DENSITY OF THE COTOTAL ENUMERATION DEGREES

JOSEPH S. MILLER AND MARIYA I. SOSKOVA

Abstract. We prove that the cototal enumeration degrees are exactly the enumeration degrees of sets with good approximations, as introduced by Lachlan and Shore [17]. Good approximations have been used as a tool to prove density results in the enumerations degrees, and indeed, we prove that the cototal enumerations degrees are dense.

1. Introduction

Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set. It has a complicated history, being intimately connected to the idea of computing with partial oracles. Equivalent forms of this reducibility have been introduced several times over the years, by Kleene [16], Myhill [21], and Selman [22], however Friedberg and Rogers [7] provided us with the most convenient way of thinking about it. A set A is *enumeration reducible* to a set B if there is a uniform way to enumerate A given any enumeration of B.

Definition 1.1. $A \leq_e B$ if and only if there is a c.e. set W such that

$$x \in A \Leftrightarrow (\exists v)[\langle x, v\rangle \in W \ \&\ D_v \subseteq B].$$

Here D_v denotes the finite set with code v in the standard coding of finite sets. The set W is called an *enumeration operator* and the pair $\langle x, v\rangle$ is called an *axiom* for x in W.

Enumeration reducibility induces a partial order, $\mathcal{D}_e$, called the enumeration degrees. This partial order attracts interest for, among other reasons, the fact that it contains a substructure that is isomorphic to the structure of the the Turing degrees: the total enumeration degrees.

Definition 1.2. A set A is *total* if $\overline{A} \leq_e A$. An enumeration degree is *total* if it contains a total set.

The structure of the Turing degrees, $\mathcal{D}_T$, properly embeds into $\mathcal{D}_e$, i.e., there are enumeration degrees that are not total. Once this was established, the questions that drove the study of the enumeration degrees were mostly aimed at understanding to what extent the two degree structures are alike and in which respects they differ. One question, in particular, that attracted a lot of interest was whether $\mathcal{D}_e$

2010 *Mathematics Subject Classification.* Primary 03D30.

Key words and phrases. good approximations,enumeration degrees, density, cototal enumeration degrees.

The first author was partially supported by grant #358043 from the Simons Foundation. The second author was partially supported by National Science Fund of Bulgaria grant #02/16 from 19.12.2016.

has minimal degrees: a degree is minimal if it is not the least degree, $\mathbf{0}_e$, but the only degree strictly below it is $\mathbf{0}_e$. In 1971, the Notices of the American Mathematical Society published two announcements: the first one by Cooper claimed the existence of a minimal enumeration degree; the second one by Gutteridge claimed that the enumeration degrees are dense. Both announcement turned out to be wrong, but also true to some extent. Gutteridge [12] proved that the enumeration degrees are downwards dense, while Cooper [6] showed that there are empty intervals of enumeration degrees.

In 1984 Cooper [5] showed that the Σ^0_2 enumeration degrees are a dense substructure of $\mathcal{D}_e$. In view of this result, he posed a challenge: identify the least level of the arithmetical hierarchy containing a set whose enumeration degree is the top of an empty interval, i.e., a minimal cover. To give a further incentive for his challenge, he added a conjecture: he believed that the Π^0_2 enumeration degrees are dense. Lachlan and Shore [17] extended Cooper's density result and at the same time significantly limited the possibilities for the existence of minimal covers: they introduced the notion of a *good approximation* and proved that degrees of sets with good approximations, *good* enumeration degree for short, cannot be or have minimal covers. They gave two examples of classes of good enumeration degrees: the total enumeration degrees and the enumeration degrees of the n-c.e.a. sets for every natural number n. On the other hand, they proved that there is a Π^0_2 set that does not have a good approximation, suggesting that Cooper's conjecture could turn out to be wrong. Finally, Calhoun and Slaman [3] resolved the question by constructing a Π^0_2 set whose enumeration degree is a minimal cover.

As a byproduct of this line of research, we obtained an important technical tool, the notion of a good approximation, and a big mystery—what is a natural characterization of the class of good enumeration degrees? The two known examples of classes of good degrees, the total enumeration degrees and the n-c.e.a. enumeration degrees, can be combined to show that the enumeration degrees of sets that are n-c.e.a. relative to any total oracle are good enumeration degrees. But does this exhaust the class of good enumeration degrees? On the other hand, good approximations were introduced as a way to extend Cooper's density results. Lachlan and Shore [17] do in fact obtain a density result using their tool: they show the density of the enumeration degrees of n-c.e.a. sets for every fixed n. A natural questions is, therefore, whether good approximations capture a class of enumeration degrees that is itself dense. Yet, without the right characterization of the good degrees, these questions did not seem approachable. It is not even immediately obvious that the good enumeration degrees are closed under join, i.e., that they form a substructure of $\mathcal{D}_e$. Nevertheless, structural properties of good enumeration degrees have been investigated. Cooper [5] introduced an enumeration jump operator that maps an enumeration degree $\mathbf{a}$ to a strictly larger total enumeration degree $\mathbf{a}'$. Griffiths [11] showed a jump inversion theorem involving the good enumeration degrees: if $\mathbf{w}$ is a good enumeration degree and $\mathbf{x}$ is an enumeration degree such that $\mathbf{x} < \mathbf{w} \leq \mathbf{x}'$, then there is a degree $\mathbf{a}$ such that $\mathbf{x} \leq \mathbf{a} < \mathbf{w}$ and $\mathbf{a}' = \mathbf{w}'$. This result was further refined by Harris [13].

We will give several useful characterizations of the good enumeration degrees by showing that they coincide with the cototal degrees, a class that has recently been shown to have many natural properties.

