LOWNESS NOTIONS, MEASURE AND DOMINATION

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Abstract. We show that positive measure domination implies uniform almost everywhere domination and that this proof translates into a proof in the subsystem WWKL_0 (but not in RCA_0) of the equivalence of various Lebesgue measure regularity statements introduced by Dobrinen and Simpson. This work also allows us to prove that low for weak 2-randomness is the same as low for Martin-Löf randomness (a result independently obtained by Nies). Using the same technique, we show that $\leq_{LR}$ implies $\leq_{LK}$, generalizing the fact that low for Martin-Löf randomness implies low for K.

1. Introduction

Dobrinen and Simpson [4] asked how difficult it is to prove, in the context of reverse mathematics, the following three statements about the Lebesgue measure μ on 2^ω. (The reader who is not familiar with the project of reverse mathematics is referred to Simpson [15] for an introduction to the subject.)

(1) G_δ-REG: For every G_δ set $P \subseteq 2^\omega$, there is an F_σ set $Q \subseteq P$ such that $\mu(Q) = \mu(P)$.
(2) G_δ-ε: For every G_δ set $P \subseteq 2^\omega$ and every $\varepsilon > 0$, there is a closed set $F \subseteq P$ such that $\mu(F) \geq \mu(P) - \varepsilon$.
(3) POS: For every G_δ set $P \subseteq 2^\omega$ such that $\mu(P) > 0$, there is a closed set $F \subseteq P$ such that $\mu(F) > 0$.

It is straightforward to show that ACA_0 proves all three statements, $\mathsf{RCA}_0 \vdash \mathsf{G}_\delta$-$\mathsf{REG} \to \mathsf{G}_\delta$-$\varepsilon$ and $\mathsf{RCA}_0 \vdash \mathsf{G}_\delta$-$\varepsilon \to \mathsf{POS}$. Dobrinen and Simpson introduced the notions of *uniformly almost everywhere* (*u.a.e.*) *domination* and *almost everywhere* (*a.e.*) *domination* and showed that these are the recursion theoretic counterparts of G_δ-REG and G_δ-ε.

Definition 1.1 (Dobrinen and Simpson [4])**.** A set $A \in 2^\omega$ is *a.e. dominating* if for almost all $X \in 2^\omega$ (with respect to the Lebesgue measure) and all functions $g \leq_T X$, there is a function $f \leq_T A$ such that f dominates g (that is, $\exists m \forall n > m\,(f(n) \geq g(n))$). $A \in 2^\omega$ is *u.a.e. dominating* if there is a single function $f \leq_T A$ such that for almost all $X \in 2^\omega$ and all functions $g \leq_T X$, f dominates g.

Theorem 1.2 (Dobrinen and Simpson [4])**.** *The following are equivalent.*

(i) *A is u.a.e. dominating.*
(ii) *For all Π^0_2 sets $P \subseteq 2^\omega$, there is a Σ^A_2 set $Q \subseteq P$ such that $\mu(Q) = \mu(P)$.*

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Theorem 1.3 (Dobrinen and Simpson [4]). *The following are equivalent.*

(i) *A is a.e. dominating.*
(ii) *For all Π^0_2 sets $P \subseteq 2^\omega$ and all $\varepsilon > 0$, there is a Π^A_1 set $F \subseteq P$ such that $\mu(F) \geq \mu(P) - \varepsilon$.*

Dobrinen and Simpson observed that $\mathsf{WKL}_0 \nvdash \mathsf{G}_\delta\text{-}\mathsf{REG}$ and asked whether any (or all) of $\mathsf{G}_\delta\text{-}\mathsf{REG}$, $\mathsf{G}_\delta\text{-}\varepsilon$ or POS implied ACA_0. They suggested finding simpler recursion theoretic equivalences of a.e. domination and u.a.e. domination to help answer this question. At that time, it was known that

$$A \text{ is complete } (A \geq_T \emptyset') \Rightarrow A \text{ is u.a.e. dominating } \Rightarrow A \text{ is high } (A' \geq_T \emptyset'').$$

The first implication is a result of Kurtz [9] while the second implication follows from Martin's Theorem [11]. Dobrinen and Simpson asked whether either of these implications reverses. Cholak, Greenberg and Miller [3] proved that the first arrow does not reverse and that even $\mathsf{G}_\delta\text{-}\mathsf{REG}$, the strongest of the measure theoretic statements, does not imply ACA_0.

Theorem 1.4 (Cholak, Greenberg and Miller [3]). *There is a (c.e.) set $A <_T \emptyset'$ such that A is u.a.e. dominating (and hence u.a.e. domination does not imply completeness). Furthermore, $\mathsf{WKL}_0 + \mathsf{G}_\delta\text{-}\mathsf{REG}$ does not imply ACA_0, and $\mathsf{RCA}_0 + \mathsf{G}_\delta\text{-}\mathsf{REG}$ does not imply the much weaker principle DNR_0.*

Binns, Kjos-Hanssen, Lerman and Solomon [2] proved that the second arrow does not reverse by constructing a high c.e. set A which is not a.e. dominating. In addition, they found a connection between a.e. domination and randomness, specifically the reducibility $\leq_{LR}$ developed by Nies [12].

