ON WORK OF BARMPALIAS AND LEWIS-PYE: A DERIVATION ON THE D.C.E. REALS

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Let α and β be (Martin-Löf) random left-c.e. reals with left-c.e. approximations $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$. To compare the rates of convergence, consider[1]

$$\frac{\partial\alpha}{\partial\beta} = \lim_{s\to\infty} \frac{\alpha-\alpha_s}{\beta-\beta_s}. \tag{1}$$

Barmpalias and Lewis-Pye [2] recently proved that this limit exists and is independent of the choice of approximations to α and β. Furthermore, they showed that $\alpha-\beta$ is random if and only if $\partial\alpha/\partial\beta \neq 1$, and that

$$\begin{aligned} \frac{\partial\alpha}{\partial\beta} &= \sup\{c\in\mathbb{Q}\colon \alpha - c\,\beta \text{ is a left-c.e. real}\} \\ &= \inf\{c\in\mathbb{Q}\colon \alpha - c\,\beta \text{ is a right-c.e. real}\} \end{aligned} \tag{2}$$

These are beautiful results that clarify the behavior of random left-c.e. reals. It has long been understood that all random left-c.e. reals are "essentially interchangeable". One of the key arguments for this heuristic was given by Kučera and Slaman [8], who showed that, up to multiplicative constants, we cannot approximate one random left-c.e. real faster than another (see Lemma 1.1). The convergence of (1) shows more: all approximations to random left-c.e. reals converge in essentially the same way. This not only solidifies our belief that that random left-c.e. reals are interchangeable, but ironically, it gives us a useful way to contrast them. For example, it follows that $\partial\alpha/\partial\beta > 1$ if and only if $\alpha-\beta$ is a random left-c.e. real and $\partial\alpha/\partial\beta < 1$ if and only if $\alpha-\beta$ is a random right-c.e. real.

This note has three main purposes. The first two go hand in hand: to give relatively short, self-contained proofs of the results of Barmpalias and Lewis-Pye, and to extend them to the d.c.e. reals. This extension is easy; the main technical breakthrough is the convergence of (1). However, extending to the d.c.e. reals gives us a clearer picture.

Fix a random left-c.e. real Ω with left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$. We will use this as the benchmark against which we measure the convergence of other d.c.e. reals. If α is a d.c.e. real with d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \frac{\partial\alpha}{\partial\Omega} = \lim_{s\to\infty}\frac{\alpha-\alpha_s}{\Omega-\Omega_s}.$$

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[1] For reasons that will become clear, we use different notation than Barmpalias and Lewis-Pye [2]. They write $\mathcal{D}(\alpha,\beta)$ instead of $\partial\alpha/\partial\beta$.

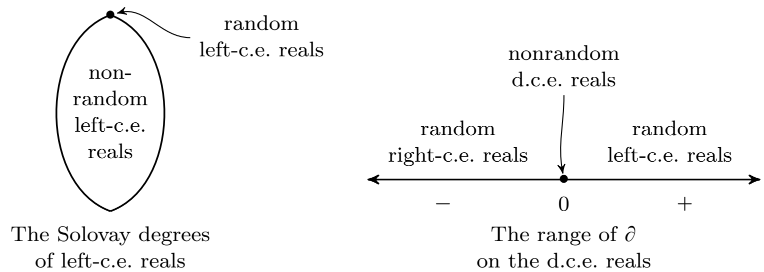

FIGURE 1.

We show that $\partial\alpha = 0$ if and only if α is nonrandom, $\partial\alpha > 0$ if and only if α is a random left-c.e. real, and $\partial\alpha < 0$ if and only if α is a random right-c.e. real. Note that implicit in this case breakdown is the fact, due to Rettinger and Zheng [12], that random d.c.e. reals must either be left-c.e. or right-c.e. (see Remark 1.4).

As we have telegraphed by our choice of notation (and the title of the paper), ∂ acts somewhat like differentiation. This should not be surprising; $\partial\alpha$ is, after all, defined as the limit of a difference quotient and is meant to capture the rate of convergence of $\{\alpha_s\}_{s\in\omega}$ to α. In fact, ∂ is a derivation on the field of d.c.e. reals.[2] In other words, ∂ preserves addition and satisfies the Leibniz law:

$$\partial(\alpha\beta) = \alpha\,\partial\beta + \beta\,\partial\alpha.$$

Furthermore, if $f\colon \mathbb{R} \to \mathbb{R}$ is a computable function that is differentiable at α, then $\partial f(\alpha) = f'(\alpha)\,\partial\alpha$. This allows us to apply basic identities from calculus, so for example, $\partial\alpha^n = n\alpha^{n-1}\,\partial\alpha$ and $\partial e^\alpha = e^\alpha\,\partial\alpha$. Since $\partial\Omega = 1$, we have $\partial e^\Omega = e^\Omega$.

The third purpose of this note is to investigate the nonrandom d.c.e. reals. Given a derivation on a field, the elements that it maps to zero also form a field: the *field of constants*. In our case, these are the nonrandom d.c.e. reals. We show that, in fact, the nonrandom d.c.e. reals form a *real closed field*. It was not even previously known that the nonrandom d.c.e. reals are closed under addition, and indeed, in Remark 3.2, we note that it is easy to prove the convergence of (1) from this fact. In contrast, it has long been known that the nonrandom left-c.e. reals are closed under addition (Demuth [5] and Downey, Hirschfeldt, and Nies [7]). While also nontrivial, this fact seems to be easier to prove. Towards understanding this difference, we show that the real closure of the nonrandom left-c.e. reals is strictly smaller than the field of nonrandom d.c.e. reals. In particular, there are nonrandom d.c.e. reals that cannot be written as the difference of nonrandom left-c.e. reals; despite being nonrandom, they carry some kind of intrinsic randomness.

We should compare the results above to the work on the Solovay degrees of left-c.e. reals. Solovay [13] introduced Solovay reducibility in his study of the halting probability of a universal prefix-free machine, the standard example of a random left-c.e. real [4]. As can be seen in Figure 1, the Solovay degrees are complementary to ∂; on the one hand, all random left-c.e. reals are Solovay equivalent [8],[3] while on the other hand, ∂ maps all nonrandom d.c.e. reals to 0 and distinguishes the random left-c.e. (and right-c.e.) reals. There is significant overlap, however, in what

[2]However, we will show that ∂ maps outside of the d.c.e. reals, so it does not make them a differential field.