Definition 1.3. A set A is *cototal* if $A \leq_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

The cototal enumeration degrees form a proper substructure of $\mathcal{D}_e$ closed under join and the enumeration jump operator, although the latter is just because the jump of an enumeration degree is always total. Gutteridge [12] proved that the cototal enumeration degrees properly extend the substructure of the total enumeration degrees. Answering a question raised by Case [4], he proved that there is a quasiminimal cototal enumeration degree. Recall that a *quasiminimal* enumeration degree is a nonzero enumeration degree that bounds no total enumeration degree apart from $\mathbf{0}_e$. Andrews et al. [1] were the first to make an in-depth study of the cototal enumeration degrees. The authors introduced a new monotone operator on $\mathcal{D}_e$, *the skip operator*. The skip of an enumeration degree $\mathbf{a}$, denoted by $\mathbf{a}^{\diamond}$, is never below $\mathbf{a}$, but unlike the jump, it is not always above the degree $\mathbf{a}$. In fact, we have that $\mathbf{a}' = \mathbf{a} \vee \mathbf{a}^{\diamond}$. The skip inversion theorem proved in [1] shows that the range of the skip operator is the upper cone with base $\mathbf{0}_e'$. As there are non-total enumeration degrees in this cone, it follows that for many enumeration degrees $\mathbf{a} \not\leq \mathbf{a}^{\diamond}$. The cototal enumeration degrees are characterized as the degrees for which $\mathbf{a} \leq \mathbf{a}^{\diamond}$.

The cototal degrees have many other characterizations. Andrews et al. [1] proved that they are the enumeration degrees of complements of maximal independent sets for infinite computable graphs. McCarthy [18] gave several characterizations: they are the enumeration degrees of complements of maximal antichains in $\omega^{<\omega}$, the enumeration degrees of (uniformly) enumeration pointed trees, and the enumeration degrees of languages of minimal subshifts. Yet another characterization comes from computable analysis. Miller [19] introduced a reducibility between points in computable metric spaces. This reducibility gave rise to the structure of the continuous degrees, which Miller showed properly embeds into $\mathcal{D}_e$ and forms a proper extension of the Turing degrees. Andrews et al. [1] showed that the image of the continuous degrees is contained in the cototal enumeration degrees. Kihara and Pauli [15] extended Miller's reducibility to capture points in any represented topological space. In terms of this extension, Kihara[1] characterized the cototal enumeration degrees as the enumeration degrees of points in sufficiently effective second countable G_δ spaces (i.e., spaces in which every closed set is G_δ).

In this article, we prove that an enumeration degree is cototal if an only if it is good. Combining the power of good approximations with our understanding of cototality, we are able to prove the density of the cototal enumeration degrees.

2. Cototal enumeration degrees

We describe in more detail three characterizations of the cototal enumeration degrees that we will use.

The first characterization is related to a notion in graph theory.

Definition 2.1. An *independent set* for a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that no pair of distinct vertices in S is connected by an edge. An independent set is *maximal* if it has no proper independent superset.

In other words, an independent set S is maximal if and only if every vertex $v \in V$ is either in S or is connected by an edge to an element of S. In the graph

[1] Private communication.

theory literature, maximal independent sets are also sometimes called *independent dominating sets*. Maximal independent sets for finite graphs have a long history in graph theory. They were introduced in relation to a chessboard problem, posed in 1842 by Carl von Jaenisch, a top chess player and theorist. He wanted to find the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. For a survey of results on independence and domination for finite graphs we refer the reader to Goddard and Henning [10]. We are interested in maximal independent sets for infinite graphs, specifically for the graph $\omega^{<\omega}$. We will say that two strings σ and τ in $\omega^{<\omega}$ are *connected* if one is the immediate predecessor of the other. It is easy to see that if A is the complement of a maximal independent set for $\omega^{<\omega}$, then $A \leq_e \overline{A}$ in a uniform way. The enumeration operator witnessing this reduction simply enumerates every node connected to any $\sigma \in \overline{A}$. In fact, every cototal enumeration degree contains such a set.

Theorem 2.2 (Andrews et al. [1])**.** *An enumeration degree is cototal if and only if it contains the complement of a maximal independent set for the graph $\omega^{<\omega}$.*

The second characterization that we will use is related to a special type of tree. Recall that a set $T \subseteq 2^{<\omega}$ is a *tree* if it is closed under initial segments. A *path* in T is an infinite binary string $p \in 2^\omega$ such that every initial segment of p is a member of T. We can represent a path p by the linearly ordered set $P \subseteq 2^{<\omega}$ of all initial segments of p. Note, that if P represents a path in a tree $T \subseteq 2^{<\omega}$, then P is a total set: a finite binary string σ is in $\overline{P}$ if and only if there is a different string $\tau \in P$ of length $|\sigma|$. We will intentionally conflate a path $p \in 2^\omega$ with its representation $P \subseteq 2^{<\omega}$. A tree $T \subseteq 2^{<\omega}$ *has no dead ends* if every string $\sigma \in T$ is an initial segment of a path p in T.

Definition 2.3. A tree $T \subseteq 2^{<\omega}$ is *enumeration pointed* (*e-pointed*) if it has no dead ends and it is enumeration reducible to each of its infinite paths. If there is a single enumeration operator witnessing these reductions, we say that T is *uniformly* e-pointed.

McCarthy [18] proved that (uniformly) e-pointed trees are universal for the cototal enumeration degrees.

Theorem 2.4 (McCarthy [18])**.** *An enumeration degree is cototal*

- *if and only if it contains an e-pointed tree;*
- *if and only if it contains a uniformly e-pointed tree.*

Finally, we introduce the jump operator and the skip operator, and give a characterization of the cototal enumeration degrees in terms of them. Fix a set A. If one were asked to give the natural analog of the halting set relative to A in the context of enumeration reducibility, one might define $K_A = \{\langle e, x\rangle \mid x \in \Gamma_e(A)\}$, where $\{\Gamma_e\}_{e<\omega}$ is a standard listing of all enumeration operators. Since enumeration reducibility translates positive information into positive information, we have that $K_A \equiv_e A$. The complement of this set is the one that actually diagonalizes against enumeration reductions to A.

Definition 2.5. Let A be a set of natural numbers.