There are several ways to formalize algorithmic randomness and we start with a measure theoretic approach due to Martin-Löf. A *Martin-Löf test relative to an oracle* A is an A-computable sequence of nested Σ^A_1 classes $U^A_0 \supseteq U^A_1 \supseteq \cdots$ such that $\mu(U^A_n) \leq 2^{-n}$. A set R is A-*random* if for every Martin-Löf test relative to A, $R \notin \bigcap_{n\in\omega} U^A_n$. This notion of randomness is often called Martin-Löf randomness (relative to A) or 1-randomness (relative to A).

Definition 1.5 (Nies [12]). $A \leq_{LR} B$ if every B-random real is A-random.

The idea of $A \leq_{LR} B$ is that A is no more useful than B in the sense that A does not "derandomize" any B-random sets.

Theorem 1.6 (Binns, Kjos-Hanssen, Lerman and Solomon [2]). *If A is a.e. dominating, then $\emptyset' \leq_{\mathrm{LR}} A$.*

Applying work of Nies [12], it follows from Theorem 1.6 that if $A \leq_T \emptyset'$ is a.e. dominating, then A is high, in fact superhigh (namely, $\emptyset'' \leq_{tt} A'$). Using the methods introduced in the present paper, Simpson [14] has generalized this corollary by removing the restriction that $A \leq_T \emptyset'$.

The proof of Theorem 1.6 actually shows that $\emptyset' \leq_{LR} A$ follows from the assumption that for every Π^0_2 class $P \subseteq 2^\omega$ such that $\mu(P) > 0$, there is a Π^A_1 class $Q \subseteq P$ such that $\mu(Q) > 0$. (This property is the recursion theoretic analogue of POS.) Kjos-Hanssen proved that this property is equivalent to what he called *positive measure (p.m.) domination* and proved the following general theorem connecting $\leq_{LR}$ with the ability to find closed subclasses of positive measure.

Theorem 1.7 (Kjos-Hanssen [8]). *$A \leq_{\mathrm{LR}} B$ if and only if every Π^A_1 class of positive measure has a Π^B_1 subclass of positive measure.*

Combining Theorem 1.7 with the well-known result of Kurtz [9] that every Π^0_2 class has a $\Sigma^{\emptyset'}_2$ subclass of the same measure, it follows that $\emptyset' \leq_{LR} A$ exactly characterizes the p.m. dominating sets.

Corollary 1.8 (Kjos-Hanssen [8]). *A is p.m. dominating if and only if $\emptyset' \leq_{\mathrm{LR}} A$.*

As this point, we have the following picture.

$$A \text{ is u.a.e. dominating} \Rightarrow A \text{ is a.e. dominating}$$
$$\Rightarrow A \text{ is p.m. dominating} \Leftrightarrow \emptyset' \leq_{LR} A$$

In Section 3, we close this circle by showing that if A is p.m. dominating, then A is u.a.e. dominating. This result is an application of a more general theorem along the lines of Theorem 1.7: every Σ^A_2 class has a Σ^B_2 subclass of the same measure if and only if $A \leq_{LR} B$ and $A \leq_T B'$. As another application, we prove that if A is low for 1-randomness then it is low for weak 2-randomness (see also Nies [13]). The main technique used in Section 3 gives us a new way to leverage the assumption that $A \leq_{LR} B$. It is first introduced in Section 2, where we show that $\leq_{LR}$ implies $\leq_{LK}$, a reducibility that compares the strength of oracles in terms of their effect on prefix-free Kolmogorov complexity.

In the remaining sections, we examine the implication of the equivalence of u.a.e. domination and p.m. domination for the reverse mathematics question of how difficult it is to prove that $\mathsf{POS} \rightarrow \mathsf{G}_\delta\text{-}\mathsf{REG}$. In Section 5, we show that RCA_0 is not strong enough to prove this implication, or even that $\mathsf{G}_\delta\text{-}\varepsilon \rightarrow \mathsf{G}_\delta\text{-}\mathsf{REG}$. In Section 7, we show that $\mathsf{WWKL}_0 \vdash \mathsf{POS} \rightarrow \mathsf{G}_\delta\text{-}\mathsf{REG}$. Notice that since WKL_0 does not prove $\mathsf{G}_\delta\text{-}\mathsf{REG}$, the fact that WWKL_0—which is weaker than WKL_0—proves this implication is not trivial. Moreover, since measure theory is very limited without WWKL_0 [16], it is reasonable to work over this system to prove the equivalence.

