UNIFORM ALMOST EVERYWHERE DOMINATION

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ABSTRACT. We explore the interaction between Lebesgue measure and dominating functions. We show, via both a priority construction and a forcing construction, that there is a function of incomplete degree that dominates almost all degrees. This answers a question of Dobrinen and Simpson, who showed that such functions are related to the proof-theoretic strength of the regularity of Lebesgue measure for G_δ sets. Our constructions essentially settle the reverse mathematical classification of this principle.

1. INTRODUCTION

1.1. **Domination.** Fast growing functions have been investigated in mathematics for over 90 years. Set theorists, for example, have investigated the structure ω^ω/Fin and the associated invariants of the continuum ever since Hausdorff constructed his (ω_1, ω_1^*)-gap [5]; today, this structure has a role to play in modern descriptive set theory.

Fast growing functions have deep connections with computability. A famous early example is that of Ackermann's function, defined in 1928 [1]. This is a computable function that grows faster than any primitive recursive function. This example was useful in elucidating the mathematical concept of computability, an understanding reflected in Church's Thesis.

In the 1960s, computability theorists became interested in functions that grow faster than all computable functions.

Definition 1.1. Let $f, g\colon \omega \to \omega$. The function f *majorizes* g if $f(n) \geqslant g(n)$ for all $n \in \omega$. If $f(n) \geqslant g(n)$ for all but finitely many n, then f *dominates* g. These are written as $f \geqslant g$ and $f \geqslant^* g$, respectively. We call f *dominant* if it dominates all (total) computable functions.

Dominant functions were explored in conjunction with Post's Program. The goal of Post's Program was to find a "sparseness" property of the complement of a c.e. set A that would ensure that A is incomplete. Yates [17] proved that even maximal c.e. sets, which have the sparsest possible complements among coinfinite c.e. sets, can be complete. This put an end to Post's Program, but not to the study of sparseness properties.

Let $p_A(n)$ be the n^{th} element of the complement of A. Having p_A dominant would certainly imply that the complement of A is sparse. On the other hand, Tennenbaum [16] and Martin [12] showed that if A is maximal, then p_A is dominant. Furthermore, Martin characterized the Turing degrees of both the dominant functions and the maximal c.e. sets. He showed that there is dominant function of degree $\mathbf{a}$ iff $\mathbf{a}$ is high (i.e., $\mathbf{0}'' \leqslant \mathbf{a}'$), and that every high c.e. degree contains a maximal set. Together, these results revealed a surprising connection between the structure of c.e. sets, the place of their Turing degree within the

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jump hierarchy, and domination properties of functions. Later research explored further connections between domination properties, algebraic properties and computational power.

In this paper, we consider the interaction between Lebesgue measure and domination. Motivated by results on dominating functions in generic extensions of set theory, Dobrinen and Simpson [3] introduced the notion of a *uniformly almost everywhere* (*a.e.*) *dominating* degree: a Turing degree $\mathbf{a}$ that computes a function $f\colon \omega \to \omega$ such that

$$\mu\left(\{Z \in 2^\omega : (\forall g \in \omega^\omega)[\, g \leqslant_T Z \implies g \leqslant^* f\,]\}\right) = 1.$$

(Here μ denotes Lebesgue measure on 2^ω.) We also call such a function f *uniformly a.e. dominating*.

A natural goal is to characterize those Turing degrees that are uniformly a.e. dominating. A function of degree $\mathbf{0}'$ that dominates almost all degrees was first constructed by Kurtz [11, Theorem 4.3]. (Kurtz used this result to exhibit a difference between the 1-generic and the (weakly) 2-generic degrees: the upward closure of the 1-generic degrees has measure one [11, Theorem 4.1], while the upward closure of the (weakly) 2-generic degrees has measure zero [11, Corollary 4.3a].) Since the collection of uniformly a.e. dominating degrees is closed upwards, Kurtz's result implies that every degree $\geqslant \mathbf{0}'$ is in the class. On the other hand, a uniformly a.e. dominating function is dominant, and so by Martin's result, every uniformly a.e. dominating degree is high. Thus, Dobrinen and Simpson asked whether either the class of complete degrees (degrees above $\mathbf{0}'$) or the class of high degrees is identical to the class of uniformly a.e. dominating degrees.

The truth lies somewhere in the middle. Binns, Kjos-Hanssen, Lerman and Solomon [2] showed that not every high degree is uniformly a.e. dominating, or even *a.e. dominating*, an apparently weaker notion also introduced by Dobrinen and Simpson [3]. They gave two proofs. First, by a direct construction, they produced a high c.e. degree that is not a.e. dominating. (A similar result was independently obtained by Greenberg and Miller, although their example was Δ^0_2, not c.e.)

Second, Binns et al. [2] showed that if A has a.e. dominating degree, then every set that is 1-random over A is 2-random. If A is also Δ^0_2, then by Nies [13], $\emptyset'$ is K-trivial over A and so A is *super-high* (i.e., $A' \geqslant_{tt} \emptyset''$). By an index set calculation, there is a c.e. set that is high but not *super-high*, hence not a.e. dominating. It is open whether $\emptyset'$ being K-trivial over $\mathbf{a} \leqslant \mathbf{0}'$ implies that $\mathbf{a}$ is (uniformly) a.e. dominating; Kjos-Hanssen has some related results.[1]

We prove that Dobrinen and Simpson's other suggested characterization of the uniformly a.e. dominating degrees also fails.

Theorem 1.2. *There is an incomplete (c.e.) uniformly a.e. dominating degree.*

We provide two proofs of this result, although only one produces a c.e. degree. In Section 2 we use a priority argument to construct an incomplete c.e. uniformly a.e. dominating degree and in Section 4 we present a more flexible forcing construction of an incomplete uniformly a.e. dominating degree.

1.2. **Domination and Reverse Mathematics.** As observed by Dobrinen and Simpson [3], uniformly a.e. dominating degrees play a role in determining the reverse mathematical strength of the fact that the Lebesgue measure is regular. For an introduction to reverse mathematics, the reader is directed to Simpson [15].

[1] Added in proof: An upcoming paper of Binns, Kjos-Hanssen, Miller and Solomon answers this question in the positive. Note that then the pseudojump inversion theorem of Jockusch and Shore [7] can be used to construct an incomplete c.e. set A such that $\emptyset'$ is K-trivial over A, which provides an alternate proof of Theorem 1.2.

Regularity means that for every measurable set P there is a G_δ set $Q \supseteq P$ and an F_σ set $S \subseteq P$ such that $\mu(S) = \mu(P) = \mu(Q)$, where a G_δ set is the intersection of countably many open sets and an F_σ set is the union of countably many closed sets. Hence the following principle is implied by the regularity of the Lebesgue measure.

G_δ-REG. *For every G_δ set $Q \subseteq 2^\omega$ there is an F_σ set $S \subseteq Q$ such that $\mu(S) = \mu(Q)$.*

Recall that the G_δ sets are exactly those that are Π^0_2 in a real parameter (that is, boldface $\boldsymbol{\Pi}^0_2$), and the F_σ sets are exactly the $\boldsymbol{\Sigma}^0_2$ sets. Hence we can consider G_δ-REG as a statement of second order arithmetic. We will see that G_δ-REG, which appears to be a natural mathematical statement, does not fall in line with the commonly occurring systems of reverse mathematics. In particular, we examine the chain

$$\mathsf{RCA}_0 \subsetneq \mathsf{DNR}_0 \subsetneq \mathsf{WWKL}_0 \subsetneq \mathsf{WKL}_0 \subsetneq \mathsf{ACA}_0.$$

Here RCA_0 is the standard base system that all of the other systems extend; WKL_0 is RCA_0 plus weak König's lemma; and ACA_0 is RCA_0 plus the scheme of arithmetic comprehension. These systems are studied extensively in [15]. The system WWKL_0 is somewhat less standard. It consists of RCA_0 plus "weak weak König's lemma", which is introduced in Yu and Simpson [18]. A large amount of basic measure theory can be proved in WWKL_0, so it is a natural system for us to be concerned with. The final system, DNR_0, is less natural from a proof-theoretic standpoint but very natural for computability theorists. It is RCA_0 plus the existence of a function that is diagonally non-recursive; see Giusto and Simpson [4] and Jockusch [6].