(1) The *skip* of A is $A^\diamond = \overline{K_A}$. The *skip* of the degree $d_e(A)$ is $d_e(A)^\diamond = d_e(A^\diamond)$.

(2) The enumeration *jump* of A is $A' = K_A \oplus \overline{K_A}$. The *jump* of the degree $d_e(A)$ is $d_e(A)' = d_e(A')$.

Theorem 2.6 (Andrews et al. [1]). *An enumeration degree* $\mathbf{a}$ *is cototal if and only if* $\mathbf{a} \leq \mathbf{a}^{\Diamond}$*, or equivalently if* $\mathbf{a}' = \mathbf{a}^{\Diamond}$.

3. Good approximations

In this section, we define good approximations and outline the properties that are useful for density results.

Definition 3.1 (Lachlan and Shore [17]). A good approximation to a set A is a computable sequence of finite sets $\{A_s\}_{s<\omega}$ with the following two properties:

G1. For every n there is a stage s such that $A \restriction n \subseteq A_s \subseteq A$;
G2. For every n there is a stage s such that for every $t \geq s$, if $A_t \subseteq A$ then $A \restriction n \subseteq A_t$.

Stages s such that $A_s \subseteq A$ are called *good stages.* Let G_A denote the set of good stages.

Equivalently, a good approximation to a set A is a uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ such that there are infinitely many stages s such that $A_s \subseteq A$ (i.e., G_A is infinite), and for every n we have that $A(n) = \lim_{s \in G_A} A_s(n)$.

If A has a good approximation $\mathbb{A} = \{A_s\}_{s<\omega}$ and $B = \Theta(A)$, then it is not necessarily true that B also has a good approximation. As mentioned earlier, there is a Π^0_2 set B with no good approximation, even though the set $B \oplus \overline{B}$ does have one, and clearly $B \leq_e B \oplus \overline{B}$. The sequence $\{B_s\}_{s<\omega}$, where $B_s = \Theta_s(A_s)$, does have infinitely many good stages: if $A_s \subseteq A$, then by the monotonicity of enumeration operators, we will have that $B_s \subseteq B$. The problem is that this sequence might have too many good stages—so many, in fact, that they violate the second property of a good approximation. Nevertheless, these approximations are useful. Soskov [24], who worked on ω-enumeration reducibility, a uniform extension of enumeration reducibility for sequences of sets, called them *correct approximations.*

Definition 3.2. Fix a good approximation $\mathbb{A} = \{A_s\}_{s<\omega}$ to a set A. We say that $\mathbb{B} = \{B_s\}_{s<\omega}$ is a *correct approximation* to a set B with respect to $\mathbb{A}$ if

C1. For every s, if $A_s \subseteq A$ then $B_s \subseteq B$;
C2. For every n there exists a stage s such that for every $t \geq s$, if $A_t \subseteq A$ then $B \restriction n \subseteq B_t$.

So if $B = \Theta(A)$, then $\{\Theta_s(A_s)\}_{s<\omega}$ is a correct approximation to B with respect to $\mathbb{A}$. Correct approximations allow us to detect equality between two sets by looking at the length of agreement between their approximations at good stages. Recall that the *length of agreement* $l(A, B, s)$ is defined as the maximal $n \leq s$ such that $A_s \restriction n = B_s \restriction n$.

Lemma 3.3 (Lachlan and Shore [17]). *Let* $\mathbb{A} = \{A_s\}_{s<\omega}$ *be a good approximation to a set* A*. Let* $\{B_s\}_{s<\omega}$ *and* $\{C_s\}_{s<\omega}$ *be correct approximations with respect to* $\mathbb{A}$ *to sets* B *and* C*. The following two statements are equivalent:*

(1) *The sequence* $\{l(B_s, C_s, s)\}_{s \in G_A}$ *is unbounded, and*
(2) $B = C$.

4. Goodness and cototality

We show that the good enumeration degrees and the cototal enumeration degrees coincide. Harris [13] proved the following property of sets with good approximations.

Proposition 4.1 (Harris [13, Proposition 4.1])**.** *If A has a good approximation, then $K_A \leq_e \overline{K_A}$.*

As $A \equiv_e K_A$, it follows that every good enumeration degree is cototal. We show that the reverse is also true.

Theorem 4.2. *An enumeration degree is cototal if and only if it contains a set with a good approximation.*

Proof. We show that every cototal enumeration degree contains a set with a good approximation. Let $\mathbf{a}$ be a cototal enumeration degree. By Theorem 2.4, we can fix a uniformly enumeration pointed tree $T \in \mathbf{a}$. Let Γ be the enumeration operator such that $T = \Gamma(P)$ for every infinite path P on T. Fix a computable listing $\{D_s\}_{s<\omega}$ of all finite nonempty downwards closed subsets of $2^{<\omega}$, ordered so that if $n < m$, then subsets of $2^{<n}$ appear before subsets of $2^{<m}$. Let $\{T_s\}_{s<\omega}$ be the uniformly computable sequence defined by $T_s = D_s \cup \bigcup_{\sigma \in D_s} \Gamma_s(\{\tau \mid \tau \subseteq \sigma\})$. We claim that $\{T_s\}_{s<\omega}$ is a good approximation to T.

First, note that if $D_s \subseteq T$ then $T_s \subseteq T$. Here we use the fact that T has no dead ends, so if $\sigma \in T$, then $\{\tau \mid \tau \preceq \sigma\} \subseteq P$ for some infinite path P on T and so $\Gamma_s(\{\tau \mid \tau \preceq \sigma\}) \subseteq \Gamma(P) = T$. This ensures that G_1 is satisfied: For a fixed natural number n, let s be such that $D_s = T \restriction n$. It follows, that $T \restriction n \subseteq T_s \subseteq T$.