Our notation is standard throughout. We use $\subseteq$ to denote the subset relation between sets (or classes), $\sqsubseteq$ to denote the initial segment relation between (finite or infinite) strings, and $|\sigma|$ to denote the length of a finite string σ. We identify a set X with the infinite string given by its characteristic function. For $X \subseteq \omega$ and $s \in \omega$, $X[s]$ denotes the string $\langle X(0), X(1), \dots, X(s-1)\rangle$. For $Y \subseteq 2^{<\omega}$, $[Y]$ denotes the open class in 2^ω of all X such that $\exists \sigma \in Y(\sigma \sqsubseteq X)$. If $Z \subseteq 2^\omega$, then $Z^c = 2^\omega \setminus Z$. Finally, if M is any machine (viewed as defining a partial function from $2^{<\omega}$ to $2^{<\omega}$), then $\mathrm{dom}(M)$ denotes the set of strings on which M converges (that is, the domain of the defined function).

2. $\leq_{LR}$ IMPLIES $\leq_{LK}$

In this section, we examine the relationship between $\leq_{LR}$ and $\leq_{LK}$, a reducibility based on an information theoretic definition of randomness. The reader who is not familiar with Kolmogorov complexity is referred to Li and Vitányi [10] for an introduction. If U is a universal prefix-free (Turing) machine and τ is a finite binary string, then the *prefix-free* (*Kolmogorov*) *complexity* of τ is defined (up to an additive constant depending on the choice of U) by

$$K(\tau) = \min\{|\sigma| \mid U(\sigma) = \tau\}.$$

We will use two basic facts from the theory of Kolmogorov complexity.

Lemma 2.1 (Kraft inequality)**.** *If $A \subseteq 2^{<\omega}$ is prefix-free, then $\sum_{\sigma \in A} 2^{-|\sigma|} \leq 1$. In particular, if M is a prefix-free Turing machine, then $\sum_{\sigma \in dom(M)} 2^{-|\sigma|} \leq 1$.*

Theorem 2.2 (Kraft–Chaitin Theorem)**.** *Let $\langle d_i, \tau_i \rangle_{i\in\omega}$ be a computable sequence of pairs such that $d_i \in \omega$, $\tau_i \in 2^{<\omega}$ and $\sum_{i\in\omega} 2^{-d_i} \leq 1$. (The range $\{\langle d_i, \tau_i\rangle : i \in \omega\}$ of such a sequence is called a Kraft–Chaitin set.) There is a prefix-free machine M and strings σ_i of length d_i such that $M(\sigma_i) = \tau_i$ for all $i \in \omega$. In particular, the universality of U implies that $K(\tau_i) \leq d_i + O(1)$.*

A is called *Levin-Chaitin random* if for all n, $K(A[n]) \geq n - O(1)$. Despite the difference in context, this notion of randomness coincides with Martin-Löf randomness defined above. Nies [12] defined a reducibility $\leq_{LK}$ similar to $\leq_{LR}$, but based on Kolmogorov complexity. The idea of this reducibility is that $A \leq_{LK} B$ if A is no more useful than B in the sense that A cannot compress information any more than B can.

Definition 2.3 (Nies [12])**.** $A \leq_{LK} B$ if $(\forall\tau)\ K^B(\tau) \leq K^A(\tau) + O(1)$.

It is straightforward to show that $A \leq_{LK} B$ implies $A \leq_{LR} B$; our goal for this section is to show that they are equivalent. Our proof will require one basic fact from real analysis.

Lemma 2.4. *Let $\langle a_i \rangle_{i\in\omega}$ be a sequence of real numbers with $0 \leq a_i < 1$, for all i. Then $\prod_{i\in\omega}(1 - a_i) > 0$ iff $\sum_{i\in\omega} a_i$ converges.*

Lemma 2.5. *For any computable function $f : \omega \to \omega$ there is a uniformly computable collection of finite sets of binary strings V_n, $n \in \omega$, such that $\mu[V_n] = 2^{-f(n)}$ and the sets $[V_n]$, $n \in \omega$, form a mutually independent family of events under μ.*

Proof. Assume that V_t has been defined for all $t < s$. Let k be the length of the longest string in $\bigcup_{t<s} V_t$ and let $V_s = \{\sigma^\frown 0^{f(s)} : \sigma \in 2^k\}$. It is clear that V_s, $s \in \omega$, has the required properties. □

Theorem 2.6. *If $A \leq_{\mathrm{LR}} B$, then $A \leq_{\mathrm{LK}} B$.*

Proof. Identifying the elements of $\omega \times 2^{<\omega}$ with natural numbers via an effective bijection, we let V_s, $s \in \omega$ be as guaranteed by Lemma 2.5 for the function $f(\langle n, \tau\rangle) = n$. This ensures that if $I \subseteq \omega \times 2^{<\omega}$, then $\mu\left(\bigcap_{s\in I}[V_s]^c\right) = \prod_{\langle n,\tau\rangle\in I}(1 - 2^{-n})$, since each V_s is independent from all of the others.