Kurtz's result that $\mathbf{0}'$ is uniformly everywhere dominating essentially shows that G_δ-REG follows from ACA_0. This relies on the following:

Theorem 1.3 (Theorem 3.2 of Dobrinen and Simpson [3])**.** *A Turing degree* $\mathbf{a}$ *is of uniformly a.e. dominating degree iff for every Π^0_2 set $Q \subseteq 2^\omega$ there is a $\Sigma^0_2(\mathbf{a})$ set $S \subseteq Q$ such that $\mu(S) = \mu(Q)$.*

Dobrinen and Simpson conjectured that G_δ-REG and ACA_0 are equivalent over RCA_0 ([3, Conjecture 3.1]). This is not true; in fact, there is an ω-model of G_δ-REG that omits $\mathbf{0}'$ and hence is not a model of ACA_0. This was discovered by B. Kjos-Hanssen after the circulation of the priority-method proof of Theorem 1.2. This proof appears to be too rigid to allow us to obtain a version with cone avoidance, but Kjos-Hanssen found a clever way to build the ω-model without such a result. His construction is presented in Section 3.

The forcing construction is flexible enough to prove cone avoidance and more. We can thus improve Kjos-Hanssen's result by showing that G_δ-REG does not imply even systems much weaker than ACA_0:

Theorem 1.4. $\mathsf{RCA}_0 + G_\delta$-REG *does not imply* DNR_0.

But although G_δ-REG seems to lack proof-theoretic strength, none of the traditional systems below ACA_0 are strong enough to prove it:

Proposition 1.5 (Remark 3.5 of Dobrinen and Simpson [3])**.** WKL_0 *does not imply* G_δ-REG.

The proposition follows easily from the fact that there is an ω-model of WKL_0 that consists of low sets; by formalizing Theorem 1.3, every ω-model of G_δ-REG must include uniformly a.e. dominating degrees, which by Martin's result are high.

Furthermore, G_δ-REG seems to be "orthogonal" to the traditional systems in that its strength is insufficient to lift one such system to the system above it:

Theorem 1.6. $\mathsf{WKL}_0 + G_\delta$-$\mathsf{REG}$ *does not imply* ACA_0*;* $\mathsf{WWKL}_0 + G_\delta$-$\mathsf{REG}$ *does not imply* WKL_0.

It remains open whether $\mathsf{DNR}_0 + G_\delta$-$\mathsf{REG}$ implies WWKL_0.

1.3. **Notation, conventions and other technicalities.** Our computability theoretic notation is not always classical or consistent, but hopefully completely understandable. Thus, $\langle \varphi_e \rangle_{e\in\omega}$ is an effective list of all Turing functionals with oracle, and we write $\varphi_e^f(x), \varphi_{e,s}^f(x)\downarrow$, etc. This notation will be used when we try to diagonalize against some oracle f (so $\varphi_e\colon \omega^\omega \to \omega^\omega$). On the other hand, for domination purposes, we write Turing functionals as $\Phi(Z;x)$ and $\Phi(Z;x)[s]$. In fact, we only need to consider a single Φ:

Lemma 1.7. *There is a partial computable functional* $\Phi\colon 2^\omega \to \omega^\omega$ *such that if*

$$\mu\,\{Z \in 2^\omega : \text{if } \Phi(Z) \text{ is total, then } \Phi(Z) \leqslant^* f\} = 1,$$

then f *is uniformly a.e. dominating.*

Proof. Let $\langle \Psi_i \rangle_{i\in\omega}$ be an effective list of partial computable functionals $2^\omega \to \omega^\omega$ and define $\Phi(0^i 1 Z) = \Psi_i(Z)$. □

We assume that Φ has the following (standard) properties (for every $s, n \in \omega$ and $Z \in 2^\omega$):

(1) $\Phi(Z;n)\downarrow\ [s]$ implies $\Phi(Z;n)[s] \leqslant s$.
(2) $\Phi(Z;n)\downarrow\ [s]$ implies $(\forall m < n)\, \Phi(Z;m)\downarrow\ [s]$.

We let $\operatorname{dom}\Phi$ be the collection of Z such that $\Phi(Z)$ is total. For $n \in \omega$, we let $D_n = \{Z \in 2^\omega : \Phi(Z;n)\downarrow\}$. For a stage $s \in \omega$, $D_n[s]$ is given the obvious meaning. For $g \in \omega^{\leqslant\omega}$, let

$$D_{[n,m)}[g] = \{Z \in 2^\omega : (\forall k \in [n,m))\, \Phi(Z;k)\downarrow\ [g(k)]\},$$

(including the case where $m = \infty$). It follows from condition (1) that if $Z \in D_{[n,m)}[g]$, then g majorizes $\Phi(Z)$ on the interval $[n,m)$.

2. A proof of Theorem 1.2 via a priority construction

In this section we prove Theorem 1.2. We build $f\colon \omega \to \omega$ by giving a computable sequence of approximations $\langle f_s \rangle_{s\in\omega}$. Assuming the limit exists, $f = \lim f_s$ is Δ^0_2. To ensure that f has c.e. degree, it is enough to require that f is approximated from below. Formally, $(\forall n)(\forall s)\ f_s(n) \leqslant f_{s+1}(n)$. This means that $W = \{\langle n,m\rangle : f(n) \geqslant m\}$ is a c.e. set; it is clear that $f \equiv_T W$.

To ensure that f is incomplete we will enumerate a c.e. set B and meet the requirement

$R_e\colon \varphi_e^f \neq B$,

for each $e \in \omega$. These requirements will be handled by incompleteness strategies. The same strategies are responsible for assigning values to f, which essentially means that they must make f large enough to be uniformly a.e. dominating. This can be accomplished if they are supplied with appropriate approximations to the measure of $\operatorname{dom}\Phi$. These approximations are given by measure guessing strategies.

We describe the incompleteness and measure guessing strategies first, in relative isolation. Then we explain the priority tree and the full construction.

2.1. **Incompleteness Strategy.** Let σ be an agent assigned the goal of ensuring that $\varphi_e^f \neq B$, for some index $e = e(\sigma)$. When σ is initialized, it chooses a *follower* $x = x(\sigma)$ that has not been used before in the construction. A typical incompleteness strategy would wait for a computation $\varphi_e^f(x)\downarrow = 0$, preserve f on the use of this computation, and enumerate x into B. The main difference is that our incompleteness strategy will be proactive: it is permitted to change the values of f to make $\varphi_e^f(x)\downarrow = 0$. Indeed, only the incompleteness agents change the values of f at all, so they are not only permitted to make these changes, it is crucial that they do so.

Three restrictions are placed on σ's ability to change the values of f. First, as already mentioned, it cannot decrease the current values of f. Second, higher priority agents (who wish to preserve diagonalizing computations) impose restraint $N = N(\sigma)$; σ is not allowed to change $f \restriction N$. The third restriction (which ensures that eventually f will be dominating) involves a rational parameter $\varepsilon = \varepsilon(\sigma)$. For σ to permanently protect a computation $\varphi_e^f(x)\downarrow = 0$ with use r, it must be the case that

$$(\lozenge) \qquad \mu\left(\operatorname{dom}\Phi \smallsetminus D_{[N,r)}[f]\right) \leqslant \varepsilon.$$

In other words, σ's action (in protecting $f \restriction r$) prevents f from majorizing $\Phi(Z)$ above N for no more than ε of all $Z \in 2^\omega$. This is the restriction that forces σ to increase the values of f.