To prove that the approximation satisfies G_2, we use compactness. If $\sigma \in T$, then there is a maximal length l_σ such that $\sigma \in \Gamma(\{\tau \in P \mid |\tau| < l_\sigma\})$ for every path P on T. So fix n and let $l = \max\{l_\sigma \mid \sigma \in T \restriction n\}$. There is a stage s_l such that if $t > s_l$, then D_t contains a binary string of length greater than or equal to l. Furthermore, as T has no dead ends and all D_t are nonempty, if $D_t \subseteq T$ then D_t contains an initial segment of an infinite path through T. So by our choice of l, we have $T \restriction n \subseteq T_t$. It follows that $T \restriction n \subseteq T_t \subseteq T$ at all good stages $t > s_l$. □

This characterization sheds light on the class of good enumeration degrees that was previously not accessible to us. For example, we now know that the good enumeration degrees form a substructure of the enumeration degrees, closed under the operations join, jump, and skip. Furthermore, we can show that this class captures more than the enumeration degrees of n-c.e.a. sets relative to total oracles. Recall, that a set A is 1-*c.e.a. relative to a total oracle* F if $A \equiv_e F$. If A is n-c.e.a. relative to F and B is c.e. in A, then $A \oplus B$ is $(n+1)$-*c.e.a. relative to* F. It follows that every n-c.e.a. set relative to a total oracle oracle F is enumeration above F and arithmetic in F. We mentioned already the result from [1] showing that every continuous enumeration degree is cototal and hence good. Miller [19] proved that every countable *Scott set*, i.e., a set of total enumeration degrees closed under join and the relation "PA in", can be realized as the set of total enumeration degrees below some nontotal continuous enumeration degree. Let $\mathbf{x}$ be a nontotal continuous enumeration degree such that the total enumeration degrees below $\mathbf{x}$ are exactly the arithmetical enumeration degrees. It follows that $\mathbf{x}$ is a good enumeration degree, but no member of $\mathbf{x}$ can be n-c.e.a. relative to any total oracle.

5. Density of the cototal enumeration degrees

The characterization proved in the previous section gives us new information about the structure of the cototal enumeration degrees. In the following theorem, we are finally able to utilize the full power of good approximations.

Theorem 5.1. *The cototal enumeration degrees are dense.*

Proof. Let U be a set of cototal enumeration degree. By Theorem 4.2, we may assume that U has a good approximation $\{U_s\}_{s<\omega}$. Let G_U denote the set of good stages in this approximation. Let V be a set of cototal enumeration degree such that $V <_e U$. Fix an approximation $\{V_s\}_{s<\omega}$ that is correct with respect to $\{U_s\}_{s<\omega}$. We will build an enumeration operator Θ such that $V <_e \Theta(U) \oplus V <_e U$. The set Θ will be built so that it satisfies the following list of requirements:

(1) To ensure that $V <_e \Theta(U) \oplus V$, we fix a computable listing of all enumeration operators $\{\Phi_i\}_{i<\omega}$ and satisfy
$$\mathcal{P}_i : \ \Theta(U) \neq \Phi_i(V);$$
(2) To ensure that $\Theta(U) \oplus V <_e U$, we fix a computable listing of all enumeration operators $\{\Psi_i\}_{i<\omega}$ and satisfy
$$\mathcal{Q}_i : \ U \neq \Psi_i(\Theta(U) \oplus V).$$

We order all requirements linearly: let $\mathcal{R}_{2n} = \mathcal{P}_n$ and let $\mathcal{R}_{2n+1} = \mathcal{Q}_n$.

To ensure that $\Theta(U) \oplus V$ is of cototal enumeration degree, by Theorem 2.2 it suffices to build Θ so that $\Theta(U)$ is the complement of a maximal independent set for the graph $\omega^{<\omega}$. We will construct Θ following these basic rules: No axiom for the empty string $\emptyset$ will be enumerated into Θ at any stage. This forces us to put every length-one string e into $\Theta(U)$. So we will enumerate into Θ axioms $\langle e, \emptyset\rangle$ for all strings e. The requirements $\mathcal{R}_i$ where $i \leq e$ will be the only ones allowed to enumerate axioms into Θ for nodes extending e. If an axiom for σ is enumerated at stage s, then the axiom is always $\langle \sigma, U_s\rangle$. Thus we can concentrate only on the good stages in the approximation to U, as these are the only stages at which we modify the set $\Theta(U)$. It will follow from the construction that each requirement acts only at finitely many good stages and contributes a finite set to $\Theta(U)$.

Let $e(\omega^{<\omega})$ denote the subgraph of $\omega^{<\omega}$ consisting of all strings extending some fixed element e. To ensure that $\Theta(U)$ is the complement of a maximal independent set, we will have a computable procedure M that tells us how to extend the finite set of strings in $e(\omega^{<\omega})$ that is enumerated into $\Theta(U)$ by the $\mathcal{R}$-strategies to a set that is the complement of a maximal independent set for $e(\omega^{<\omega})$. Given a finite subset F of $e(\omega^{<\omega})$, the procedure defines an infinite computable set $M(F)$ which serves this purpose. We inductively define the set $M_n(F)$ of strings of length n in $M(F)$. $M_0(F) = \emptyset$, $M_1(F) = \{e\}$. Suppose that we have defined $M_n(F)$. We add to $M_{n+1}(F)$ all strings σ of length $n+1$ extending e such that:

(1) $\sigma \in F$ or
(2) $\sigma^- \notin M_n(F)$, where σ^- is the immediate predecessor of σ.

We summarize some properties of this procedure in the following lemma.

Lemma 5.2. *Let F be a finite subset of $e(\omega^{<\omega})$.*

(1) *$\overline{M(F)}$ is a maximal independent set for $e(\omega^{<\omega})$.*
(2) *$M_2(F) \subseteq F$.*

(3) *If G is a finite subset of $e(\omega^{<\omega})$ such that $F \subseteq G \subseteq M(F)$, then $M(F) = M(G)$.*

Proof. To see that $\overline{M(F)}$ is independent, let τ and σ be two connected strings. We may assume that τ has length n and σ has length $n+1$. From the second point of the construction, it follows that if $\tau \notin M_n(F)$ then $\sigma \in M_{n+1}(F)$. To see that $\overline{M(F)}$ is maximal, fix any string $\tau \in M_n(F)$. There are only finitely many strings in F, hence there are, in fact, infinitely many strings σ extending τ such that $\sigma \notin M_{n+1}(F)$.