Let U^A be a universal prefix-free machine relative to A and define

$$I = \{\langle|\sigma|, \tau\rangle \colon U^A(\sigma) = \tau\}.$$

Then I is A-c.e., so $P = \bigcap_{s\in I}[V_s]^c$ is a Π^A_1 class. Note that $\sum_{\langle n,\tau\rangle\in I} 2^{-n} \leq \sum_{\sigma\in\mathrm{dom}(U)} 2^{-|\sigma|} \leq 1$ by the Kraft inequality. Also, $\langle 0, \tau\rangle$ is not in I for any τ. So by Lemma 2.4, $\mu(P) = \prod_{\langle n,\tau\rangle\in I}(1 - 2^{-n}) > 0$. Therefore by Theorem 1.7, there is a Π^B_1 class $Q \subseteq P$ such that $\mu(Q) > 0$.

Define $J = \{\langle n, \tau\rangle \colon [V_{\langle n,\tau\rangle}] \cap Q = \emptyset\}$. Note that J is a B-c.e. set since Q^c is generated by a B-c.e. set of strings, $V_{\langle n,\tau\rangle}$ is a finite set of strings, and $[V_{\langle n,\tau\rangle}] \cap Q = \emptyset$ if and only if $[V_{\langle n,\tau\rangle}]$ is covered by a finite set of basic intervals from Q^c. Also, by the comments in the first paragraph of this proof, $\prod_{\langle n,\tau\rangle\in J}(1-2^{-n}) = \mu\left(\bigcap_{s\in J}[V_s]^c\right) \geq \mu(Q) > 0$. Therefore by Lemma 2.4, $\sum_{\langle n,\tau\rangle\in J} 2^{-n}$ converges. Furthermore, we claim that $I \subseteq J$. If $\langle n, \tau\rangle \in I$, then $[V_{\langle n,\tau\rangle}] \cap P = \emptyset$. Since $Q \subseteq P$, $[V_{\langle n,\tau\rangle}] \cap Q = \emptyset$ and hence $\langle n, \tau\rangle \in J$.

Since $\sum_{\langle n,\tau\rangle\in J} 2^{-n}$ converges, fix $c\in\omega$ such that this sum is bounded by 2^c. Then $\widehat{J}=\{\langle n+c,\tau\rangle\colon \langle n,\tau\rangle\in J\}$ is a Kraft–Chaitin set relative to B. Therefore by the Kraft–Chaitin Theorem,

$$\langle n,\tau\rangle\in J \implies \langle n+c,\tau\rangle\in\widehat{J} \implies K^B(\tau)\le n+c+O(1)\le n+O(1).$$

Since $I\subseteq J$, we have $\langle K^A(\tau),\tau\rangle\in J$ for each $\tau\in 2^\omega$. Thus $K^B(\tau)\le K^A(\tau)+O(1)$. In other words, $A\le_{LK} B$. □

Corollary 2.7. *$A\le_{\mathrm{LR}} B$ if and only if $A\le_{\mathrm{LK}} B$.*

Proof. As noted previously, $A\le_{LK} B$ implies $A\le_{LR} B$. Theorem 2.6 supplies the other implication. □

We offer one application of Theorem 2.6 based on a special case of $\le_{LR}$ and $\le_{LK}$. A is *low for* 1*-randomness* if $A\le_{LR}\emptyset$, that is, if every random (in the measure theoretic sense) remains random relative to A. Similarly, A is called *low for K* if $A\le_{LK}\emptyset$, that is, every string contains as much information relative to A as it does with no oracle.

Corollary 2.8 (Nies [12][1])**.** *A is low for 1-randomness if and only if A is low for K.*

Proof. This corollary follows from Corollary 2.7 by setting $B=\emptyset$. □

3. Preserving Measure

In this section, we show that p.m. domination implies u.a.e. domination, thereby showing the equivalence of the three domination notions introduced in Section 1.

Lemma 3.1. *If $A\le_T B'$ and $A\le_{\mathrm{LR}} B$, then every Π^A_1 class has a Σ^B_2 subclass of the same measure.*

Proof. The proof will be similar to that of Theorem 2.6. Identifying now the elements of $2^{<\omega}\times 2^{<\omega}$ with natural numbers via an effective bijection, we let $\{V_s\}_{s\in\omega}$ be as guaranteed by Lemma 2.5 for the function $f(\langle\sigma,\tau\rangle)=|\tau|$. As before, if $I\subseteq 2^{<\omega}\times 2^{<\omega}$, then $\mu\left(\bigcap_{s\in I}[V_s]^c\right)=\prod_{\langle\sigma,\tau\rangle\in I}(1-2^{-|\tau|})$.

Let X be a Π^A_1 class. Assume, without loss of generality, that $X\ne\emptyset$. Let $S^A\subseteq 2^{<\omega}$ be a prefix-free A-c.e. set of strings such that $X=2^\omega\smallsetminus[S^A]$; note that S^A does not contain the empty string. Let $I=\{\langle\sigma,\tau\rangle\colon \tau\in S^A \text{ with use } \sigma\}$. Consider the Π^A_1 class $P=\bigcap_{s\in I}[V_s]^c$. Note that $\sum_{\langle\sigma,\tau\rangle\in I}2^{-|\tau|}=\sum_{\tau\in S^A}2^{-|\tau|}\le 1$ by the Kraft inequality. So by Lemma 2.4, $\mu(P)=\prod_{\langle\sigma,\tau\rangle\in I}(1-2^{-|\tau|})>0$. Therefore by Theorem 1.7, there is a Π^B_1 class $Q\subseteq P$ such that $\mu(Q)>0$.