The first two restrictions place no significant burden on σ, but the third is more demanding. In fact, σ cannot hope to meet the third restriction without help because it does not know what $\operatorname{dom}\Phi$ is. To approximate it, we supply σ with two useful pieces of information: a rational $q = q(\sigma)$ and a natural number $M = M(\sigma)$ such that:

(1) $q \leqslant \mu(\operatorname{dom}\Phi)$.
(2) $\mu(D_M) \leqslant q + \varepsilon/2$.

In the full construction, these parameters are provided by a measure guessing agent. If σ is on the true path, then the values of q and M that are supplied to σ will meet conditions (1) and (2).

We are now ready to describe the behavior of σ. The possible states of σ are active, meaning that it is currently imposing restraint to protect a computation $\varphi_e^f(x)$, and passive. When σ is initialized, it is passive and it has restraint $r(\sigma) = 0$. If σ ever becomes active, it will remain so unless it is reset. This happens if the execution ever moves left of σ, or if condition (2) proves to be false for either σ or a higher priority active agent. The details of the full construction are below.

Say that σ is visited at stage $s \in \omega$. If either σ or a higher priority agent for R_e is currently active, then there is nothing to do. Otherwise, σ searches for a string $g \in s^{<s}$ of length M or greater that has the following (computable) properties:

(1) $g \supset f_s \restriction N$;
(2) $(\forall n \in [N, |g|))\ f_s(n) \leqslant g(n)$;
(3) $\mu(D_{[N,|g|)}[g]) > q - \varepsilon/2$; and
(4) $\varphi_{e,s}^g(x)\downarrow = 0$.

If there is such a string g, then σ lets $f_{s+1} \supset g$ and $r(\sigma) = |g|$. It enumerates x into B and declares itself active. If there is no such g, then σ does nothing and remains passive.

This completes the description of the incompleteness strategy. We prove below that if σ is on the true path and it ever becomes satisfied, then ($\lozenge$) holds. Because agents that are not on the true path might also attempt to protect computations, what we actually prove is stronger: if an agent ever becomes active (hence is imposing restraint), either (◊) holds or the agent is eventually reinitialized (so that its restraint is removed).

Remark 2.1. Unlike many tree constructions, it is important that at most one node on each level (i.e. at most one node per requirement) imposes restraint. Say a node at level e ensures that f dominates except for a set of size at most ε_e. We will argue that f dominates almost everywhere, using the fact that $\lim_{e\to\infty}\sum_{e'>e}\varepsilon_{e'}=0$. If several nodes on the same level e were to impose restraint, then ε_e must be counted more than once, making the calculation incorrect. This is why we stipulated that if σ is visited at some stage s and if at the same stage, some $\sigma' <_L \sigma$ on the same level is active, then σ does not act. Of course, we are making use of the fact that σ''s success is also σ's.

2.2. **Measure Guessing Strategy.** Measure guessing agents change neither f nor B and they impose no restraint on other agents. Their only function is to provide the values of q and M to the incompleteness agents at the next higher level. A measure guessing agent τ is initialized with a rational parameter $\delta=\delta(\tau)$. Its primary job is to find a rational q that approximates the measure of $\operatorname{dom}\Phi$ from below to within δ. This is done as follows. Divide the interval $[0,1]$ into subintervals of length δ. When τ is visited at stage s, it compares, for each $n\leqslant s$, the measure of $D_n[s]$ with that of $D_n[t]$, where t was the previous stage at which τ was visited. If the measure of some D_n has crossed the threshold from one subinterval I' to one on its right I, then (for the least such n) τ guesses that $q=\min I$ approximates the measure of $\operatorname{dom}\Phi$. Assume that τ is visited infinitely often and $\min I$ is the largest approximation guessed infinitely often. Then $\mu(D_n)\geqslant\min I$ for all $n\in\omega$ and $\mu(D_n)>\max I$ for finitely many n. Therefore, $\mu(\operatorname{dom}\Phi)\in I$.

We give the details. Let $d=\lceil 1/\delta\rceil$. The outcomes of τ will be of the form $\langle q,M\rangle\in\mathbb{Q}\times\omega$, where $q\in\{0,\delta,2\delta,\dots,d\delta\}$. When τ is first initialized, its outcome is $\langle 0,0\rangle$. Say that τ is visited at stage $s\in\omega$ and that the previous visit occurred at stage $t<s$. To provide a guess, τ looks for $n\leqslant s$ and $b\leqslant d$ such that $\mu(D_n[t])<b\delta$ but $\mu(D_n[s])\geqslant b\delta$. For the greatest such b (or equivalently, the b corresponding to the least such n), τ lets $q=b\delta$. Otherwise, τ lets $q=0$. Finally, τ takes the least M such that $\mu(D_M[s])<q+\delta$. Because $\mu(D_n[s])$ is monotonically decreasing as a function of n, for all $n\geqslant M$ we also have $\mu(D_n[s])<q+\delta$. The outcome of τ at stage s is $\langle q,M\rangle$.

Remark 2.2. Suppose that τ has outcome $\langle q,M_0\rangle$ at stage s_0 and outcome $\langle q,M_1\rangle$ at $s_1>s_0$. Further suppose that whenever τ is visited at a stage t between s_0 and s_1, its outcome at t is of the form $\langle q',M'\rangle$ with $q'\leqslant q$. Then $M_1=M_0$.

2.3. **The Priority Tree.** As usual, agents are organized on a tree, with the children of an agent representing its potential outcomes. Write $\alpha\subset\beta$ to mean that β is a proper extension of α. Each agent comes with a linear ordering $<_L$ on its children. We extend $<_L$ to other nodes as follows: say that α is *to the left* of β and write $\alpha<_L\beta$ if there are $\rho\subseteq\alpha$ and $\nu\subseteq\beta$ such that ρ and ν have the same parent and $\rho<_L\nu$. Write $\alpha<\beta$ if either $\alpha\subset\beta$ or $\alpha<_L\beta$. This is the total ordering lexicographically induced on the tree by the ordering we impose on the children of agents. If $\alpha<\beta$, then we say that α *has higher priority* than β.

The even levels of the priority tree are devoted to measure guessing agents and the odd levels to incompleteness agents. A measure guessing agent τ at level $2k$ is supplied with the parameter $\delta(\tau)=3^{-k}/2$. As described above, its outcomes have the form $\langle q,M\rangle\in\mathbb{Q}\times\omega$, where q is restricted to rationals of the form $b\delta(\tau)$. The outcomes are ordered first by q and then by M, with larger numbers *to the left of* smaller numbers.

An incompleteness agent $\sigma=\tau^\frown\langle q,M\rangle$ at level $2k+1$ has parameters $e(\sigma)=k$ and $\varepsilon(\sigma)=3^{-k}=2\delta(\tau)$. We also obviously set $q(\sigma)=q$ and $M(\sigma)=M$. The two final parameters, the follower $x(\sigma)$ and the restraint $N(\sigma)$ imposed by stronger nodes, are determined when σ is *initialized*. To initialize σ at stage $s \in \omega$, set its state to passive, let the restraint σ imposes $r(\sigma) = 0$ and choose a follower $x(\sigma) \in \omega$ that has not yet been assigned in the construction. Furthermore, set

$$N(\sigma) = \max\{\, r(\sigma') : \sigma' <_L \sigma \text{ is active at stage } s\,\}.$$

The children of σ are $\sigma^\frown\mathsf{active} <_L \sigma^\frown\mathsf{passive}$.