To see that $M_2(F) \subseteq F$, we note that if $|\sigma| = 2$, then $\sigma^- = e$ and hence $\sigma^- \in M_1(F)$. It follows that only the first clause of the inductive definition will apply to strings of length 2 in $e(\omega^{<\omega})$.

Finally, let $F \subseteq G \subseteq M(F)$. We show that $M_n(F) = M_n(G)$ by induction on n. The statement is clear for $n = 0$ and $n = 1$. Suppose that $M_n(F) = M_n(G)$ and fix $\sigma \in e(\omega^{<\omega})$ of length $n+1$. Note that $\sigma^- \notin M_n(F)$ if and only if $\sigma^- \notin M_n(G)$, so if σ is in either set for the second reason in the construction, it would automatically be in the other as well. If $\sigma \in F$, then as $F \subseteq G$ we have that $\sigma \in G$. On the other hand if $\sigma \in G$, as $G \subseteq M(F)$, it follows that $\sigma \in M_{n+1}(F)$. Therefore, $M_{n+1}(F) = M_{n+1}(G)$. $\square$

We are ready to describe the construction of Θ.

Construction. At stage 0, we set $\Theta_0 = \emptyset$. Suppose that we have constructed Θ_s. For every e, we will denote by $F_{e,s}$ the set of all strings in $\Theta_s(U_s) \cap e(\omega^{<\omega})$. At stage $s+1$, we add the axiom $\langle s, \emptyset\rangle$ to Θ_{s+1} and consider all requirements $\mathcal{R}_e$ where $e < s$ in turn. We have two cases:

(1) If $\mathcal{R}_e = \mathcal{P}_i$, then we activate the strategy for $\mathcal{P}_i$: We will denote by $l_{e,s}$ the length of agreement $l(\Theta_s(U_s), \Phi_{i,s}(V_s), s)$. For every $n < l_{e,s}$ such that $n \in U_s$, we add the axiom $\langle en, U_s\rangle$ to Θ_{s+1}. Intuitively, we are threatening to code U into $\Theta(U)$.
(2) If $\mathcal{R}_e = \mathcal{Q}_i$, then we activate the strategy for $\mathcal{Q}_i$: We will denote by $l_{e,s}$ the length of agreement $l(U_s, \Psi_i(\Theta_s(U_s) \oplus V_s), s)$. For every $n < l_{e,s}$, we check if there is an axiom $\langle n, E \oplus D\rangle \in \Psi_{i,s}$, such that $D \subseteq V_s$, $\emptyset \notin E$, and such that if $k\sigma \in E$ and $k < e$ then $k\sigma \in M(F_{k,s})$. If there is such an axiom, then we pick $\langle n, E \oplus D\rangle \in \Psi_{i,s}$ with least code (in some fixed computable coding of all possible axioms) and enumerate the axioms $\langle y, U_s\rangle$ into Θ_{s+1} for all $y \in E$ that start with numbers greater than or equal to e. Intuitively, we are threatening to make $\Theta(U)$ computable.

In both cases, we end by activating the procedure M for $F_{e,s}$: for every $\sigma \in M(F_{e,s})$ such that σ has code less than s, enumerate the axiom $\langle \sigma, U_s\rangle$ into Θ_{s+1}. $\dashv$

Let $\Theta = \bigcup_{s<\omega} \Theta_s$. We prove that Θ satisfies all requirements. The essence of the proof is contained in the following lemma.

Lemma 5.3. *For every e there is a good stage s_e such that at all stages $t \in G_U$ such that $t > s_e$:*

(1) *The length of agreement $l_{e,t}$ is bounded above by l_{e,s_e}.*
(2) *The strategy for $\mathcal{R}_e$ does not enumerate axioms for elements that are not already in $\Theta_{s_e}(U_{s_e})$.*
(3) *$M(F_{e,t}) = M(F_{e,s_e})$.*

Proof. We prove the lemma by simultaneous induction on e. Suppose the lemma is true for $i < e$. Let $s > \max(s_i : i < e)$ be a good stage in the approximation to U. Once again we have two cases depending on $\mathcal{R}_e$.

Suppose that $\mathcal{R}_e = \mathcal{P}_i$. There are only finitely many strings of the form en that are enumerated into $\Theta(U)$ by strategies different from $\mathcal{R}_e$. By construction, strategies with lower priority than $\mathcal{R}_e$ do not enumerate axioms for such strings at any stage. Strategies of higher priority can only enumerate such elements if they add axioms to Θ at good stages. By our choice of s, all axioms for these elements have already been enumerated by stage s. Let n be the largest such that en is added to $\Theta(U)$ by a higher priority strategy. Suppose towards a contradiction that the length of agreement $l_{e,t}$ is unbounded on good stages $t \in G_U$. It follows by Lemma 3.3 that $\Theta(U) = \Phi_i(V)$. On the other hand, we have that for all $m > n$,

$$m \in U \Leftrightarrow em \in \Theta(U).$$

Indeed, if $em \in \Theta(U)$ and $m > n$, then a valid axiom $\langle m, U_t\rangle$ was enumerated into Θ by $\mathcal{R}_e$. By Lemma 5.2, the procedure $M(F_{e,s})$ does not add axioms for elements of length 2, so the axiom must have been added in our attempt to code U. As the axiom is valid, it follows that $U_t \subseteq U$ and by construction $m \in U_t$. On the other hand, if $m \in U$ then by the second property of a good approximation, $m \in U_t$ at all sufficiently large good stages $t \in G_U$. As by assumption $\{l_{e,t}\}_{t \in G_U}$ is unbounded, there will be a good stage t at which $\mathcal{R}_e$ adds an axiom for m to Θ_t. Thus, we have that