[3]In fact, Rettinger and Zheng [14, 12] extended Solovay reducibility to the d.c.e. reals and showed that their notion retains this basic property, putting all randoms in the top degree.

the two approaches tell us about the random left-c.e. reals. For example, in their work on Solovay degrees, Downey, Hirschfeldt, and Nies [7] showed that a left-c.e. real β is random if and only if

> for every left-c.e. real α, there is a $c \in \omega$ and a left-c.e. real γ such that $c\beta = \alpha + \gamma$.

This follows easily from the work above: If β is random, then $\partial\beta > 0$. So given any left-c.e. real α, take c large enough that $c\,\partial\beta > \partial\alpha$. Then let $\gamma = c\beta - \alpha$ and note that $\partial\gamma > 0$, so it is left-c.e. For the other direction, if β is not random and α is, then for any c and any left-c.e. real γ, we have $\partial(c\beta) = 0 < \partial\alpha + \partial\gamma$.

1. Preliminaries

We assume that the reader is familiar with the basics of computability theory and effective randomness. See Downey and Hirschfeldt [6] and Nies [10] for background, including past work on random left-c.e. reals.

1.1. **Left-c.e. reals.** Let $\{\alpha_s\}_{s\in\omega}$ be a computable nondecreasing sequence of rationals converging to α. We say that $\{\alpha_s\}_{s\in\omega}$ is a *left-c.e. approximation* of the *left-c.e. real* α.[4] We define *right-c.e.* approximations and reals similarly. It is easy to see that a real is computable if and only if it is both a left-c.e. and a right-c.e. real.

As we have already hinted, the random left-c.e. reals are an interesting class. The key steps in understanding this class were made by Chaitin [4], Solovay [13], Calude, Hertling, Khoussainov, and Wang [3], and Kučera and Slaman [8]. Together, they showed that the following are equivalent:

- α is a random left-c.e. real,
- α is the halting probability of a universal prefix-free machine,
- Any left-c.e. approximation to α converges at least as slowly as any left-c.e. approximation to any other left-c.e. real.

The last of these conditions is made precise in the next lemma. It is somewhat stronger than saying that α is "Solovay complete", but since we do not need Solovay reducibility below, we will leave this hair unsplit.

Lemma 1.1 (Kučera and Slaman [8])**.** *Let α and β be a left-c.e. reals with left-c.e. approximations $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$. If β is random, then there is a $c \in \omega$ such that*

$$(\forall k)\ \alpha - \alpha_k \leqslant c\,(\beta - \beta_k)\,.$$

Proof. We define a Martin-Löf test $\{U_n\}_{n\in\omega}$. Fix n. We will build U_n in stages. At stage t, we will define $s(t)$ and put $[\beta_{s(t)}, \beta_{s(t)} + 2^{-n}(\alpha_{t+1} - \alpha_t)]$ into U_n. First, let $s(0) = 0$ and put $[\beta_0, \beta_0 + 2^{-n}(\alpha_1 - \alpha_0)]$ into U_n. At stage $t+1$, define $s(t+1) > s(t)$ such that $\beta_{s(t+1)}$ is no longer in the previous interval added to U_n. In other words, we have $\beta_{s(t+1)} > \beta_{s(t)} + 2^{-n}(\alpha_{t+1} - \alpha_t)$. Add the corresponding interval to U_n and complete the stage. Note that $\mu(U_n) \leqslant \sum_{t\in\omega} 2^{-n}(\alpha_{t+1} - \alpha_t) = 2^{-n}(\alpha - \alpha_0)$, so $\{U_n\}_{n\in\omega}$ is a Martin-Löf test (perhaps offset by a constant).

By assumption, β is random, so take n such that $\beta \notin U_n$. For this n, we add infinitely many intervals to U_n. Note that these intervals are all disjoint. In particular,

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[4]There is not broad agreement in the literature on what to call left-c.e. reals. They are often called "c.e. reals", as in Downey, Hirschfeldt, and Nies [7], or "left computable", as in Ambos-Spies, Weihrauch, and Zheng [1]. Several other names have been used, including "lower semicomputable". Both Downey and Hirschfeldt [6] and Nies [10] use "left-c.e.", so perhaps a consensus is forming.

for any k, we add disjoint intervals of lengths $2^{-n}(\alpha_{k+1}-\alpha_k), 2^{-n}(\alpha_{k+2}-\alpha_{k+1}), \ldots$ between $\beta_{s(k)}$ and β. Therefore, $\beta - \beta_k \geqslant \beta - \beta_{s(k)} \geqslant 2^{-n}(\alpha - \alpha_k)$. ◻

The next lemma is the main technical tool used in the rest of the paper.

Lemma 1.2 (Barmpalias and Lewis-Pye [2]). *Let α and β be left-c.e. reals with left-c.e. approximations $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$. If β is random, then*

$$\lim_{s\to\infty} \frac{\alpha-\alpha_s}{\beta-\beta_s} \text{ exists.}$$

Proof. Assume, for a contradiction, that the limit fails to exists. By Lemma 1.1, $\limsup_{s\to\infty}(\alpha-\alpha_s)/(\beta-\beta_s) < \infty$. On the other hand, all of the terms in the sequence are non-negative, so $\liminf_{s\to\infty}(\alpha-\alpha_s)/(\beta-\beta_s) \geqslant 0$. Therefore, there must be $c, d \in \mathbb{Q}$ such that

$$\liminf_{s\to\infty} \frac{\alpha-\alpha_s}{\beta-\beta_s} < c < d < \limsup_{s\to\infty} \frac{\alpha-\alpha_s}{\beta-\beta_s}.$$

In particular, there are infinitely many s such that $\alpha_s - d\beta_s < \alpha - d\beta$ and infinitely many t such that $\alpha_t - c\beta_t > \alpha - c\beta$. Fix such stages $s < t$. So

$$\alpha_t - c\beta_t > \alpha - c\beta = \alpha - d\beta + (d-c)\beta > \alpha_s - d\beta_s + (d-c)\beta.$$

Rearranging, we have

$$\beta < \frac{\alpha_t - \alpha_s + d\beta_s - c\beta_t}{d-c}.$$

The idea of the proof is to use such upper bounds to cover β with a Solovay test. The difficulty is that we cannot effectively determine which stages s and t satisfy our requirements, so we guess and update our guesses dynamically.