Define $J=\{\langle\sigma,\tau\rangle\colon [V_{\langle\sigma,\tau\rangle}]\cap Q=\emptyset\}$. As in the proof of Theorem 2.6, J is a B-c.e. set, $I\subseteq J$, and $\prod_{\langle\sigma,\tau\rangle\in J}(1-2^{-|\tau|})=\mu\left(\bigcap_{s\in J}[V_s]^c\right)\ge\mu(Q)>0$. Therefore by Lemma 2.4, $\sum_{\langle\sigma,\tau\rangle\in J}2^{-|\tau|}$ converges.

By assumption $A\le_T B'$, so let $\{A_s\}_{s\in\omega}$ be a B-computable sequence approximating A. Define

$$T_s=\{\langle\sigma,\tau\rangle\in J\colon (\exists t\ge s)\ \tau\in S_t^{A_t} \text{ with use } \sigma\}$$

[1]Yet another proof—one based on work of Hirschfeldt, Nies and Stephan [7]—can be found in Nies [13].

and let $U_s = \{\tau\colon (\exists\sigma)\ \langle\sigma,\tau\rangle \in T_s\}$ be the projection of T_s onto the second coordinate. $\{T_s\}_{s\in\omega}$ and $\{U_s\}_{s\in\omega}$ are B-computable (nested) sequences of B-c.e. sets. We claim that $Y = \bigcup_{s\in\omega}[U_s]^c$ is the desired Σ^B_2 class.

We claim that $S^A \subseteq U_s$ for all s, so $Y \subseteq X$. Suppose $\tau \in S^A$ and fix the use σ of this computation. Then $\langle\sigma,\tau\rangle \in I$ and hence $\langle\sigma,\tau\rangle \in J$. Because A_s is a B-computable approximation to A, it follows that $\forall s \exists t \geq s(\tau \in S^{A_t}_t$ with use $\sigma)$. In other words, $\langle\sigma,\tau\rangle \in T_s$ for all s, and hence $\tau \in U_s$ for all s as required.

For each $\langle\sigma,\tau\rangle \in T_0 \smallsetminus I$, there is a last stage t such that σ is a prefix of A_t, otherwise $\langle\sigma,\tau\rangle$ would be in I. Then $\langle\sigma,\tau\rangle \notin T_s$ for any $s > t$. Fix $\varepsilon > 0$. Take n large enough that $\sum_{\langle\sigma,\tau\rangle\in J,\, \langle\sigma,\tau\rangle\geq n} 2^{-|\tau|} < \varepsilon$ and take s large enough that $\langle\sigma,\tau\rangle \in T_0 \smallsetminus I$ and $\langle\sigma,\tau\rangle < n$ implies $\langle\sigma,\tau\rangle \notin T_s$. Then,

$$\mu(X \smallsetminus [U_s]^c) \leq \sum_{\tau\in U_s\smallsetminus S^A} 2^{-|\tau|} \leq \sum_{\langle\sigma,\tau\rangle\in T_s\smallsetminus I} 2^{-|\tau|} \leq \sum_{\langle\sigma,\tau\rangle\in J,\, \langle\sigma,\tau\rangle\geq n} 2^{-|\tau|} < \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so $\mu(X) = \mu(Y)$. □

Theorem 3.2. *The following are equivalent:*

(i) $A \leq_T B'$ *and* $A \leq_{\mathrm{LR}} B$,
(ii) *Every* Π^A_1 *class has a* Σ^B_2 *subclass of the same measure,*
(iii) *Every* Σ^A_2 *class has a* Σ^B_2 *subclass of the same measure.*

Proof. (i) $\Longrightarrow$ (ii) is Lemma 3.1.

(ii) $\Longrightarrow$ (iii): Let W be a Σ^A_2 class. So $W = \bigcup_{i\in\omega} X_i$ for Π^A_1 classes $\{X_i\}_{i\in\omega}$. Consider the Π^A_1 class $X = \{0^i1^\frown\alpha\colon i \in \omega \text{ and } \alpha \in X_i\}$. By (ii), there is a Σ^B_2 class $Y \subseteq X$ such that $\mu(Y) = \mu(X)$. For each i, let $Y_i = \{\alpha\colon 0^i1^\frown\alpha \in Y\}$. So, Y_i is a Σ^B_2 class and $Y_i \subseteq X_i$ for all i. Clearly $\mu(Y_i) \leq \mu(X_i)$. If $\mu(Y_i) < \mu(X_i)$ for some i, then $\mu(Y) = \sum_{i\in\omega} 2^{i+1}\mu(Y_i) < \sum_{i\in\omega} 2^{i+1}\mu(X_i) = \mu(X)$, which is a contradiction. Therefore, $\mu(Y_i) = \mu(X_i)$ for all i. Let $Z = \bigcup_{i\in\omega} Y_i$. So Z is a Σ^B_2 class and $Z \subseteq W$. Furthermore, $\mu(W \smallsetminus Z) \leq \sum_{i\in\omega} \mu(X_i \smallsetminus Y_i) = 0$, so $\mu(Z) = \mu(W)$.