2.4. **Full Construction.** Let $f_0(n) = 0$ for all $n \in \omega$. The construction proceeds in stages. The preliminary phase of stage $s \in \omega$ involves reevaluating, and possibly *resetting*, currently active incompleteness agents. Reset agents must be reinitialized the next time they are visited. Say that $\sigma = \tau^\frown\langle q, M\rangle$ is active at stage s. If $\mu(D_M[s]) > q + \varepsilon(\sigma)/2$, then σ acted based on a false assumption and it could be the case that σ is forcing $f \restriction r(\sigma)$ to remain prohibitively small. Therefore, we reset σ. We also reset all previously initialized incompleteness agents of lower priority than σ (to allow them to recompute their restraints the next time they are visited).

Remark 2.3. Suppose that τ lies on the true path and that $\sigma = \tau^\frown\langle q, M\rangle$ is active at stage s. Further suppose that τ's guess is found to be incorrect at s (in other words, $\mu(D_M[s]) > q + \delta(\tau)$). Then the next time that τ is accessible, its new outcome lies to the left of σ and so σ is reset. It would seem that this mechanism would suffice and that explicit resetting is unnecessary. However, unlike many tree constructions, we need to be concerned with the restraint imposed by nodes that lie to the left of the true path. Such unwarranted restraint may prevent f from sufficiently dominating, and so needs to be reset when found incorrect.

During the main phase of stage s, we execute the strategies of finitely many agents on the priority tree, following a path of length at most s. This is done in substages $t \leqslant s$. We begin at substage $t = 0$ by visiting the root node $\alpha_0 = \lambda$. Say that we are visiting an agent α_t at substage t. First, reset any incompleteness agents σ such that $\alpha_t <_L \sigma$. (Note that if σ is reset and $\sigma < \sigma'$, then $\alpha_t <_L \sigma'$, so σ' is also reset.)

Case 1: *α_t is a measure guessing agent.* If the outcome of α_t at stage s is $\langle q, M\rangle$, then let $\alpha_{t+1} = \alpha_t^\frown\langle q, M\rangle$ and end the substage.

Case 2: *α_t is an incompleteness agent.* If α_t has never been visited before or has been reset since the last time it was visited, then it is initialized. If α_t is currently active, then end the substage and set $\alpha_{t+1} = \alpha_t^\frown\mathsf{active}$. Similarly, if there is a higher priority agent for R_e that is active at stage s, then set $\alpha_{t+1} = \alpha_t^\frown\mathsf{passive}$ and end the substage. Otherwise, execute the incompleteness strategy for α_t at stage s. If α_t becomes active (so that changes are made to f and B), then end stage s entirely. Otherwise, let $\alpha_{t+1} = \alpha_t^\frown\mathsf{passive}$ and end the substage.

This continues until substage $t = s$ is completed or until stage s is explicitly ended because an incompleteness agent becomes active. Finally, for any $x < \operatorname{dom} f_s$, if not expressly altered by us during the stage, we let $f_{s+1}(x) = f_s(x)$. This completes the construction.

2.5. **Verification.** Inductively define the *true path* to be the leftmost path visited infinitely often. In particular:

- The root node λ is on the true path.
- If ρ is on the true path and ν is the leftmost child of ρ that is visited infinitely often (if such exists), then ν is on the true path.

It is clear that if ρ is on the true path, then there is a stage $s \in \omega$ after which no agent left of ρ is ever visited.

Claim 2.4. *If σ is an incompleteness agent on the true path, then there is a stage $s \in \omega$ at which σ is initialized and after which it will never be reset.*

Proof. Take a stage $t \in \omega$ large enough that no agent left of σ will ever again be visited. By induction, we may also assume that t is large enough that the agents $\sigma' \subset \sigma$ have all been initialized for the final time (and will never be reset). None of these σ' can become active after stage t, or else the execution would move left of σ.

Although no $\sigma' <_L \sigma$ can become active after stage t, they can be reset in the preliminary phase of the construction and this will reset σ. But only active agents become reset and only finitely many $\sigma' <_L \sigma$ are active at stage t. Therefore, there is a stage $t' \geqslant t$ after which no agents left of σ are ever reset.

This leaves only one way that $\sigma = \tau^\frown\langle q,M\rangle$ can be reset at any stage $t'' \geqslant t'$: if σ is active at stage t'' and $\mu(D_M[t'']) > q+\varepsilon(\sigma)/2$. But if this is the case, then $\langle q,M\rangle$ cannot be the outcome of τ after stage t'', contradicting the fact that σ is on the true path. Therefore, σ is never reset after stage t'. But σ is visited infinitely often, so there is a stage $s \in \omega$ at which σ is initialized and after which it will never be reset. □

Claim 2.5. *The true path is infinite.*

Proof. We prove that there is no last node on the true path. First, consider an incompleteness agent σ on the true path. By Claim 2.4, there is a last stage t at which σ is initialized. After stage t, σ may become active at most once, so one of the outcomes of σ is eventually permanent.

Now consider a measure guessing agent τ on the true path. The first coordinate of the outcome of τ is taken from the finite set $Q = \{b\delta(\tau) : 0 \leqslant b \leqslant \lceil 1/\delta(\tau)\rceil\}$. Let q be the greatest element of Q that occurs as the first coordinate of the outcome infinitely often. Assume that no greater first coordinate occurs after stage $s \in \omega$. Let $\langle q,M\rangle$ be the outcome of τ at some stage $\geqslant s$. By Remark 2.2, if $\langle q',M'\rangle$ is the outcome of τ at some other stage $\geqslant s$, then either $q' < q$ or $q' = q$ and $M' = M$. Therefore, either $\langle q,M\rangle <_L \langle q',M'\rangle$ or $\langle q,M\rangle = \langle q',M'\rangle$, and the second case occurs infinitely often. Hence $\tau^\frown\langle q,M\rangle$ is on the true path. □

Remark 2.6. Let τ be a measure guessing node, and suppose that τ and $\tau^\frown\langle q,M\rangle$ are on the true path. Then $\mu(D_n) \geqslant q$ for all $n \in \omega$ and $\mu(D_M) \leqslant q+\delta(\tau)$. Therefore, $\mu(\operatorname{dom}\Phi) \in [q,q+\delta(\tau)]$.

We are primarily interested in the incompleteness agents that are eventually permanently active. Let the set of all such agents be $G = \{\sigma_0,\sigma_1,\dots\}$, with $\sigma_0 < \sigma_1 < \cdots$. The fact that we can thus enumerate G relies on the following:

Fact 2.7. The collection of nodes that lie either on, or to the left of the true path that are ever visited has order type ω under $<$. This is because for each node α on the true path, only finitely many nodes to the left of α are ever visited.

Claim 2.8. *Assume that σ is initialized at stage $s \in \omega$ and is never reset after stage s. Suppose that $\sigma' < \sigma$. Then if σ' is active at s, it remains permanently so (hence $\sigma' \in G$); otherwise, σ' never becomes active after s (hence $\sigma' \notin G$).*

Proof. First assume that σ' is active at stage s. If σ' is ever reset, then every lower priority agent is reset, including σ. But this never happens, so $\sigma' \in G$.

Now suppose that σ' is not active at stage s. It follows that $\sigma'^\frown\mathsf{passive} < \sigma$ (as σ is accessible at stage s). If σ' becomes active at some later stage, then $\sigma'^\frown\mathsf{active}$ would be accessible. But this would reset σ because $\sigma'^\frown\mathsf{active}$ lies to the left of σ. □

For all $i \in \omega$, let N_i and r_i denote the final values of $N(\sigma_i)$ and $r(\sigma_i)$, respectively.

Claim 2.9. *For all $i \in \omega$:*

(a) *Once σ_i becomes permanently active, f cannot change below r_i.*
(b) $N_{i+1} = r_i$.

Proof. (a) Assume that σ_i is permanently active after stage $s \in \omega$. From s onwards, σ_i imposes restraint r_i on weaker agents, so such agents do not change $f \upharpoonright r_i$. Any action by a stronger agent is impossible after the last stage s_i at which σ_i is initialized, and $s_i < s$.