At stage t of the construction, first search for the largest $u < t$ such that $\alpha_u - c\beta_u \geqslant \alpha_t - c\beta_t$. If no such u exists, let $u = -1$. Now take the largest $s \in (u, t]$ minimizing $\alpha_s - d\beta_s$. We say that t is *absorbed* by s and we tentatively guess that s and t will give us an upper bound of β as described above (even though we may know better, for example, when $s = t$). We would like to add the interval

$$\left(\beta_s, \frac{\alpha_t - \alpha_s + d\beta_s - c\beta_t}{d-c}\right) \tag{3}$$

to the Solovay test, but this might cost too much, so we act more conservatively. First note that if $s = t$, then (3) is the empty interval (β_s, β_s), so we can "add" it to the Solovay test for free. Now consider $s < t$. Let $v \in [s, t)$ be the largest stage that has previously been absorbed by s. (It is not hard to see from our choice of s that s must have absorbed itself, so v is well-defined.) We claim that $\alpha_v - c\beta_v \geqslant \alpha_{t-1} - c\beta_{t-1}$. If not, then it must be the case that $s \leqslant v < t-1$ and

$$\alpha_v - c\beta_v < \alpha_{t-1} - c\beta_{t-1} < \alpha_t - c\beta_t.$$

(If the second inequality were false, then we would have picked $u = t-1$ and $s = t$.) The fact that both v and t are absorbed by s implies that $t-1$ should have also been absorbed by s, contradicting the choice of v.

Now assume inductively that our Solovay test contains the interval

$$\left(\beta_s, \frac{\alpha_v - \alpha_s + d\beta_s - c\beta_v}{d-c}\right).$$

We extend this to the desired interval from (3), which adds measure

$$\begin{aligned}\frac{(\alpha_t - c\beta_t) - (\alpha_v - c\beta_v)}{d-c} &\leqslant \frac{(\alpha_t - c\beta_t) - (\alpha_{t-1} - c\beta_{t-1})}{d-c} \\ &\leqslant \frac{(\alpha_t - \alpha_{t-1}) - c(\beta_t - \beta_{t-1})}{d-c} \leqslant \frac{\alpha_t - \alpha_{t-1}}{d-c}.\end{aligned}$$

Hence the total weight of the Solovay test is bounded by $\alpha/(d-c)$.

What remains is to prove that β is captured by the Solovay test. Pick s_0 to be the largest stage minimizing $\alpha_{s_0} - d\beta_{s_0}$, and $t_0 > s_0$ to be the least stage maximizing $\alpha_{t_0} - c\beta_{t_0}$ among stages greater than s_0. Note that t_0 is absorbed by s_0, so the corresponding interval is in the Solovay test. Also, it must be the case that $\alpha_{s_0} - d\beta_{s_0} < \alpha - d\beta$ and $\alpha_{t_0} - c\beta_{t_0} > \alpha - c\beta$, so β is contained in this interval. Now, pick $s_1 \geqslant t_0$ to be the greatest stage minimizing $\alpha_{s_1} - d\beta_{s_1}$ and $t_1 > s_1$ to be the least stage maximizing $\alpha_{t_1} - c\beta_{t_1}$. Again, β is contained in the corresponding interval, which in turn, is in the Solovay test. Continuing in this way, β fails the Solovay test, which is a contradiction. □

1.2. **D.c.e. reals.** If β and γ are left-c.e. reals, we call $\alpha = \beta - \gamma$ a *d.c.e. real.*[5] Let $\{\beta_s\}_{s\in\omega}$ and $\{\gamma_s\}_{s\in\omega}$ be left-c.e. approximations of β and γ, respectively. If we set $\alpha_s = \beta_s - \gamma_s$, then not only do we have $\lim_{s\to\infty} \alpha_s = \alpha$, but the *variation* of the approximation is finite, i.e.,

$$\begin{aligned}\sum_{s\in\omega} |\alpha_{s+1} - \alpha_s| &= \sum_{s\in\omega} |(\beta_{s+1} - \beta_s) - (\gamma_{s+1} - \gamma_s)| \\ &\leqslant \sum_{s\in\omega} |\beta_{s+1} - \beta_s| + \sum_{s\in\omega} |\gamma_{s+1} - \gamma_s| = \beta + \gamma < \infty.\end{aligned}$$

This characterizes the d.c.e. reals.

Proposition 1.3 (Ambos-Spies, Weihrauch, and Zheng [1])**.** *A real α is d.c.e. if and only if it is the limit of a computable sequence $\{\alpha_s\}_{s\in\omega}$ of rationals such that*

$$\sum_{s\in\omega} |\alpha_{s+1} - \alpha_s| < \infty.$$

In this case, we call $\{\alpha_s\}_{s\in\omega}$ a d.c.e. approximation *of α.*

Proof. We proved one direction above. Now assume that α is the limit of a sequence $\{\alpha_s\}_{s\in\omega}$ with finite variation. Let $\beta = \alpha_0 + \sum\{\alpha_{s+1} - \alpha_s \colon \alpha_{s+1} - \alpha_s \geqslant 0\}$ and $\gamma = \sum\{\alpha_s - \alpha_{s+1} \colon \alpha_{s+1} - \alpha_s < 0\}$. Since $\{\alpha_s\}_{s\in\omega}$ has finite variation, both β and γ are finite. It should be clear that they are left-c.e. reals and that $\alpha = \beta - \gamma$. □

It is evident that the d.c.e. reals are closed under addition and subtraction and not too hard to see that they form a field [1]. Ng [9] and Raichev [11] independently proved that they actually form a *real closed field*; this just means that the real roots of a polynomial whose coefficients are d.c.e. reals must also be d.c.e. reals.

Rettinger and Zheng [12] observed that d.c.e. approximations of random reals are severely limited.

[5] D.c.e. is short for "difference of computably enumerable", which is admittedly an imperfect name because it is too easy to confuse d.c.e. *reals* with d.c.e. *sets*. As with "left-c.e.", various other terms have been used in the literature. Many sources, including Ambos-Spies, Weihrauch, and Zheng [1], call them "weakly computable" real numbers, which is not particularly descriptive. On the other hand, Downey and Hirschfeldt [6] call them "left-d.c.e.", while admitting that "d.l.c.e." would make somewhat more sense. Indeed, Nies [10] calls them "difference left-c.e.".

Remark 1.4 (Rettinger and Zheng [12]). Let $\{\alpha_s\}_{s\in\omega}$ be a d.c.e. approximation of α. Consider the Solovay test $\{[\alpha_s,\alpha_{s+1}]\colon \alpha_s<\alpha_{s+1}\}$; note that it has finite weight because $\{\alpha_s\}_{s\in\omega}$ has finite variation. If there are infinitely many s such that $\alpha_s<\alpha$ and infinitely many t such that $\alpha_t>\alpha$, then α would be covered by the test, hence it would be nonrandom.