(iii) $\Longrightarrow$ (i): Suppose that every Σ^A_2 class has a Σ^B_2 subclass of the same measure. First, we show that $A \leq_{LR} B$. By Theorem 1.7, it suffices to show that if P is a Π^A_1 class of positive measure, then P has a Π^B_1 subclass of positive measure. By assumption, P has a Σ^B_2 subclass $Q = \bigcup_{i\in\omega} Q_i$ of positive (in fact the same) measure. At least one of the Π^B_1 classes $Q_i \subseteq Q \subseteq P$ must have positive measure.

Next, we show that $A \leq_T B'$. Let $\sigma_n = 0^n1$ and consider the Σ^A_1 class $U = \bigcup_{n\in A}[\sigma_n]$. Since U is a Σ^A_1 (and hence a Π^A_2) class, by (iii) there is a Π^B_2 class Q such that $U \subseteq Q$ and $\mu(Q) = \mu(U) = \sum_{n\in A} 2^{-(n+1)}$. We claim that $n \in A$ if and only if $[\sigma_n] \subseteq Q$. If $n \in A$, then $[\sigma_n] \subseteq U \subseteq Q$. On the other hand, if $n \notin A$ and $[\sigma_n] \subseteq Q$, then $\mu(Q) \geq \sum_{i\in A} 2^{-(i+1)} + 2^n > \mu(U)$ which is a contradiction. Writing $Q = \bigcap_{k\in\omega} Q_k$ where each Q_k is Σ^B_1, we have

$$n \in A \Leftrightarrow [\sigma_n] \subseteq Q \Leftrightarrow \forall k([\sigma_n] \subseteq Q_k).$$

Since $[\sigma_n] \subseteq Q_k$ is a Σ^B_1 relation, these equivalences show that A is Π^B_2. However, the same argument with the Σ^A_1 class $\bigcup_{n\notin A}[\sigma_n]$ shows that $\overline{A}$ is Π^B_2 as well, and hence $A \leq_T B'$. □

We cannot remove the condition that $A \leq_T B'$ from Theorem 3.2. Indeed, there is a B for which uncountably many A satisfy $A \leq_{LR} B$ (see Barmpalias, Lewis, and Soskova [1]), whereas for each B there are only countably many A with $A \leq_T B'$.

Corollary 3.3. *For all B, the following are equivalent:*

1. *B is uniformly almost everywhere dominating,*
2. *B is almost everywhere dominating,*
3. *B is positive measure dominating, and*
4. *$\emptyset' \leq_{\mathrm{LR}} B$.*

Proof. As noted in Section 1, we have (1) implies (2), (2) implies (3), and (3) if and only if (4). It remains to show that (4) implies (1). Suppose $\emptyset' \leq_{LR} B$. Since $\emptyset' \leq_T B'$, Theorem 3.2 tells us that every $\Sigma_2^{\emptyset'}$ class has a Σ_2^B subclass of the same measure. By Theorem 1.2, to show that B is uniformly almost everywhere dominating, it suffices to show that every Π_2^0 class has a Σ_2^B subclass of the same measure. Fix a Π_2^0 class P. By Kurtz [9], P contains a $\Sigma_2^{\emptyset'}$ subclass $\hat{P}$ such that $\mu(\hat{P}) = \mu(P)$. But, $\hat{P}$ contains a Σ_2^B subclass Q of the same measure and hence $Q \subseteq \hat{P} \subseteq P$ and $\mu(Q) = \mu(\hat{P}) = \mu(P)$ as required. □

Our second corollary of Theorem 3.2 involves the notions of low for weak 2-randomness and low for weak 2-random tests. A *generalized Martin-Löf test* is a computable nested sequence of Σ_1^0 classes $U_0 \supseteq U_1 \supseteq \cdots$ such that $\mu(\bigcap_{i\in\omega} U_i) = 0$. That is, a generalized Martin-Löf test is a Martin-Löf test with the restriction that $\mu(U_i) \leq 2^{-i}$ loosened. Note that if $\{U_i\}_{i\in\omega}$ is a generalized Martin-Löf test, then $\bigcap_{i\in\omega} U_i$ is a Π_2^0 class of measure 0, and conversely, that any Π_2^0 class of measure 0 can be viewed as a generalized Martin-Löf test. A set X is *weakly 2-random* if $X \notin \bigcap_{i\in\omega} U_i$ for all generalized Martin-Löf tests. Notice that all weakly 2-random sets are 1-random.