$$U \leq_e \Theta(U) \leq_e V,$$

contradicting the fact that $V <_e U$. It follows that the length of agreement measured at good stages is bounded, say by a number l. Let $s_e > s$ be a good stage in the approximation to U such that at all good stages $t \geq s_e$ we have that $U \restriction l \subseteq U_t$ and $l_{e,t} \leq l_{e,s_e}$. The stage s_e satisfies the first two statements, as the strategy $\mathcal{R}_e$ will not have a reason to enumerate axioms for any element that it had not already enumerated an axiom for at stage s_e. The third statement also follows: after stage s_e, the only way in which we add elements to $e(\omega^{<\omega}) \cap \Theta(U)$ is through the procedure M, performed at good stages t. By a routine induction on the good stages $t > s_e$, we can show that $F_{e,s_e} \subseteq F_{e,t} \subseteq M(F_e, s_e)$, and hence by Lemma 5.2 we have that $M(F_{e,t}) = M(F_{e,s_e})$.

The second case we need to consider is $\mathcal{R}_e = \mathcal{Q}_i$. Towards a contradiction, suppose that the length of agreement $l_{e,t}$ is unbounded on good stages $t \in G_U$. It follows by Lemma 3.3 that $U = \Psi_i(\Theta(U) \oplus V)$. By induction, we have that at good stages $t > s$, the set $\bigcup_{k<e} M(F_{k,t}) = \bigcup_{k<e} M(F_{k,s}) = M_e$ remains constant. Furthermore, by construction and the actions of the procedure M, it follows that $M_e = \bigcup_{k<e} k(\omega^{<\omega}) \cap \Theta(U)$. On the other hand, by construction we have that $n \in \Psi_i(\Theta(U) \oplus V)$ if and only if $n \in \Psi_i\Big(\big(M_e \cup \bigcup_{k \geq e} k(\omega^{<\omega})\big) \oplus V\Big)$. Indeed, one direction of this equivalence follows from the fact that $\Theta(U) \subseteq M_e \cup \bigcup_{k \geq e} k(\omega^{<\omega})$. The other direction is ensured by the actions of the strategy for $\mathcal{Q}_i$. But now, as $M_e \cup \bigcup_{k \geq e} k(\omega^{<\omega})$ is a computable set, we have that

$$U = \Psi_i\left(\Big(M_e \cup \bigcup_{k \geq e} k(\omega^{<\omega})\Big) \oplus V\right) \leq_e V,$$

contradicting that $V <_e U$ once again. Hence, in this case as well, the length of agreement measured at good stages is bounded, say by a number l. Let m be such that for every $n < l$ the least valid axiom $\langle n, E \oplus D\rangle \in \Psi$ that puts n in the set $\Psi_i\Big(\big(M_e \cup \bigcup_{k\geq e} k(\omega^{<\omega})\big)\oplus V\Big) \upharpoonright l$ has the property that $D \subseteq V \upharpoonright m$. Fix a good stage $s_e > s$ such that at all good stages $t \geq s_e$ we have that $V \upharpoonright m \subseteq V_t$ (we can pick such a stage by the second property of correct approximations) and such that $l_{e,t} \leq l_{e,s_e}$. Note that by the end of stage s_e, all elements in $\Psi_i\Big(\big(M_e \cup \bigcup_{k\geq e} k(\omega^{<\omega})\big) \oplus V\Big) \upharpoonright l$ that will ever be added to $\Psi_i(\Theta(U) \oplus V)$ already have valid axioms. It follows that the strategy $\mathcal{Q}_i$ will not have any reason to enumerate valid axioms for elements different than the ones for which it enumerated axioms by stage s_e. In particular, once again we will have that after stage s_e, the only way in which we add elements to $e(\omega^{<\omega}) \cap \Theta(U)$ is through the procedure M performed at good stages t, and hence $M(F_{e,t}) = M(F_{e,s_e})$. Thus s_e satisfies all three statements. $\square$

An immediate corollary of Lemma 3.3 and part (1) of the previous lemma is that every requirement $\mathcal{R}_e$ is satisfied. Parts (2) and (3) of the lemma, along with the procedures M performed at every good stage, ensure that $\Theta(U) = \bigcup_e M(F_{e,s_e})$ and hence $\overline{\Theta(U)} = \bigcup_e \overline{M(F_{e,s_e})}$. By Lemma 5.2, each set $\overline{M(F_{e,s_e})}$ is a maximal independent set for $e(\omega^{<\omega})$, so $\overline{\Theta(U)}$ is a maximal independent set for $\omega^{<\omega}$. $\square$

6. Further properties of the cototal enumeration degrees

6.1. **Embedding countable partial orders in intervals of cototal enumeration degrees.** A variation of Cooper's density theorem was used by Bianchini [2] to show that nontrivial intervals of Σ^0_2 contain copies of all countable partial orders. Recently, Slaman and Sorbi [23] proved that the same is true for any nontrivial initial segment of the enumeration degrees. We describe how to modify the construction in Theorem 5.1 to prove the following.

Theorem 6.1. *If $\mathbf{v} < \mathbf{u}$ are cototal enumeration degrees, then one can embed every countable partial order in the cototal degrees in the interval $(\mathbf{v}, \mathbf{u})$.*

Proof. Let $V <_e U$ be cototal sets in $\mathbf{v}$ and $\mathbf{u}$ respectively, and suppose that U has a good approximation $\{U_s\}_{s<\omega}$. We follow the same method for embedding countable partial orders as is always used (see for instance [23]). We build a sequence $\{A_i\}_{i<\omega}$ of sets with the following properties:

(1) The A_i are cototal, uniformly in i;
(2) $\bigoplus_{i<\omega} A_i \leq_e U$;
(3) For every i we have $A_i \not\leq_e V \oplus (\bigoplus_{j\neq i} A_j)$, i.e., the sequence $\{V \oplus A_i\}_{i<\omega}$ is *computably independent*.