Now assume that α is random. We know that all but finitely many of the elements of the approximation fall on the same side of α. Assume, for the sake of argument, that there is an $s* \in \omega$ such that $(\forall s \geqslant s*)\ \alpha_s<\alpha$. Then $\alpha^*_s=\max_{s*\leqslant t\leqslant s}\alpha_t$ is a left-c.e. approximation of α, so α is a left-c.e. real. Similarly, if we assume that almost all elements of the approximation are greater than α, then α is a right-c.e. real. Note that α cannot be both a left-c.e. real and a right-c.e. real or it would be computable, and hence not random. So if we know that α is a random left-c.e. real, then we know that $\alpha_s<\alpha$ for almost all s. □

Proposition 1.5 (Rettinger and Zheng [12]). *Random d.c.e. reals are either left-c.e. reals or right-c.e. reals.*

We finish with what is essentially the converse of Remark 1.4: nonrandom d.c.e. reals have "properly" d.c.e. approximations.

Lemma 1.6. *Let α be a nonrandom d.c.e. real. There is a d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$ of α such that there are infinitely many s for which $\alpha_s<\alpha$ and infinitely many t for which $\alpha_t>\alpha$.*

Proof. Let $\{\alpha^*_s\}_{s\in\omega}$ be a d.c.e. approximation of α. Let $\{[c_n,d_n]\}_{n\in\omega}$ be a Solovay test that covers α, viewed as a sequence of rational intervals. We define our new approximation of α as follows. At stage s, check if α^*_s is contained in some *unused* interval $[c_n,d_n]$ for $n\leqslant s$. If so, mark that interval *used* and let $\alpha_{4s}=\alpha_{4s+3}=\alpha^*_s$, $\alpha_{4s+1}=c_n$, and $\alpha_{4s+2}=d_n$. Otherwise, let $\alpha_{4s}=\cdots=\alpha_{4s+3}=\alpha^*_s$.

Note that the variation of $\{\alpha_s\}_{s\in\omega}$ is bounded by the variation of $\{\alpha^*_s\}_{s\in\omega}$ plus the extra variation added when intervals are used. When an interval $[c_n,d_n]$ is used, it adds $2|d_n-c_n|$ to the variation. Each interval in the Solovay test is used at most once, so the contribution of all such intervals is bounded by twice the weight of the test. So $\{\alpha_s\}_{s\in\omega}$ has finite variation, which implies that it converges. Since there is a subsequence converging to α, this must be the limit. Therefore, $\{\alpha_s\}_{s\in\omega}$ is a d.c.e. approximation of α.

Now note that if an interval in the Solovay test contains α, then it will eventually be used. If such an interval is used at stage s, then $\alpha_{4s+1}<\alpha$ and $\alpha_{4s+2}>\alpha$. Since there are infinitely many such intervals, the lemma is proved. □

2. A derivation on the d.c.e. reals

As before, fix a random left-c.e. real Ω with left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$.

Definition 2.1. If α is a d.c.e. real with d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha=\lim_{s\to\infty}\frac{\alpha-\alpha_s}{\Omega-\Omega_s}.$$

To justify this definition, we must prove that the limit is independent of the choice of approximation. Before we have done so, it will be convenient to write $\partial\{\alpha_s\}$ instead of $\partial\alpha$. In light of the results from the previous section, the basic properties of $\partial\{\alpha_s\}$ are now fairly easy to prove.

First, note that we get linearity, a fact also observed by Barmpalias and Lewis-Pye [2]: if α and β are d.c.e. reals with d.c.e. approximations $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$, respectively, then

$$\begin{aligned}\partial\{\alpha_s+\beta_s\} &= \lim_{s\to\infty}\frac{(\alpha+\beta)-(\alpha_s+\beta_s)}{\Omega-\Omega_s}\\ &= \lim_{s\to\infty}\frac{\alpha-\beta_s}{\Omega-\Omega_s}+\lim_{s\to\infty}\frac{\beta-\beta_s}{\Omega-\Omega_s}=\partial\{\alpha_s\}+\partial\{\beta_s\}.\end{aligned}$$

Similarly, if c is rational, then $\partial\{c\alpha_s\} = c\,\partial\{\alpha_s\}$.

Lemma 2.2. *Let α be a d.c.e. real with d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$.*

(a) *$\partial\{\alpha_s\}$ converges.*
(b) *If $\partial\{\alpha_s\} > 0$, then α is a left-c.e. real.*
(c) *If $\partial\{\alpha_s\} < 0$, then α is a right-c.e. real.*
(d) *If $\alpha = 0$, then $\partial\{\alpha_s\} = 0$.*
(e) *If $\{\alpha^*_s\}_{s\in\omega}$ is another d.c.e. approximation of α, then $\partial\{\alpha_s\} = \partial\{\alpha^*_s\}$.*

Proof. As in the proof of Proposition 1.3, let β and γ be left-c.e. reals with left-c.e. approximations $\{\beta_s\}_{s\in\omega}$ and $\{\gamma_s\}_{s\in\omega}$ such that $\alpha_s = \beta_s - \gamma_s$ for all s. Then $\alpha = \beta - \gamma$ and $\partial\{\alpha_s\} = \partial\{\beta_s\} - \partial\{\gamma_s\}$. Both $\partial\{\beta_s\}$ and $\partial\{\gamma_s\}$ converge by Lemma 1.2, so $\partial\{\alpha_s\}$ also converges. This proves (a).

For (b), note that if $\partial\{\alpha_s\} > 0$, then there is an $s* \in \omega$ such that $(\forall s \geqslant s*)\ \alpha_s < \alpha$. Hence by the argument in Remark 1.4, α is a left-c.e. real. Part (c) is proved similarly.

To prove (d), assume that $\alpha = 0$ but $\partial\{\alpha_s\} \neq 0$. Pick an integer c such that $\partial\{\Omega_s + c\alpha_s\} = \partial\{\Omega_s\} + c\,\partial\{\alpha_s\} = 1 + c\,\partial\{\alpha_s\} < 0$. But $\{\Omega_s + c\alpha_s\}_{s\in\omega}$ is a d.c.e. approximation of $\Omega + c\cdot 0 = \Omega$, so by part (c), Ω is a right-c.e. real. This implies that Ω is computable, which is a contradiction.