We say that A is *low for weak 2-randomness* if every set X that is weakly 2-random is also weakly 2-random relative to A. In other words, if $X \notin \bigcap_{i\in\omega} U_i$ for all generalized Martin-Löf tests, then $X \notin \bigcap_{i\in\omega} V_i^A$ for all generalized Martin-Löf tests relative to A. Because weak 2-randomness has been defined in terms of tests, it is possible to give a more uniform version of this condition. A is *low for weak 2-random tests* if for every generalized Martin-Löf test $\bigcap_{i\in\omega} V_i^A$ relative to A, there is a generalized Martin-Löf test $\bigcap_{i\in\omega} U_i$ such that $\bigcap_{i\in\omega} V_i^A \subseteq \bigcap_{i\in\omega} U_i$. It follows immediately that if A is low for weak 2-random tests, then A is low for weak 2-randomness.

Corollary 3.4. *If A is low for 1-randomness, then A is low for weak 2-random tests.*

Proof. Suppose that A is low for 1-randomness, that is, $A \leq_{LR} \emptyset$. Since every low for 1-random set is low (that is, $A' \leq_T \emptyset'$, in fact, even $A' \leq_{tt} \emptyset'$), A satisfies the conditions in Theorem 3.2(i) with $B = \emptyset$. Therefore, every Σ_2^A class has a Σ_2^0 subclass of the same measure. In particular, every Π_2^A class of measure 0 is contained in a Π_2^0 class of measure 0. In other words, every generalized Martin-Löf test relative to A is contained in a generalized Martin-Löf test as required. □

Downey, Nies, Weber and Yu [5] proved one implication between low for 1-randomness and low for weak 2-randomness.

Theorem 3.5 (Downey, Nies, Weber and Yu [5])**.** *If A is low for weak 2-randomness, then A is low for 1-randomness.*

Combining Corollary 3.4 and Theorem 3.5 together with the fact that low for weak 2-random tests implies low for weak for 2-randomness yields the following corollary.

Corollary 3.6. *For any set A, the following conditions are equivalent:*

(1) *A is low for 1-randomness,*
(2) *A is low for weak 2-random tests, and*
(3) *A is low for weak 2-randomness.*

Corollary 3.6 can also be proved using the *golden run* machinery of Nies [12]. This was discovered independently, and earlier, by Nies and a proof along these lines is given in Nies [13].

4. Measure Definitions in Reverse Mathematics

In the remainder of this paper, we consider the reverse mathematics question of how difficult it is to prove $\mathsf{POS} \to \mathsf{G}_\delta\text{-}\mathsf{REG}$. We begin with definitions of codes for open, closed, G_δ and F_σ subsets of $2^{\mathbb{N}}$ in RCA_0. (We switch from ω to $\mathbb{N}$ as it is standard to use $\mathbb{N}$ to denote the first order part of any given model of second order arithmetic.)

A *code for an open set in* $2^{\mathbb{N}}$ is a set $O \subseteq 2^{<\mathbb{N}}$. We can assume without loss of generality that O is prefix free. We write $X \in [O]$ (and say that X is in the set coded by O) if there is a string $\tau \in O$ such that $t \sqsubseteq X$. It is often useful to think of an open set as the union of a sequence of clopen sets. For $t \in \mathbb{N}$, we let $O_t = \{\tau \in O \mid |\tau| < t\}$ and note that $[O] = \bigcup_t [O_t]$.

Equivalently, we can specify an open set by a Σ^0_1 formula (allowing parameters) $\exists s \varphi(x)$, where $\varphi(x)$ contains only bounded quantifiers. In this context, we say that X is in the coded open set if $\exists s \varphi(X[s])$. Later it will be convenient to think of the collection of strings satisfying (or enumerated by) such a formula even though this collection need not be a set in RCA_0. We use the term Σ^0_1 class of strings (or simply Σ^0_1 class, relying on context to differentiate between this notion of class and the one used in the context of sets of reals) to denote the collection of strings satisfying a particular Σ^0_1 formula. This terminology allows us to use set notation for such collections, although any such statement is understood as standing for the appropriate translation of the defining formulas. If O is the Σ^0_1 class of strings corresponding to the formula $\exists s \varphi(x)$, then $O_t = \{\tau \mid |\tau| < t \wedge \exists s < t\, \varphi(\tau)\}$. As above, each O_t is clopen and $[O] = \bigcup_t [O_t]$. In this context, we cannot assume that the Σ^0_1 class of strings O is prefix free. However, abusing notation, we can assume (by removing strings from O_t in a uniform manner) that the finite sets O_t are prefix free.

