 1-randomness [10]. In other words, $A \equiv_{LR} \emptyset$. This shows that X and Y are a minimal pair in the LR-degrees. □

6. Remarks on the K-degrees

Recall that $X \leq_K Y$ means that $K(X \upharpoonright n) \leq^+ K(Y \upharpoonright n)$. Miller and Yu [19] proved that if $X \in 2^\omega$ is 1-random and $X \leq_K Y$, then $Y \leq_{LR} X$, which is equivalent to $Y \leq_{LK} X$. Below we give a more direct proof that $X \leq_K Y$ implies $Y \leq_{LK} X$ on the 1-randoms. We use the following lemma.

Bounding Lemma (Miller and Yu [18])**.** *If $\sum_{n\in\omega} 2^{-g(n)} < \infty$ and $g \leq_T X$ with use n, then $K(X \upharpoonright n) \leq^+ n + g(n)$.*

It turns out that the initial segment complexity of X codes the behavior of K^X in a fairly simple way. Fix a pairing function, i.e., an effective bijection $\langle \cdot, \cdot \rangle \colon \omega^2 \to \omega$. We may assume that $\langle n, m\rangle$ is greater than or equal to both n and m. We also apply the pairing function to strings, having identified them with natural numbers.

Lemma 6.1. *If X is 1-random, then $K^X(\sigma) =^+ \min_{s\in\omega} K(X \upharpoonright \langle \sigma, s\rangle) - \langle \sigma, s\rangle$.*

Proof. By the ample excess lemma, there is a $c \in \omega$ with $\sum_{n\in\omega} 2^{n-K(X\upharpoonright n)} \leq 2^c$. Let $W = \{\langle K(X \upharpoonright \langle\sigma, s\rangle) - \langle \sigma, s\rangle + c + k + 1, \sigma\rangle \colon \sigma \in 2^{<\omega} \text{ and } s, k \in \omega\}$. Note that W is X-c.e. (which is the purpose of k). Also,

$$\sum_{\langle d,\sigma\rangle\in W} 2^{-d} = \sum_{n\in\omega}\sum_{k\in\omega} 2^{-K(X\upharpoonright n)+n-c-k-1} = \sum_{n\in\omega} 2^{n-K(X\upharpoonright n)-c} \leq 1,$$

so W is a Kraft–Chaitin set relative to X. This implies that

$$K^X(\sigma) \leq^+ \min_{s\in\omega} K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle.$$

For the other direction, let

$$g(\langle\sigma, s\rangle) = \begin{cases} K^X_s(\sigma) & \text{if } s = 0 \text{ or } K^X_s(\sigma) < K^X_{s-1}(\sigma), \\ \langle\sigma, s\rangle & \text{otherwise.}\end{cases}$$

Here, $K^X_s(\sigma)$ is the stage s approximation of $K^X(\sigma)$, which by standard conventions on the use function, can only depend on $X \upharpoonright s \preccurlyeq X \upharpoonright \langle\sigma, s\rangle$.[3] Therefore, $g \leq_T X$ with use n. Next we want to bound $\sum_{n\in\omega} 2^{-g(n)}$. Note that $\sigma \in 2^{<\omega}$ contributes less than $\sum_{k\in\omega} 2^{-K^X(\sigma)-k} + \sum_{s\in\omega} 2^{-\langle\sigma,s\rangle} = 2\cdot 2^{-K^X(\sigma)} + \sum_{s\in\omega} 2^{-\langle\sigma,s\rangle}$ to the sum. Therefore,

$$\sum_{n\in\omega} 2^{-g(n)} \leq 2\sum_{\sigma\in 2^{<\omega}} 2^{-K^X(\sigma)} + \sum_{\langle\sigma,s\rangle\in\omega} 2^{-\langle\sigma,s\rangle} \leq 2 + 2 < \infty,$$

by Kraft's inequality. Applying the bounding lemma, $K(X \upharpoonright n) \leq^+ n + g(n)$. For $\sigma \in 2^{<\omega}$, choose the least s such that $K^X_s(\sigma) = K^X(\sigma)$. Then $K^X(\sigma) = g(\langle\sigma, s\rangle) \geq^+ K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle$. Hence $K^X(\sigma) \geq^+ \min_{s\in\omega} K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle$. □

Theorem 6.2 ([19]+[12])**.** *If X is 1-random and $X \leq_K Y$, then $Y \leq_{LK} X$.*

Proof. Immediate from the lemma. □

[3] This is the only place we use the fact that $\langle m, n\rangle \geq \max\{m, n\}$. We could do away with this restriction on our pairing function by simply stipulating that the use of $K^X_s(\sigma)$ is at most $\langle\sigma, s\rangle$.

The theorem allows us to apply results about the LK-degrees to the K-degrees. For example, Nies [21] proved that if $X \equiv_{LK} Y$, then $X' \equiv_T Y'$ (in fact, the jumps are truth-table equivalent). This means that if $X \equiv_K Y$ and X is 1-random (hence Y is too), then $X' \equiv_T Y'$. So 1-random K-degrees are countable. It is not hard to produce continuum many reals $X \in 2^\omega$ such that $K(X \upharpoonright n) =^+ n/2$, thus not all K-degrees are countable.

The previous section also tells us something about the K-degrees.

Theorem 6.3. *The cone above a 2-random in the K-degrees is countable.*

Proof. Let $X \in 2^\omega$ be 2-random. By Proposition 2.1, X is low for Ω. Assume $X \leq_K Y$, then $Y \leq_{LK} X$, so $Y \leq_T X'$ by Theorem 5.1. □

It is known that some 1-random reals have uncountable upper cones in the K-degrees [18]. On the other hand, it is open whether there are maximal K-degrees.

Open Question (See [17])**.** Is there a maximal K-degree? Is every 2-random K-degree maximal?

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