Finally, to prove (e), note that $\partial\{\alpha_s\} - \partial\{\alpha^*_s\} = \partial\{\alpha_s - \alpha^*_s\} = 0$, because $\{\alpha_s - \alpha^*_s\}_{s\in\omega}$ is a d.c.e. approximation of 0. □

Theorem 2.3. *Let α be a d.c.e. real.*

(a) *$\partial\alpha$ converges and does not depend on the d.c.e. approximation of α.*
(b) *$\partial\alpha = 0$ if and only if α is not random.*
(c) *$\partial\alpha > 0$ if and only if α is a random left-c.e. real.*
(d) *$\partial\alpha < 0$ if and only if α is a random right-c.e. real.*
(e) $\partial\alpha = \sup\{c \in \mathbb{Q} \colon \alpha - c\,\Omega$ *is left-c.e.*$\}$
$= \inf\{c \in \mathbb{Q} \colon \alpha - c\,\Omega$ *is right-c.e.*$\}$.

Proof. Part (a) is immediate from the previous lemma. Now assume that α is not random. Let $\{\alpha_s\}_{s\in\omega}$ be the approximation guaranteed by Lemma 1.6. So there are infinitely many s for which $\alpha - \alpha_s > 0$ and infinitely many t for which $\alpha - \alpha_t < 0$. This implies that $\partial\alpha = 0$.[6] On the other hand, if α is random, then by Proposition 1.5, it must be either a left-c.e. real or a right-c.e. real. Assume that α is a random left-c.e. real. By Lemma 1.1, there is a $c \in \omega$ such that

$$(\forall s)\ \Omega - \Omega_s \leqslant c\,(\alpha - \alpha_s)\,.$$

[6] An alternate proof might appeal to those familiar with Solovay reducibility: we can show that if $\partial\alpha \neq 0$, then we can extract good approximations of Ω from good approximations of α; hence, if α were not random, then we could derandomize Ω.

This implies that $\partial\alpha > 1/c > 0$. Similarly, if α is a random right-c.e. real, then $\partial\alpha < 0$. This proves part (b) and the "if" directions of parts (c) and (d). The "only if" directions also follow. For example, if $\partial\alpha > 0$, then α is random by (b) and left-c.e. by the previous lemma.

Finally, (e) follows from parts (c) and (d) and the fact that $\partial(\alpha - c\Omega) = \partial\alpha - c$. ☐

We have now recovered the work of Barmpalias and Lewis-Pye [2] that was discussed in the introduction. Note that we have lost nothing by working with Ω as a fixed benchmark; it is easy to see that if β is a random d.c.e. real, then

$$\frac{\partial\alpha}{\partial\beta} = \frac{\partial\alpha/\partial\Omega}{\partial\beta/\partial\Omega}.$$

Therefore, $\partial\alpha/\partial\beta$ is not ambiguous: it can either be defined as in equation (1), or as a ratio of derivations as in Definition 2.1.

Next, we show that ∂ is a derivation on the field of d.c.e. reals; in other words, that it respects addition and satisfies the Leibniz law.

Theorem 2.4. *Let α, β be d.c.e. reals.*

(a) $\partial(\alpha + \beta) = \partial\alpha + \partial\beta$.
(b) $\partial(\alpha\beta) = \alpha\,\partial\beta + \beta\,\partial\alpha$.

Proof. We proved (a) above. The proof for (b) is standard and simple:

$$\begin{aligned}\partial(\alpha\beta) &= \lim_{s\to\infty} \frac{\alpha\beta - \alpha_s\beta_s}{\Omega - \Omega_s} \\ &= \lim_{s\to\infty} \alpha\left(\frac{\beta - \beta_s}{\Omega - \Omega_s}\right) + \lim_{s\to\infty} \beta_s\left(\frac{\alpha - \alpha_s}{\Omega - \Omega_s}\right) = \alpha\,\partial\beta + \beta\,\partial\alpha.\end{aligned}$$ ☐

We also get the following version of the chain rule.

Theorem 2.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a computable function. If f is differentiable at the d.c.e. real α, then*

(a) *$f(\alpha)$ is d.c.e., and*
(b) *$\partial f(\alpha) = f'(\alpha)\,\partial\alpha$.*

Proof. Let $\{\alpha_s\}_{s\in\omega}$ be a d.c.e. approximation of α. If f were sufficiently nice, then $\{f(\alpha_s)\}_{s\in\omega}$ would be a d.c.e. approximation of $f(\alpha)$. In particular, it would be enough to assume that f is Lipschitz in some neighborhood of α, which is true for any continuously differentiable function. For the stated generality, assume only that f is differentiable at α. Hence it is continuous at α and there is an $\varepsilon > 0$ and a $c \in \omega$ such that

$$(\forall x \in \mathbb{R})\ |\alpha - x| < \varepsilon \implies |f(\alpha) - f(x)| < c|\alpha - x|.$$

Fix N large enough that $(\forall n \geqslant N)\ |\alpha - \alpha_n| < \varepsilon$. Let $n(0) = N$. If $n(s)$ has been defined, let $n(s+1) > n(s)$ be chosen so that $|f(\alpha_{n(s+1)}) - f(\alpha_{n(s)})| < c|\alpha_{n(s+1)} - \alpha_{n(s)}|$. Note that $n(s+1)$ exists because $\{\alpha_s\}_{s\in\omega}$ converges to α and f is continuous at α. In this way, we get an approximation $\{f(\alpha_{n(s)})\}_{s\in\omega}$ of $f(\alpha)$. It is a d.c.e. approximation because its variation is at most c times the variation of $\{\alpha_{n(s)}\}_{s\in\omega}$. This proves (a).

For (b), let $\{\alpha_s^*\}_{s\in\omega} = \{\alpha_{n(s)}\}_{s\in\omega}$ be the d.c.e. approximation of α from the previous paragraph. Then

$$\begin{aligned}\partial f(\alpha) &= \lim_{s\to\infty} \frac{f(\alpha) - f(\alpha_s^*)}{\Omega - \Omega_s} \\ &= \left(\lim_{s\to\infty} \frac{f(\alpha) - f(\alpha_s^*)}{\alpha - \alpha_s^*}\right)\left(\lim_{s\to\infty} \frac{\alpha - \alpha_s^*}{\Omega - \Omega_s}\right) = f'(\alpha)\,\partial\alpha.\end{aligned}$$

□

The previous theorem allows us to apply basic identities from calculus, so for example, $\partial e^{\Omega} = e^{\Omega}$.