Here $\bigoplus_{i\in X} A_i$ is the set $\bigcup_{i\in X}\{i\} \times A_i$.

If $L = (\omega, \leq_L)$ is a computable partial ordering, then we can embed it in the interval $(\mathbf{v}, \mathbf{u})$ by mapping an element of the partial ordering i to the enumeration degree $\mathbf{v} \vee d_e(\bigoplus_{j\leq_L i} A_j)$. That this map preserves the order follow from the following straightforward claim:

Claim 6.2. If X and Y are computable, then $\bigoplus_{i\in X} A_i \leq_e V \oplus \bigoplus_{j\in Y} A_j \Leftrightarrow X \subseteq Y$.

To prove that this embedding has range consisting only of cototal enumeration degrees we must show:

Claim 6.3. If X is a computable set, then $\bigoplus_{i\in X} A_i$ is cototal.

The second claim also has a simple proof. Because X is computable and the A_i are cototal uniformly in i, we have $\bigoplus_{i\in X} A_i \leq_e \bigoplus_{i\in X} \overline{A}_i \equiv_e \bigoplus_{i\in X} \overline{A}_i \cup \bigoplus_{i\notin X} \mathbb{N} = \overline{\bigoplus_{i\in X} A_i}$. Finally, we note that Mostowski [20] proved that there is a computable partial order in which one can embed any other countable partial order.

To construct the sequence $\{A_i\}_{i<\omega}$, we build an enumeration operator Θ such that $\Theta(U)$ is the complement of a maximal independent set for $\omega^{<\omega}$, just like in Theorem 5.1. We let $A_i = \Theta(U) \cap \mathbb{N}^{[i]}(\omega^{<\omega})$; in other words, A_i will contain all strings that extends $\langle i,k\rangle$ for some natural number k and that are in $\Theta(U)$. In this way, we have that no two different elements of the sequence intersect, and so $\bigoplus_{i<\omega} A_i \equiv_e \bigcup_{i<\omega} A_i = \Theta(U)$. To shorten notation, let $B_i = \bigcup_{j\neq i} A_j$. We will ensure that the following requirements are satisfied:

$$\mathcal{P}_{i,k}: \quad A_i \neq \Psi_k(B_i \oplus V).$$

We order the requirements in a list $\mathcal{R}_0, \mathcal{R}_1 \dots$. To satisfy one requirement we will combine the actions that we had for the two types of requirements $\mathcal{P}$ and $\mathcal{Q}$ in the proof of Theorem 5.1 as follows.

Construction. At stage 0 we set $\Theta_0 = \emptyset$. Suppose that we have constructed Θ_s. For every e, we will denote by $F_{e,s}$ the set of all strings in $\Theta_s(U_s) \cap e(\omega^{<\omega})$. We further set $A_{i,s} = \bigcup_{j\in\omega} F_{\langle i,j\rangle,s}$ and $B_{i,s} = \bigcup_{j\neq i} A_{j,s}$. At stage $s+1$, we add the axiom $\langle s, \emptyset\rangle$ to Θ_{s+1} and consider all requirements $\mathcal{R}_e$ where $e < s$ in turn.

Suppose that $e = \langle i,k\rangle$, i.e., $\mathcal{R}_e = \mathcal{P}_{i,k}$. Let $l_{e,s} = l(A_{i,s}, \Psi_k(B_{i,s} \oplus V_s), s)$. For every $n < l_{e,s}$ such that $n \in U_s$, we add the axiom $\langle\langle i,k\rangle n, U_s\rangle$ to Θ_{s+1}. At the same time, for every $n < l_{e,s}$ we check if there is an axiom $\langle n, E \oplus D\rangle \in \Psi_{k,s}$ such that $D \subseteq V_s$, if $j\sigma \in E$ then $j \notin \mathbb{N}^{[i]}$, and if $j \leq e$ then $j\sigma \in M(F_{j,s})$. If there is such an axiom, then we pick $\langle n, E\oplus D\rangle \in \Psi_{i,s}$ with least code (in some fixed computable coding of all possible axioms) and enumerate the axiom $\langle y, U_s\rangle$ into Θ_{s+1} for all members y of E that start with numbers greater than e.

We end by activating the procedure M for $F_{e,s}$: for every $\sigma \in M(F_{e,s})$ such that σ has code less than s, enumerate the axiom $\langle \sigma, U_s\rangle$ into Θ_{s+1}. $\dashv$

Once again the essence of the proof is contained in the following lemma:

Lemma 6.4. *For every e there is a good stage s_e such that at all stages $t \in G_U$ such that $t > s_e$:*

(1) *The length of agreement $l_{e,t}$ is bounded above by l_{e,s_e}.*
(2) *The strategy for $\mathcal{R}_e$ does not enumerate axioms for elements that are not already in $\Theta_{s_e}(U_{s_e})$.*
(3) $M(F_{e,t}) = M(F_{e,s_e})$.

The proof of this lemma is essentially the same as the proof of Lemma 5.3, except for the following modification: If we assume that the sequence $\{l_{e,t}\}_{t\in G_u}$ is unbounded, where $e = \langle i,k\rangle$, then we have:

- Modulo a finite set, $n \in U$ if and only if the length 2 string $\langle i,k\rangle n$ is in A_i, and so $U \leq_e A_i$;
- $A_i = \Psi_k(B_{i,s} \oplus V_s)$;
- $\Psi_k(B_{i,s} \oplus V_s) = \Psi_k\Big(\big(M_e \cup \bigcup_{j\geq e} j(\omega^{<\omega})\big) \oplus V\Big)$, which is enumeration reducible to V because the set $M_e = \bigcup_{j<e} F_{j,s}$ is computable.

Once again this would contradict the assumption that $V <_e U$.