(b) At stage s_i, the agents $\sigma < \sigma_i$ that are active are exactly $\sigma_0, \dots, \sigma_{i-1}$, and their restraints have reached their final values. Thus σ_i defines $N_i = \max\{r_j : j < i\}$ at stage s_i. When σ_i later becomes active, it imposes a permanent restraint r_i, which is greater than N_i. It follows that $r_0 < r_1 < \cdots$, and so $N_{i+1} = r_i$. □

Claim 2.10. *G is infinite.*

Proof. We can enumerate $\langle \varphi_e \rangle_{e\in\omega}$ in such a way that there are infinitely many e such that for all $t \in \omega$, for all $x \leqslant t$ and all $g \in (t+1)^{\leqslant t}$ we have $\varphi^g_{e,t}(x)\downarrow = 0$; we retroactively assume that we used such an enumeration. We will show that for each such e, G contains an agent working for R_e.

Pick such an e and let $\sigma = \tau^\frown\langle q, M\rangle$ be the agent of length $2e+1$ on the true path. Assume that the final initialization of σ occurs at stage $s \in \omega$.

Case 1: *An agent $\sigma' <_L \sigma$ for R_e is active at stage s.* If σ' is ever reset, then σ would also be reset. This is impossible, so $\sigma' \in G$.

Case 2: *No such σ' exists.* No $\sigma' <_L \sigma$ becomes active after stage s, so as long as σ remains passive, its full strategy will be executed every time it is visited. At stage s, a follower x is chosen and the final restraint N is determined. By Claim 2.9, $f \upharpoonright N$ is fixed after stage s.

We know that $\langle q, M\rangle$ is the correct outcome of τ, so $(\forall n)\ \mu(D_n) \geqslant q$ (recall that $\langle D_n\rangle$ is a decreasing sequence.) Let $v = \max\{N, M\}$. There is a t_0 such that $\mu(D_v[t_0]) > q - \varepsilon(\sigma)/2$. For any string $g \in \omega^{v+1}$ extending $f \upharpoonright N$ such that $g(n) \geqslant t_0$ for all $n \in [N, v+1)$, we have $\mu(D_{[N,|g|)}[g]) > q - \varepsilon(\sigma)/2$.

Consider a stage $t \geqslant \max\{t_0, x, v+1\}$ at which σ is accessible. Let $g = f \upharpoonright N^\frown\langle t\rangle^{v+1-N}$. Of course $f_t(n) \leqslant t$ for all n, so by the assumptions on e, $\varphi^g_{e,t}(x)\downarrow = 0$. Thus g satisfies all the conditions that make it eligible to be picked as a new initial segment of f. It follows that if σ did not act before stage t, then it does so and becomes permanently active. □

Claim 2.11. *$f = \lim_s f_s$ exists.*

Proof. Combining Claims 2.9(b) and 2.10, the intervals $\{[N_i, r_i)\}_{i\in\omega}$ partition ω. Furthermore, by Claim 2.9(a), f is stable on $[0, r_i)$ once σ_i becomes permanently active. Therefore, $\lim_s f_s(n)$ converges for all $n \in \omega$. □

Claim 2.12. *For all i, $\mu\left(\operatorname{dom}\Phi \smallsetminus D_{[N_i,r_i)}[f]\right) \leqslant \varepsilon(\sigma_i)$.*

Proof. Assume for a contradiction that $\mu\left(\operatorname{dom}\Phi \smallsetminus D_{[N_i,r_i)}[f]\right) > \varepsilon(\sigma_i)$. Let $\sigma_i = \tau^\frown\langle q, M\rangle$. Take $s \in \omega$ to be the stage at which σ_i becomes permanently active and let $g \in \omega^{<\omega}$ be the string that was used at that activation. So $r_i = |g|$ and $g \subset f$. This implies that $D_{[N_i,r_i)}[g] = D_{[N_i,r_i)}[f]$. But of course, $\operatorname{dom}\Phi \subseteq D_M$. Therefore, $\mu\left(D_M \smallsetminus D_{[N_i,r_i)}[g]\right) > \varepsilon(\sigma_i)$.

By the definition of the incompleteness strategy, $\mu(D_{[N_i,r_i)}[g]) > q - \varepsilon(\sigma_i)/2$. Also $r_i > M$, so $D_{[N_i,r_i)}[g] \subseteq D_M$. Together with the conclusion of the previous paragraph, we have $\mu(D_M) > q + \varepsilon(\sigma_i)/2$. But then $\mu(D_M[t]) > q + \varepsilon(\sigma_i)/2$, for any sufficiently large $t \in \omega$. Therefore, σ_i would be reset at the first phase of stage t, which is a contradiction. □

Claim 2.13. *f is uniformly a.e. dominating.*

Proof. Fix $e \in \omega$. The construction ensures that at most one incompleteness agent at each level can be active at a time; hence at most one can belong to G. Thus there is an $i \in \omega$ large enough that $(\forall j \geqslant i)\ |\sigma_j| \geqslant 2e+1$. Furthermore, $\sum_{j \geqslant i} \varepsilon(\sigma_j) \leqslant \sum_{k \geqslant e} 3^{-k} = 3^{-e+1}/2$ for this choice of i. By Claims 2.9(b) and 2.10, the intervals $\{[N_j, r_j)\}_{j \geqslant i}$ partition $[N_i, \infty)$. Therefore, if $Z \in \bigcap_{j \geqslant i} D_{[N_j, r_j)}[f]$, then f majorizes $\Phi(Z)$ above N_i. By Claim 2.12,

$$\mu\left(\operatorname{dom}\Phi \smallsetminus \bigcap_{j \geqslant i} D_{[N_j,r_j)}[f]\right) \leqslant \sum_{j \geqslant i} \mu\left(\operatorname{dom}\Phi \smallsetminus D_{[N_j,r_j)}[f]\right) \leqslant \frac{3^{-e+1}}{2}.$$

In other words, the set of $Z \in \operatorname{dom}\Phi$ such that f fails to dominate $\Phi(Z)$ has measure at most $3^{-e+1}/2$. But $e \in \omega$ was arbitrary, so f is uniformly a.e. dominating. □

Claim 2.14. *$f <_T \mathbf{0}'$.*

Proof. It is sufficient to prove that $B \not\leqslant_T f$. Fix an index $e \in \omega$.

Case 1: There is an R_e agent $\sigma_i \in G$. Let $s \in \omega$ be the last stage at which σ_i becomes active and let $x_i = x(\sigma_i)[s]$. By Claim 2.9(a), this is done via $g = f_{s+1} \upharpoonright r_i = f \upharpoonright r_i$. Because σ_i is activated, we know that $x_i \in B$ and $\varphi^g_{e,s}(x_i) = 0$. Therefore, $\varphi^f_e(x_i) = 0 \neq B(x_i)$.

Case 2: There is no agent for R_e in G. Let $\sigma = \tau^\frown\langle q, M\rangle$ be the incompleteness agent of length $2e+1$ on the true path. Assume that σ is initialized for the last time at stage $s \in \omega$. Let $x = x(\sigma)[s]$ and $N = N(\sigma)[s]$. Note that $x \notin B$, because σ does not become active after stage s (else $\sigma \in G$, so we would be in Case 1). Assume, for a contradiction, that $\varphi^f_e(x) = 0$. Take $g \in \omega^{<\omega}$ such that $|g| > \max\{M, N\}$, g is an initial segment of f, and $\varphi^g_e(x) = 0$.