As already noted, ∂ does not make the d.c.e. reals into a differential field; it is straightforward to show that ∂ maps outside of the d.c.e. reals, though we do not know its range.

Proposition 2.6. *If β is a positive Δ^0_2 real, then there is a left-c.e. real α such that $\partial\alpha = \beta$.*

Proof. Let $\{\beta_s\}_{s\in\omega}$ be an approximation of β; we may assume that is consists only of positive rationals. Define a left-c.e. approximation $\{\alpha_s\}_{s\in\omega}$ as follows: let $\alpha_0 = 0$ and $\alpha_{s+1} = \alpha_s + \beta_s(\Omega_{s+1} - \Omega_s)$. The fact that $\{\beta_s\}_{s\in\omega}$ is bounded above implies that $\alpha = \lim_{s\to\infty} \alpha_s$ is finite. We must show that $\partial\alpha = \beta$. Fix $\varepsilon > 0$ and take N such that $(\forall s \geqslant N)\ |\beta - \beta_s| < \varepsilon$. Then, for any $n \geqslant N$,

$$\begin{aligned}|\beta(\Omega - \Omega_n) - (\alpha - \alpha_n)| &\leqslant \sum_{s\geqslant n} |\beta(\Omega_{s+1} - \Omega_s) - (\alpha_{s+1} - \alpha_s)| \\ &= \sum_{s\geqslant n} |\beta - \beta_s|(\Omega_{s+1} - \Omega_s) \leqslant \varepsilon(\Omega - \Omega_n).\end{aligned}$$

For such n,

$$\left|\beta - \frac{\alpha - \alpha_n}{\Omega - \Omega_n}\right| \leqslant \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so $\partial\alpha = \beta$. □

In the same way, every negative Δ^0_2 real is $\partial\alpha$ for some right-c.e. real α. So the range of ∂ contains the Δ^0_2 reals, which is a proper superset of the d.c.e. reals.

Question 2.7. What is the range of ∂ on the d.c.e. reals?

3. The field of nonrandom d.c.e. reals

We finish with an exploration of the nonrandom d.c.e. reals, in part as an application of the work above. First, it is easy to see that if ∂ is a derivation on a field, then its kernel—in this case the nonrandom d.c.e. reals—is also a field. It is called the *field of constants*. With a little more work, we can show:

Corollary 3.1. *The nonrandom d.c.e. reals form a real closed field.*

Proof. Let α and β be nonrandom d.c.e. reals. Then $\partial(\alpha + \beta) = \partial\alpha + \partial\beta = 0$, so $\alpha + \beta$ is not random. It is similarly easy to see that $\alpha - \beta$, $\alpha\beta$ and α/β are not random. So the nonrandom d.c.e. reals form a field.

Now let $p(x)$ be a polynomial whose coefficients are nonrandom d.c.e. reals. Assume that α is a real root of $p(x)$. As mentioned above, the d.c.e. reals form a real closed field [9, 11], so α must be a d.c.e. real. We need to show that α is nonrandom.

We may assume that α has multiplicity one as a root of $p(x)$; otherwise, we could replace $p(x)$ with the greatest common divisor of $p(x)$ and $p'(x)$, which also has coefficients in the field of nonrandom d.c.e. reals. This ensures that $p'(\alpha) \neq 0$. Now note that $\partial p(\alpha) = p'(\alpha)\, \partial\alpha$. (This does not follow from Theorem 2.5 because $p(x)$ may not be a computable function, but it can be shown by an easy induction using parts (a) and (b) of Theorem 2.4.) Therefore, we have

$$\partial\alpha = \frac{\partial p(\alpha)}{p'(\alpha)} = \frac{\partial 0}{p'(\alpha)} = 0,$$

so α is nonrandom. □

The nonrandom d.c.e. reals were not previously known to be a field. In particular, it was not previously known that the sum of nonrandom d.c.e. reals is nonrandom. This was, however, known for left-c.e. reals. It was first claimed by Demuth [5] and later independently proved by Downey, Hirschfeldt, and Nies [7].

Remark 3.2. The fact that the sum of nonrandom d.c.e. reals is itself nonrandom is not, apparently, a trivial generalization of the corresponding fact for left-c.e. reals. To back up this claim, we use the fact to give a short (albeit circular) proof of Lemma 1.2. As in the actual proof, if $\lim_{s\to\infty}(\alpha - \alpha_s)/(\beta - \beta_s)$ does not exist, then there are rationals $c < d$ such that there are infinitely many s for which $\alpha_s - d\beta_s < \alpha - d\beta$ and infinitely many t for which $\alpha_t - c\beta_t > \alpha - c\beta$. Note that if $\alpha_s - d\beta_s < \alpha - d\beta$, then

$$\alpha_s - c\beta_s = \alpha_s - d\beta_s + (d - c)\beta_s < \alpha - d\beta + (d - c)\beta = \alpha - c\beta.$$

Similarly, if $\alpha_t - c\beta_t > \alpha - c\beta$, then $\alpha_t - d\beta_t > \alpha - d\beta$. Therefore, by Remark 1.4, both $\alpha - c\beta$ and $\alpha - d\beta$ are nonrandom, so their difference $(d - c)\beta$ is nonrandom. But this implies that β is nonrandom, which is a contradiction. □

This leads to a natural question: why is it (apparently) harder to prove things about nonrandom d.c.e. reals than nonrandom left-c.e. reals? One immediate answer is that there are nonrandom d.c.e. reals that can only be expressed as a difference of *random* left-c.e. reals. Although they are nonrandom, such d.c.e. reals have an intrinsic randomness. This property can also be captured by looking at the variation of d.c.e. approximations.

Definition 3.3. Call a d.c.e. real α *variation nonrandom* if it has a d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$ such that the variation $\sum_{s\in\omega} |\alpha_{s+1} - \alpha_s|$ is not random. Otherwise, call α *variation random.*

Proposition 3.4. *The following are equivalent for a d.c.e. real* α*:*

- α *is variation nonrandom,*
- *There are nonrandom left-c.e. reals* β *and* γ *such that* $\alpha = \beta - \gamma$*.*

Proof. First, assume that α is variation nonrandom, as witnessed by the d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$. Let $\alpha*$ be the variation of this approximation, with the natural left-c.e. approximation $\{\alpha^*_s\}_{s\in\omega}$. Following Proposition 1.3, let $\beta_{s+1} = \alpha_{s+1} - \alpha_s$ if this is positive; otherwise let $\gamma_{s+1} = \alpha_s - \alpha_{s+1}$. Let $\beta_0 = \alpha_0$, and set all remaining values of β_s and γ_s to 0. Thus β and γ are left-c.e. reals and $\alpha = \beta - \gamma$. Note that

$$\beta_{s+1} - \beta_s \leqslant |\alpha_{s+1} - \alpha_s| = \alpha^*_{s+1} - \alpha^*_s,$$

for all s. So $\beta - \beta_s \leqslant \alpha^* - \alpha^*_s$, which means that $\partial\beta \leqslant \partial\alpha^* = 0$. But $\partial\beta \geqslant 0$ since β is left-c.e., so β is not random. A similar argument works for γ.