This proves that every requirement $\mathcal{R}_e$ is satisfied and $\Theta(U)$ is the complement of a maximal independent set for $\omega^{<\omega}$. To finish the proof of this theorem, we note that for every i the set A_i is cototal: $\sigma \in A_i$ if and only if σ starts with a number of the form $\langle i, j\rangle$ and σ is connected to a member of $\overline{A}_i$. $\square$

6.2. **Jumps of cototal enumeration degrees.** We end with some notes on the jumps of cototal enumeration degrees.

Proposition 6.5. *If $\mathbf{x}$ is cototal, then the interval $(\mathbf{x}, \mathbf{x}')$ consists entirely of cototal enumeration degrees.*

Proof. If $\mathbf{x}$ is cototal, then $\mathbf{x}^\Diamond = \mathbf{x}'$. If $\mathbf{w} \in (\mathbf{x}, \mathbf{x}')$, then $\mathbf{x}^\Diamond \leq \mathbf{w}^\Diamond$ and hence $\mathbf{w} \leq \mathbf{w}^\Diamond$, proving that $\mathbf{w}$ is cototal. $\square$

This proposition allows us to restate Griffiths' theorem for jumps of good enumeration degrees in the context of cototal enumeration degrees.

Theorem 6.6 (Griffiths [11])**.** *If $\mathbf{x}$ is cototal and $\mathbf{w}$ is such that $\mathbf{x} < \mathbf{w} \leq \mathbf{x}'$, then there is a (cototal) degree $\mathbf{a}$ such that $\mathbf{x} < \mathbf{a} < \mathbf{w}$ and $\mathbf{a}' = \mathbf{w}'$.*

Ganchev and Sorbi [8] proved that for every enumeration degree $\mathbf{w}$ there is an initial interval of enumeration degrees such that every nonzero element in that interval has the same jump as $\mathbf{w}$. Their proof is based on the following notion introduced by Kalimullin [14].

Definition 6.7 (Kalimullin [14])**.** A pair of sets of natural numbers $\{A, B\}$ forms a *$\mathcal{K}$-pair relative to $\langle X\rangle$* if there is a set $W \leq_e X$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. The pair $\{A, B\}$ is a *nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$* if, in addition, $A \not\leq_e X$ and $B \not\leq_e X$.

Here we write $\langle X\rangle$ to indicate that X is being treated as an enumeration oracle. Andrews et al. [1] investigated the skips of $\mathcal{K}$-pairs relative to an oracle. They proved the following:

Proposition 6.8 (Andrews et al. [1])**.** *If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$ and X is of cototal enumeration degree, then*

$$(A \oplus X)^\Diamond \equiv_e B \oplus X^\Diamond \quad \textit{and} \quad (B \oplus X)^\Diamond \equiv_e A \oplus X^\Diamond.$$

Even though relativization in the enumeration degrees does not always work, in this case we are able to prove that Ganchev and Sorbi's theorem relativizes above any cototal enumeration degree. We also extend Griffiths' theorem.

Theorem 6.9. *If $\mathbf{x}$ is cototal and $\mathbf{w}$ is such that $\mathbf{x} < \mathbf{w} \leq \mathbf{x}'$, then there is an enumeration degree $\mathbf{a}$ such that $\mathbf{x} < \mathbf{a} \leq \mathbf{w}$ and every degree in the interval $(\mathbf{x}, \mathbf{a}]$ has the same jump as $\mathbf{w}$.*

Proof. Fix $W \in \mathbf{w}$. Let L_{K_W} be the set of all finite binary strings σ such that σ is lexicographically smaller than $K_W \upharpoonright |\sigma|$. Let R_{K_W} be the complement of L_{K_W}. It is easy to check that $L_{K_W} \leq_e K_W$, $\{L_{K_W}, R_{K_W}\}$ is a $\mathcal{K}$-pair (relative to $\langle\emptyset\rangle$), and $L_{K_W} \oplus R_{K_W} \equiv_e W'$; proofs are given by Ganchev and Soskova [9].

If $L_{K_W} \leq_e X$ or $R_{K_W} \leq_e X$, then $L_{K_W} \oplus R_{K_W} \leq_e X'$ and so $W' \equiv_e X'$. In this case, every element in $[\mathbf{x}, \mathbf{w}]$ has the same jump as $\mathbf{w}$. So we may assume that $\{L_{K_W} \oplus X, R_{K_W} \oplus X\}$ is a nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$. By Proposition 6.8,

$(L_{K_W} \oplus X)^{\Diamond} \equiv_e R_{K_W} \oplus X^{\Diamond}$. As $L_{K_W} \oplus X$ is of cototal enumeration degree by Proposition 6.5, it follows that $(L_{K_W} \oplus X)^{\Diamond} \geq_e L_{K_W}$. Therefore,

$$W' \geq_e (L_{K_W} \oplus X)' \geq_e (L_{K_W} \oplus X)^{\Diamond} \geq_e L_{K_W} \oplus R_{K_W} \equiv_e W',$$

and so $(L_{K_W} \oplus X)^{\Diamond} \equiv_e W'$.

We claim that $\mathbf{a} = d_e(L_{K_W} \oplus X)$ satisfies the theorem. Take C with $d_e(C)$ in the interval $(\mathbf{x}, \mathbf{a}]$. Every degree in this interval is cototal, hence $C' \equiv_e C^{\Diamond}$. Kalimulin [14] showed that if $\{A, B\}$ is a $\mathcal{K}$-pair relative to $\langle X \rangle$ and $Y \leq_e A$, then $\{Y, B\}$ is also a $\mathcal{K}$-pair relative to $\langle X \rangle$. Therefore, $\{C, R_{K_W} \oplus X\}$ is a nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$. By Proposition 6.8, we have

$$C' \equiv_e C^{\Diamond} \equiv_e R_{K_W} \oplus X^{\Diamond} \equiv_e (L_{K_W} \oplus X)^{\Diamond} \equiv_e W'. \qquad \square$$

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(Miller, Soskova) Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

E-mail address, Miller: jmiller@math.wisc.edu

E-mail address, Soskova: msoskova@math.wisc.edu