By Claim 2.8, $\sigma' < \sigma$ is active at stage s iff $\sigma' \in G$. Choose $i \in \omega$ such that $\sigma_{i-1} < \sigma < \sigma_i$. In particular, $N = N_i$. Since σ is on the true path, we have $\sigma \subset \sigma_j$ for all $j \geqslant i$. This shows that $(\forall j \geqslant i)\ |\sigma_j| > 2e+1$. Now take $m \in \omega$ large enough that $r_m \geqslant |g|$. Then, $\sum_{j \in [i,m]} \varepsilon(\sigma_j) < \sum_{k > e} 3^{-k} = 3^{-e}/2$. By the same argument as given in Claim 2.13, $\mu\left(\operatorname{dom}\Phi \smallsetminus D_{[N,r_m)}[f]\right) < 3^{-e}/2$. Therefore, $\mu\left(\operatorname{dom}\Phi \smallsetminus D_{[N,|g|)}[g]\right) < 3^{-e}/2$. We know that $\mu(\operatorname{dom}\Phi) \geqslant q$. This proves that $\mu(D_{[N,|g|)}[g]) > q - 3^{-e}/2$.

Let $t > s$ be a stage at which σ is accessible that is large enough so that $\varphi^g_{e,t}(x)\downarrow = 0$. There is nothing stopping σ from acting at stage t, which is the desired contradiction. □

3. Reverse mathematics I: avoiding cone avoidance

Although the above c.e. construction (Section 2) does not seem to generalize to yield a cone avoidance result, Kjos-Hanssen showed that it does have a reverse mathematical consequence.

Theorem 3.1 (Kjos-Hanssen)**.** *There is an ω-model of $\mathsf{RCA}_0 + G_\delta$-$\mathsf{REG}$ that does not contain $\mathbf{0}'$. Hence G_δ-REG does not imply ACA_0 over RCA_0.*

Proof. We construct an ideal of Turing degrees that (as an ω-model) satisfies G_δ-REG but does not contain $\mathbf{0}'$. The ideal is the downward closure of an increasing sequence $\mathbf{a}_1 < \mathbf{h}_1 < \mathbf{a}_2 < \mathbf{h}_2 < \dots$. We let $\mathbf{a}_1 = \mathbf{0}$ and let $\mathbf{h}_1$ be the c.e. degree given by Theorem 1.2. The degree $\mathbf{h}_1$ is high. In the structure $\mathscr{D}[\mathbf{h}_1, \mathbf{h}_1']$ we can find some $\mathbf{a}_2$ that is low($\mathbf{h}_1$) and that joins $\mathbf{0}'$ to $\mathbf{h}_1' = \mathbf{0}''$ (Posner and Robinson [14]). Now in the structure $\mathscr{D}[\mathbf{a}_2, \mathbf{a}_2'] =$

$\mathscr{D}[\mathbf{a}_2, \mathbf{0}'']$, a relativized version of Theorem 1.2 yields a degree $\mathbf{h}_2 < \mathbf{0}''$ that is uniformly almost everywhere dominating over $\mathbf{a}_2$. We cannot have $\mathbf{h}_2 \geqslant \mathbf{0}'$ because $\mathbf{h}_2 \geqslant \mathbf{a}_2$ and $\mathbf{h}_2 < \mathbf{0}''$.

We now repeat. Again, using a relativized version of [14], we get an $\mathbf{a}_3 \in \mathscr{D}[\mathbf{h}_2, \mathbf{0}''']$ that is low($\mathbf{h}_2$) and joins $\mathbf{0}''$ to $\mathbf{0}'''$; and an $\mathbf{h}_3 \in \mathscr{D}[\mathbf{a}_3, \mathbf{0}''')$ that is uniformly a.e. dominating over $\mathbf{a}_3$. As before, $\mathbf{h}_3$ is not above $\mathbf{0}''$. But as $\mathbf{h}_3 \geqslant \mathbf{a}_2$ and $\mathbf{0}' \vee \mathbf{a}_2 = \mathbf{0}''$ we cannot have $\mathbf{h}_3 \geqslant \mathbf{0}'$. The process now repeats itself to get the rest of the sequence. □

4. A PROOF OF THEOREM 1.2 VIA A FORCING CONSTRUCTION

In this section we introduce a forcing notion that produces a uniformly a.e. dominating function and that allows us to obtain cone avoidance and more.

4.1. **The notion of forcing.** We approximate a function f^G. A *condition* is a pair $\langle f, \varepsilon \rangle$ where $f \in \omega^{<\omega}$ and ε is a positive rational. The idea is that $\mathbf{p} = \langle f, \varepsilon \rangle$ states that f is an initial segment of f^G and further $\mathbf{p}$ makes an *ε-promise*: the collection of $Z \in \operatorname{dom} \Phi$ such that f^G fails to majorize $\Phi(Z)$ from $|f|$ onwards has size $< \varepsilon$. Thus, an extension $g \supset f$ *respects the ε-promise* if

$$\mu\left(\operatorname{dom} \Phi \setminus D_{[|f|,|g|)}[g]\right) < \varepsilon.$$

However, this is not a good definition of a partial ordering on the conditions; we can have g keep the ε-promise of $\langle f, \varepsilon \rangle$ and h keep the δ-promise of $\langle g, \delta \rangle$ but fail to respect the ε-promise of $\langle f, \varepsilon \rangle$. Thus, the relation would not be transitive. A simple modification ensures that every h that keeps the δ-promise of $\langle g, \delta \rangle$ also keeps the ε-promise of $\langle f, \varepsilon \rangle$. We say that a condition $\langle g, \delta \rangle$ *extends* another condition $\langle f, \varepsilon \rangle$ if $f \subset g$, $\delta \leqslant \varepsilon$ and further, if $f \neq g$, then

$$\mu\left(\operatorname{dom} \Phi \setminus D_{[|f|,|g|)}[g]\right) + \delta < \varepsilon.$$

Lemma 4.1. *The extension relation is transitive.*

Proof. Suppose that $\langle g, \delta \rangle$ extends $\langle f, \varepsilon \rangle$ and is extended by $\langle h, \gamma \rangle$; we show that $\langle h, \gamma \rangle$ extends $\langle f, \varepsilon \rangle$. If either $f = g$ or $g = h$, then this is easy. Otherwise, the point is that

$$D_{[|f|,|h|)}[h] = D_{[|f|,|g|)}[g] \cap D_{[|g|,|h|)}[h]$$

and so

$$\mu\left(\operatorname{dom} \Phi \setminus D_{[|f|,|h|)}[h]\right) \leqslant \mu\left(\operatorname{dom} \Phi \setminus D_{[|f|,|g|)}[g]\right) + \mu\left(\operatorname{dom} \Phi \setminus D_{[|g|,|h|)}[h]\right) \leqslant (\varepsilon - \delta) + (\delta - \gamma) = \varepsilon - \gamma,$$

as required. □

Notation. We let $\mathbb{P}$ be the collection of all conditions. For a condition $\mathbf{p} = \langle f, \varepsilon \rangle$ we write $f^{\mathbf{p}} = f$ and $\varepsilon^{\mathbf{p}} = \varepsilon$. We also let $n^{\mathbf{p}} = |f^{\mathbf{p}}|$.

Lemma 4.2. *For all $n < \omega$, the set $\{\mathbf{p} \in \mathbb{P} : n^{\mathbf{p}} > n\}$ is dense in $\mathbb{P}$.*

Proof. Let $\mathbf{p} \in \mathbb{P}$. Let $n > n^{\mathbf{p}}$. For large enough s,

$$\mu(D_n \setminus D_n[s]) < \varepsilon^{\mathbf{p}}.$$

Now take $g \in \omega^n$ extending $f^{\mathbf{p}}$ such that $D_n[s] \subset D_{[n^{\mathbf{p}},n)}[g]$ (for example, by defining $g(m) = s$ for $m \geqslant n^{\mathbf{p}}$). As $\operatorname{dom} \Phi \subset D_n$, we get that $\mu\left(\operatorname{dom} \Phi \setminus D_{[n^{\mathbf{p}},n)}[g]\right) < \varepsilon^{\mathbf{p}}$. We can then pick some small δ so that $\langle g, \delta \rangle$ extends $\mathbf{p}$. □

If $G \subset \mathbb{P}$ is generic (from now, by the word "generic" we mean, "sufficiently generic for the given argument"), then we let

$$f^G = \bigcup_{\mathbf{p} \in G} f^{\mathbf{p}}.$$

The following is a corollary of Lemma 4.2:

Corollary 4.3. *If G is generic, then $f^G \in \omega^\omega$.*

We now show that the ε-promises are kept.