Now assume assume that $\alpha = \beta - \gamma$, where β and γ are nonrandom left-c.e. reals with left-c.e. approximations $\{\beta_s\}_{s\in\omega}$ and $\{\gamma_s\}_{s\in\omega}$. Let $\alpha_s = \beta_s - \gamma_s$, so $\{\alpha_s\}_{s\in\omega}$ is a d.c.e. approximation of α. As before, let $\alpha*$ be the variation of this approximation and $\{\alpha^*_s\}_{s\in\omega}$ the natural left-c.e. approximation to α^*. Then

$$\alpha^*_{s+1} - \alpha^*_s = |\alpha_{s+1} - \alpha_s| \leqslant (\beta_{s+1} - \beta_s) + (\gamma_{s+1} - \gamma_s),$$

for all s. So $\alpha^* - \alpha^*_s \leqslant (\beta - \beta_s) + (\gamma - \gamma_s)$. This means that $\partial\alpha^* \leqslant \partial\beta + \partial\gamma = 0$, so α^* is nonrandom. □

Next, we will show that variation randomness is a nontrivial notion. So as promised, there is a nonrandom d.c.e. real that cannot be written as the difference of nonrandom left-c.e. reals.

Theorem 3.5. *There is a nonrandom, variation random d.c.e. real.*

Proof. Let $\{\beta_{0,s}\}_{s\in\omega}, \{\beta_{1,s}\}_{s\in\omega}, \dots$ be an effective list of rational sequences that contains d.c.e. approximations of every d.c.e. real, with every possible variation. This is possible because we can pad a partial computable sequence of rationals by repeating the last value until a new convergence is seen, and this process does not change the variation.

We build a nonrandom d.c.e. real α such that, for each $e \in \omega$:

R_e: If $\{\beta_{e,s}\}_{s\in\omega}$ is a d.c.e. approximation of α,
then its variation is random.

The construction uses the infinite injury priority method. Strategies are organized on a tree, with the eth level containing strategies for R_e. Each node on the tree has outcomes $\infty < \cdots < w_2 < w_1 < w_0$. Strategies will update the global value of α as they are executed; we start with $\alpha = 1/2$. To each node σ on the priority tree, we assign a rational parameter $\varepsilon_\sigma > 0$, in an effective way, such that the total sum of these parameters is bounded by 1. They will be used to meet the global requirements that α is nonrandom and d.c.e.

We are ready to describe the behavior of a node σ on level e of the tree. Let $\varepsilon = \varepsilon_\sigma$. The goal of σ is to make sure that, at any stage, the error in the current approximation to the variation of $\{\beta_{e,s}\}_{s\in\omega}$ is at least ε times the error in the current approximation of Ω. That will ensure that the variation is random. To force the variation to increase, σ will move α back and forth, subject to restraints imposed by other nodes, each time waiting for $\beta_{e,s}$ to get close to α before moving again.

If σ is visited at stage s, it runs the following algorithm, picking up where it left off after the last visit:

(1) Impose the restraint $(\alpha - \varepsilon/2, \alpha + \varepsilon/2)$. Let $t = 0$.
(2) End the substage with outcome ∞.
(3) Let (a, b) be the intersection of all current restraints. Let c be the current value of α (which will be in the interval (a, b)). Pick $n \in \omega$ and a rational $\delta < b - c$ such that $n\delta = \varepsilon(\Omega_{t+1} - \Omega_t)$. Run the following loop n times:
   (a) Let σw_m be the rightmost unvisited child. Move α to c, if it is not already there. Establish the restraint $(\alpha - \delta/8, \alpha + \delta/8)$.
   (b) If $\beta_{e,s}$ is within $\delta/8$ of α, then cancel the restraint from (3a) and all restraints imposed by nodes extending σw_m, including itself, (these nodes will never again be visited); continue the algorithm. Otherwise,

end the substage with outcome w_m; the next time σ is visited, repeat this step.

(c) Let σw_m be the rightmost unvisited child. Move α to $c + \delta$. Establish the restraint $(\alpha - \delta/8, \alpha + \delta/8)$.

(d) If $\beta_{e,s}$ is within $\delta/8$ of α, then cancel the restraint from (3c) and all restraints imposed by nodes extending σw_m; continue the algorithm. Otherwise, end the substage with outcome w_m; the next time σ is visited, repeat this step.

(4) Execute steps (3a) and (3b) one more time.

(5) Increment t and go to (2).

At stage s of the construction, we execute the algorithm above, starting at the root node and following the outcomes until we get to a node at level s of the tree. Let α_s be the value of α at the end of the stage. Note that α always respects all current restraints. In particular, any new restraints that are imposed while we wait in steps (3b) or (3d) for σ are canceled before we move α again for the sake of σ.

We must show that $\{\alpha_s\}_{s\in\omega}$ is a d.c.e. approximation. Let us look at how much σ can move α. Fix a value of t and the corresponding n and δ from step (3). When we transition from (3b) to (3c), we move α by at most $9\delta/8$. The same holds for the transition from (3d) to (3a). There are a total of $2n$ such transitions for t, so α is moved by at most

$$2n \cdot 9\delta/8 = 9/4 \cdot n\delta = 9/4 \cdot \varepsilon(\Omega_{t+1} - \Omega_t),$$

where $\varepsilon = \varepsilon_\sigma$. Over all t, the algorithm for σ moves α by at most $9/4 \cdot \varepsilon\Omega \leqslant 9/4 \cdot \varepsilon_\sigma$. So in total, α is moved by at most $9/4$. Therefore, $\{\alpha_s\}_{s\in\omega}$ is a d.c.e. approximation converging to a d.c.e. real, which of course we call α.

Next, it is not hard to see that α is nonrandom. Each σ that is visited imposes a restraint in step (1). Put the *closure* of this restraint into a Solovay test; it has length ε_σ, so the total weight of the test is bounded by 1. If σ is on the true path, this restraint is never canceled, hence all future approximations of α must respect it. This means that (in the limit) α must be in the closure of the restraint. There are infinitely many nodes on the true path, so α is covered by the Solovay test.

Finally, we must prove that each R_e is satisfied. Assume that $\{\beta_{e,s}\}_{s\in\omega}$ is a d.c.e. approximation of α. Let β^*_e be the variation of $\{\beta_{e,s}\}_{s\in\omega}$, and let $\{\beta^*_{e,s}\}_{s\in\omega}$ be its natural left-c.e. approximation. Let σ be the node at level e of the true path and let $\varepsilon = \varepsilon_\sigma$. Fix t and the corresponding n and δ. Every time we leave (3b), $\beta_{e,s}$ is within $\delta/8$ of α, which is within $\delta/8$ of c. Every time we leave (3d), $\beta_{e,s}$ is within $\delta/8$ of α, which is within $\delta/8$ of $c + \delta$. So every transition adds at least $\delta/2$ to the variation of $\{\beta_{e,s}\}_{s\in\omega}$. By assumption, the algorithm for σ does not get stuck in steps (3b) or (3d), so there are $2n$ such transitions. Therefore, at least $2n \cdot \delta/2 = n\delta = \varepsilon(\Omega_{t+1} - \Omega_t)$ is added to β^*_e for this t. But t is always less than s, the current stage, so this increase in the variation happens after stage t. This means that

$$\beta^*_e - \beta^*_{e,t} \geqslant \varepsilon(\Omega - \Omega_t),$$

for all t. Therefore, $\partial\beta^*_e \geqslant \varepsilon > 0$, so β^*_e is random and R_e is satisfied. □

We finish by arguing that the nonrandom, variation random d.c.e. reals cannot be generated in any reasonably way from nonrandom left-c.e. reals. This is because the variation nonrandom reals form a robust class with a lot of closure. We will see that it is a real closed field, making it the real closure of the nonrandom left-c.e.

reals. Furthermore, the field of variation nonrandom d.c.e. reals is closed under the application of sufficiently well-behaved computable functions.

Lemma 3.6. *Assume that $\alpha_1, \dots, \alpha_n$ are variation nonrandom d.c.e. reals and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a computable function. Let $\beta = f(\alpha_1, \dots \alpha_n)$. If either*

(a) *f is Lipschitz in a neighborhood of $(\alpha_1, \dots \alpha_n)$, or*
(b) *f is differentiable at $(\alpha_1, \dots \alpha_n)$,*

then β is variation nonrandom

Proof. (a) Let $\{\alpha_{1,s}\}_{s\in\omega}, \dots, \{\alpha_{n,s}\}_{s\in\omega}$ be d.c.e. approximations of $\alpha_1, \dots, \alpha_n$ that have nonrandom variations $\alpha_1^*, \dots, \alpha_n^*$. Let $\{\beta_s\}_{s\in\omega}$ be an approximation of β such that β_s is within 2^{-s-1} of $f(\alpha_{1,s}, \dots, \alpha_{n,s})$. By the Lipschitz assumption, there is a $c \in \omega$ such that

$$\begin{aligned} |\beta_{s+1} - \beta_s| &\leqslant 2^{-s} + |f(\alpha_{1,s+1}, \dots, \alpha_{n,s+1}) - f(\alpha_{1,s}, \dots, \alpha_{n,s})| \\ &\leqslant 2^{-s} + c\,\|(\alpha_{1,s+1}, \dots, \alpha_{n,s+1}) - (\alpha_{1,s}, \dots, \alpha_{n,s})\|_2 \\ &\leqslant 2^{-s} + c\,|\alpha_{1,s+1} - \alpha_{1,s}| + \cdots + c\,|\alpha_{n,s+1} - \alpha_{n,s}|. \end{aligned}$$

This proves that $\{\beta_s\}_{s\in\omega}$ has finite variation; call it β^*. Furthermore, assuming the natural approximations for β^* and $\alpha_1^*, \dots, \alpha_n^*$, we have

$$\beta^* - \beta_s^* \leqslant 2^{-s+1} + c\,(\alpha_1^* - \alpha_{1,s}^*) + \cdots + c\,(\alpha_n^* - \alpha_{n,s}^*).$$

Using the fact that $\{-2^{-s+1}\}_{s\in\omega}$ is a d.c.e. approximation of 0, we have $\partial\beta^* \leqslant \partial(0 + c\alpha_1^* + \cdots + c\alpha_n^*) = 0$, so β is a variation nonrandom d.c.e. real.

The argument for (b) is similar, but now the d.c.e. approximation to β must be defined using the method in the proof of Theorem 2.5(a). □

Proposition 3.7. *The variation nonrandom d.c.e. reals form a real closed field.*

Proof. Closure under addition and subtraction are obvious. Multiplication and division are computable and locally Lipschitz, so by the previous lemma, the variation nonrandom d.c.e. reals form a field.

Now let $p(x)$ be a polynomial whose coefficients are variation nonrandom d.c.e. reals. Assume that α is a real root of $p(x)$. We need to show that α is variation nonrandom. As in Corollary 3.1, we may assume that α has multiplicity one as a root of $p(x)$, so $p'(\alpha) \neq 0$. We now essentially follow the proof of Theorem 2.9 in Raichev [11]. Say that $p(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_n x^n$ and let $f(x, y_0, \dots, y_n) = y_0 + y_1 x + \cdots + y_n x^n$. So $f(\alpha, \gamma_1, \dots, \gamma_n) = 0$ and $(\partial_x f)(\alpha, \gamma_1, \dots, \gamma_n) = p'(x) \neq 0$. By the implicit function theorem, there is an open rational ball V containing $(\gamma_1, \dots, \gamma_n)$ and an open rational interval U containing α such that $f(x, y_1, \dots, y_n)$ has a unique root $g(y_1, \dots, y_n) \in U$ for every $(y_1, \dots, y_n) \in V$. Furthermore, g is continuously differentiable, hence Lipschitz in a neighborhood of $(\gamma_1, \dots, \gamma_n)$. By taking V to be small enough to ensure that $g(y_1, \dots, y_n)$ is a multiplicity one root of $f(x, y_1, \dots, y_n)$ for every $(y_1, \dots, y_n) \in V$, it is not hard to see that $g\colon V \to \mathbb{R}$ is computable. Therefore, α is a variation nonrandom d.c.e. real. □

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