Lemma 4.4. *Let $\mathbf{p} \in \mathbb{P}$, and suppose that $\mathbf{p} \in G$ and that G is generic. Then*

$$\mu\left(\operatorname{dom}\Phi \setminus D_{[n^{\mathbf{p}},\omega)}\left[f^G\right]\right) \leqslant \varepsilon^{\mathbf{p}}.$$

Proof. The sequence $\langle D_{[n^{\mathbf{p}},m)}[f^G]\rangle_{m>n^{\mathbf{p}}}$ decreases with m and

$$D_{[n^{\mathbf{p}},\omega)}\left[f^G\right] = \bigcap_{m>n^{\mathbf{p}}} D_{[n^{\mathbf{p}},m)}\left[f^G\right].$$

So it is enough to prove that $\mu\left(\operatorname{dom}\Phi \setminus D_{[n^{\mathbf{p}},m)}\left[f^G\right]\right) \leqslant \varepsilon^{\mathbf{p}}$, for all $m > n^{\mathbf{p}}$. For any m, there is a $\mathbf{q} \in G$ extending $\mathbf{p}$ such that $n^{\mathbf{q}} \geqslant m$. By the definition of our partial ordering,

$$\mu\left(\operatorname{dom}\Phi \setminus D_{[n^{\mathbf{p}},n^{\mathbf{q}})}\left[f^{\mathbf{q}}\right]\right) < \varepsilon^{\mathbf{p}}.$$

But $D_{[n^{\mathbf{p}},n^{\mathbf{q}})}\left[f^{\mathbf{q}}\right] \subseteq D_{[n^{\mathbf{p}},m)}\left[f^G\right]$ because $f^{\mathbf{q}} \subset f^G$, which completes the proof. □

The following is immediate.

Lemma 4.5. *For all $\varepsilon > 0$, the set $\{\mathbf{p} \in \mathbb{P} : \varepsilon^{\mathbf{p}} < \varepsilon\}$ is dense in $\mathbb{P}$.* □

As a corollary,

Corollary 4.6. *If $G \subset \mathbb{P}$ is generic, then f^G is uniformly almost everywhere dominating.*

4.2. **Cone avoidance, etc.** We show that if G is generic, then indeed f^G has no special properties beyond domination. The following is the crucial technical lemma. Consider the proof that if g is Cohen generic over A and A is not computable, then g does not compute A. If some condition $\tau \in 2^{<\omega}$ forces that $\varphi_e^g = A$ (and in particular is total), then $A = \bigcup_{\sigma \supset \tau} \varphi_e^\sigma$ is computable because the collection of extensions of τ is computable. We would like to do the same, but our partial ordering is not computable. This difficulty is overcome as follows: given $\mathbf{p} \in \mathbb{P}$, we can make a promise ε^* much tighter than $\varepsilon^{\mathbf{p}}$ and find a rational q sufficiently close to $\operatorname{dom}\Phi$ such that every sufficiently long string $g \supset f^{\mathbf{p}}$ respecting the ε^*-promise satisfies $\mu(D_{[n^{\mathbf{p}},|g|)}[g]) > q$ and every string satisfying the latter (computable) condition respects the $\varepsilon^{\mathbf{p}}$-promise. We can now imitate the diagonalization argument (and more): if $\mathbf{p}$ forces that $\varphi_e^{f^G} = A$, then we compute A by examining φ_e^g for strings g satisfying the middle condition above. We argue that this must give us all of A, for otherwise we could extended $\mathbf{p}$ to keep the ε^*-promise and avoid $\varphi_e^{f^G} = A$.

Lemma 4.7. *Let $\mathbf{p} \in \mathbb{P}$. Then there is a c.e. set*

$$S \subset \{f^{\mathbf{q}} : \mathbf{q} \leqslant \mathbf{p}\}$$

and a $\mathbf{p}^* \leqslant \mathbf{p}$ such that $\{\mathbf{q} \leqslant \mathbf{p}^* : f^{\mathbf{q}} \in S\}$ is dense below $\mathbf{p}^*$.

Proof. Find some $n > n^{\mathbf{p}}$ such that $\mu\left(D_n \setminus \operatorname{dom}\Phi\right) < \varepsilon^{\mathbf{p}}/2$; also find a rational $q < \mu(\operatorname{dom}\Phi)$ such that $\mu(\operatorname{dom}\Phi) - q < \varepsilon^{\mathbf{p}}/2$. Let

$$S = \left\{g \in \omega^{<\omega} : g \supset f^{\mathbf{p}},\ |g| > n, \text{ and } \mu(D_{[n^{\mathbf{p}},|g|)}[g]) > q\right\}$$

It is clear that S is c.e. Let $g \in S$; we show that for some $\mathbf{q} \leqslant \mathbf{p}$ we have $f^{\mathbf{q}} = g$. We have $\mu(D_{[n^{\mathbf{p}},|g|)}[g]) > \mu(\operatorname{dom}\Phi) - \varepsilon^{\mathbf{p}}/2$ and $\mu(D_{|g|}) < \mu(\operatorname{dom}\Phi) + \varepsilon^{\mathbf{p}}/2$; together we get $\mu\left(D_{|g|} \setminus D_{[n^{\mathbf{p}},|g|)}[g]\right) < \varepsilon^{\mathbf{p}}$. Of course, $\operatorname{dom}\Phi \subset D_{|g|}$ and so $\mu\left(\operatorname{dom}\Phi \setminus D_{[n^{\mathbf{p}},|g|)}[g]\right) < \varepsilon^{\mathbf{p}}$.

Next, let $\mathbf{p}^* = \langle f^{\mathbf{p}}, \delta\rangle$ where $\delta < \varepsilon^{\mathbf{p}}$ (so $\mathbf{p}^* \leqslant \mathbf{p}$) and $\delta < \mu(\operatorname{dom}\Phi) - q$. Suppose that $\mathbf{q} \leqslant \mathbf{p}^*$ and $n^{\mathbf{q}} > n$. Then from

$$\mu\left(\operatorname{dom}\Phi \setminus D_{[n^{\mathbf{p}},n^{\mathbf{q}})}[f^{\mathbf{q}}]\right) < \delta$$

we can conclude that $\mu(D_{[n^{\mathbf{p}},n^{\mathbf{q}})}[f^{\mathbf{q}}]) > q$, so $f^{\mathbf{q}} \in S$. □

Lemma 4.8. *If A is noncomputable and $G \subset \mathbb{P}$ is generic over A, then $f^G \not\geqslant_T A$.*

Proof. Let $\Psi\colon \omega^\omega \to 2^\omega$ be a Turing functional. We show that the union of

$$E_0 = \{\mathbf{p} \in \mathbb{P} : \Psi(f^{\mathbf{p}}) \perp A\} \quad \text{and}$$
$$E_1 = \{\mathbf{p} \in \mathbb{P} : (\exists x)(\forall \mathbf{q} \leqslant \mathbf{p})\ \Psi(f^{\mathbf{q}},x)\uparrow\}$$

is dense in $\mathbb{P}$. Of course if $G \cap (E_0 \cup E_1) \neq \emptyset$, then $\Psi(f^G) \neq A$.

Let $\mathbf{p} \in \mathbb{P}$, and take S and $\mathbf{p}^*$ given by Lemma 4.7. If there are $g, g' \in S$ such that $\Psi(g) \perp \Psi(g')$, then one of them is incompatible with A, so $\mathbf{p}$ has an extension in E_0. If $\bigcup_{g\in S}\Psi(g)$ is total, then it is computable, hence different from A. Again, $\mathbf{p}$ has an extension in E_0.

Otherwise, for some $x \in \omega$, we have $\Psi(g,x)\uparrow$ for all $g \in S$. This implies that $\mathbf{p}^* \in E_1$: for all $\mathbf{q} \leqslant \mathbf{p}^*$, $f^{\mathbf{q}}$ has an extension in S, and so $\Psi(f^{\mathbf{q}},x)\uparrow$. □

Lemma 4.9. *If $G \subset \mathbb{P}$ is generic, then f^G does not have PA-degree.*

Proof. Let $\psi\colon \omega \to 2$ be a partial computable function that has no total computable extension. We show that f^G does not compute a 0-1 valued total extension of ψ.

Let $\Theta\colon \omega^\omega \to 2^\omega$ be a Turing functional. We show that the union of

$$E_0 = \{\mathbf{p} \in \mathbb{P} : (\exists x \in \operatorname{dom}\psi)\ \Theta(f^{\mathbf{p}},x)\downarrow \neq \psi(x)\} \quad \text{and}$$
$$E_1 = \{\mathbf{p} \in \mathbb{P} : (\exists x)(\forall \mathbf{q} \leqslant \mathbf{p})\ \Theta(f^{\mathbf{q}},x)\uparrow\}$$

is dense in $\mathbb{P}$. Of course if $G \cap (E_0 \cup E_1) \neq \emptyset$, then $\Theta(f^G)$ is not a total extension of ψ.

Let $\mathbf{p} \in \mathbb{P}$; take S and $\mathbf{p}^*$ given by Lemma 4.7. If there is a $g \in S$ such that $\Theta(g) \perp \psi$, then $\mathbf{p}$ has an extension in E_0. If for some x, $\Theta(g,x)\uparrow$ for all $g \in S$, then $\mathbf{p}^* \in E_1$.

One of the above must be the case; otherwise, we could compute a completion of ψ as follows: for each x, search for a $g \in S$ such that $\Theta(g,x)\downarrow$. For the first such g found, let $h(x) = \Theta(g,x)$. Then h is computable, and must extend ψ. □

In fact, the same proof gives us somewhat more:

Lemma 4.10. *If $G \subset \mathbb{P}$ is generic, then f^G does not have DNR-degree.*

Proof. Let $\langle \varphi_e\rangle_{e\in\omega}$ be an enumeration of all partial computable functions from ω to ω.

Let $\Psi\colon \omega^\omega \to \omega^\omega$ be a Turing functional. We show that the union of

$$E_0 = \{\mathbf{p} \in \mathbb{P} : (\exists e)\ \Psi(f^{\mathbf{p}},e)\downarrow = \varphi_e(e)\downarrow\} \quad \text{and}$$
$$E_1 = \{\mathbf{p} \in \mathbb{P} : (\exists x)(\forall \mathbf{q} \leqslant \mathbf{p})\ \Psi(f^{\mathbf{q}},x)\uparrow\}$$

is dense in $\mathbb{P}$. Of course if $G \cap (E_0 \cup E_1) \neq \emptyset$, then $\Psi(f^G)$ is not DNR.

Let $\mathbf{p} \in \mathbb{P}$, and take S and $\mathbf{p}^*$ given by Lemma 4.7. If there is a $g \in S$ and $e \in \omega$ such that $\Psi(g,e)\downarrow = \varphi_e(e)\downarrow$, then $\mathbf{p}$ has an extension in E_0. If there is an x such that $\Psi(g,x)\uparrow$ for all $g \in S$, then $\mathbf{p}^* \in E_1$.

Otherwise, define a total function $h\colon \omega \to \omega$ as follows: for each x, search for a $g \in S$ such that $\Psi(g,x)\downarrow$. Let $h(x) = \Psi(g,x)$ for the first such g discovered. Then h is computable and DNR, which is impossible. □

4.3. **Relativization.** Let $B \subset \omega$. All the results of this section relativize to working above B. Namely, we can define a notion of forcing $\mathbb{P}_B$; all is exactly as above, except that instead of D_n and $\operatorname{dom}\Phi$ we use $\{Z : \Phi(B \oplus Z, n)\downarrow\}$ and $\{Z : \Phi(B \oplus Z) \text{ is total}\}$. With exactly the same proofs, we see that a generic yields a function f^G that is uniformly almost everywhere dominating over B. Lemma 4.7 now becomes the following:

Lemma 4.11. *Let $\mathbf{p} \in \mathbb{P}_B$. Then there is a set S, c.e. in B, such that $S \subset \{f^{\mathbf{q}} : \mathbf{q} \leqslant \mathbf{p}\}$ and some $\mathbf{p}^* \leqslant \mathbf{p}$ such that $\{\mathbf{q} \leqslant \mathbf{p}^* : f^{\mathbf{q}} \in S\}$ is dense below $\mathbf{p}^*$.*

These are the analogous corollaries:

Lemma 4.12. *Suppose that $A \nleqslant_T B$ and that $G \subset \mathbb{P}_B$ is generic over A. Then $B \oplus f^G \ngeqslant_T A$.*

Lemma 4.13. *Suppose that B does not have PA-degree and that $G \subset \mathbb{P}_A$ is generic over B. Then $B \oplus f^G$ does not have PA-degree.*

Lemma 4.14. *Suppose that B is not DNR, and that $G \subset \mathbb{P}_B$ is generic. Then $B \oplus f^G$ is not DNR.*

5. Reverse mathematics II

The above forcing argument directly yields the results concerning the proof-theoretic strength of G_δ-REG.

Recall from Simpson [15] that $M \subseteq 2^\omega$ is an ω-model of RCA_0 iff it forms an ideal in the Turing degrees, and it is an ω-model of WKL_0 iff it is a *Scott system*: i.e., a Turing ideal such that for all $A \in M$, there is a $B \in M$ of PA-degree relative to A. Similarly, Yu and Simpson [18] proved that a Turing ideal $M \subseteq 2^\omega$ is an ω-model of WWKL_0 iff for all $A \in M$, there is a $B \in M$ that is sufficiently random over A (it is enough that B is 1-random relative to A by a result of Kučera [10]).

Proof of Theorem 1.4. An ideal of Turing degrees that models G_δ-REG but does not include any DNR degrees is easily built using Lemma 4.14. □

Proof of Theorem 1.6. For the first part, we can inductively construct an ω-model of $\mathsf{WKL}_0 + G_\delta$-REG that avoids $\mathbf{0}'$ by alternatively appealing to Lemma 4.12 and to the fact that a similar cone avoidance lemma holds for obtaining paths through trees, hence for PA-degrees (Jockusch and Soare [8, Theorem 2.5]).

For the second part, a similar construction yields an ω-model of $\mathsf{WWKL}_0 + G_\delta$-REG that does not satisfy WKL_0, this time using Lemma 4.13 and the following claim, which essentially appears in Yu and Simpson [18].

Claim 5.1. *Suppose that B does not have PA-degree. If A is sufficiently random over B, then $A \oplus B$ does not have PA-degree.*

Proof. This is from [18, Page 172]. Let E and F be disjoint c.e. sets that cannot be separated by any set computable in B. By relativizing a result from Jockusch and Soare [9], the measure of

$$S = \{Z : (\exists Y \leqslant_T Z \oplus B)\ E \subseteq Y \wedge F \cap Y = \emptyset\}$$

is zero. This is the collection of sets Z such that $Z \oplus B$ computes a separator of E and F. If A is sufficiently random over B, then $A \notin S$, meaning that it satisfies the claim. (In fact, since S is a $\Sigma^0_3(B)$-class, it suffices for A to be (weakly) 2-random relative to B [11].) □

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