In systems weaker than ACA_0, we cannot assume that bounded increasing sequences of rationals converge. Therefore, rather than assuming that open sets have definite measures, we work with comparative statements such as $\mu(O) \geq q$ for $q \in \mathbb{Q}$. To define these notions in RCA_0, let O be a (prefix-free) code for an open set. For $t \in \mathbb{N}$, define $\mu(O_t) = \sum_{\tau \in O_t} 2^{-|\tau|}$, and for $q \in \mathbb{Q}$, define

$$\mu(O) \leq q \Leftrightarrow \forall t\, (\mu(O_t) \leq q)$$
$$\mu(O) > q \Leftrightarrow \exists t\, (\mu(O_t) > q)$$
$$\mu(O) \geq q \Leftrightarrow \forall r \in \mathbb{Q}(r < q \to \mu(O) > r)$$

Thus, $\mu(O) \leq q$ is a Π^0_1 statement (with parameter O), $\mu(O) > q$ is a Σ^0_1 statement, and $\mu(O) \geq q$ is a Π^0_2 statement. However, if $\lim_{t\to\infty} \mu(O_t)$ is irrational, then $\mu(O) \geq q \Leftrightarrow \mu(O) > q$, and hence $\mu(O) \geq q$ is a Σ^0_1 expression.

We specify a *closed set* by giving a code O for its complement as an open set and we write $X \in [O]^c$ if for all $\tau \in O$, $\tau \not\sqsubseteq X$. (Equivalently, we can specify a closed set by a Π^0_1 formula $\forall s \varphi(x)$ and say that X is in the closed set if $\forall s \varphi(X[s])$.) We say $\mu([O]^c) \geq q$ if $\mu([O]) \leq 1 - q$, and similarly for the other inequalities.

A *code for a* G_δ set is a sequence $G = \langle G_k \mid k \in \mathbb{N}\rangle$ such that each G_k is a code for an open set and we write $X \in [G]$ if for every k, there is a string $\tau_k \in G_k$ such that $\tau_k \sqsubseteq X$. We frequently abuse notation and simply write $G = \bigcap_{k\in\mathbb{N}} G_k$. (Equivalently, we can specify a G_δ set by a Π^0_2 formula $\forall n \exists s \varphi(x)$ and say that X is in the coded set if $\forall n \exists s \varphi(X[s])$.)

To define our measure inequalities for G, we form the sequence of open sets $\langle G^n \mid n \in \mathbb{N}\rangle$ where $G^n = \bigcap_{k=0}^n G_k$. Notice that $G^1 \supseteq G^2 \supseteq \cdots$ and that classically, $\mu(G) = \lim_n \mu(G^n)$. For all $q \in \mathbb{Q}$, we define

$$\mu(G) \leq q \;\Leftrightarrow\; \forall r \in \mathbb{Q}(r > q \rightarrow \exists n(\mu(G^n) \leq r)$$
$$\mu(G) \geq q \;\Leftrightarrow\; \forall n(\mu(G^n) \geq q)$$

Thus, $\mu(G) \leq q$ is a Π^0_3 statement and $\mu(G) \geq q$ is a Π^0_2 statement. However, if $\lim_{n\to\infty} \mu(G^n)$ is irrational, then $\mu(G) \leq q \Leftrightarrow \exists n(\mu(G^n) \leq q)$ and hence $\mu(G) \leq q$ is a Σ^0_2 statement.

A *code for an* F_σ set is also sequence $F = \langle F_n \mid n \in \mathbb{N}\rangle$ such that each F_n is a code for an open set. F codes the union of the closed sets $[F_n]^c$: $X \in [F]$ if there is an n such that $X \in [F_n]^c$. (Equivalently, we can specify an F_σ set by a Σ^0_2 formula $\exists n \forall s \varphi(x)$ and say that X is in the coded set if $\exists n \forall s(\varphi(X[s]))$.) We define the measure inequalities for an F_σ set from the measure inequalities for its G_δ complement.

When working in subsystems below ACA_0, we regard a measure theoretic statement such as $\mu(G) = \mu(F)$ as an abbreviation for the sentence stating that for all $q \in \mathbb{Q}$, $\mu(G) \geq q$ if and only if $\mu(F) \geq q$. That is, we do not assume that the measures converge to reals in the models for the weak subsystems.

5. Working in REC

In this section we work in REC, the ω-model consisting of the computable sets. A G_δ set in this model is called a *computable* G_δ set. Our goal is to show that $\mathsf{REC} \not\models \mathsf{G}_\delta\text{-}\varepsilon \rightarrow \mathsf{G}_\delta\text{-}\mathsf{REG}$ and hence that $\mathsf{RCA}_0 \nvdash \mathsf{G}_\delta\text{-}\varepsilon \rightarrow \mathsf{G}_\delta\text{-}\mathsf{REG}$. Therefore, $\mathsf{RCA}_0 \nvdash \mathsf{POS} \rightarrow \mathsf{G}_\delta\text{-}\mathsf{REG}$.

First we show that $\mathsf{REC} \not\models \mathsf{G}_\delta\text{-}\mathsf{REG}$. This follows from the existence of a computable G_δ with measure different from that of every computable F_σ set, which in turn, follows easily from the existence of a set that is Π^0_2 but not Σ^0_2. Recall that if G is a computable G_δ set and $q \in \mathbb{Q}$, then $\mu(G) \geq q$ is a Π^0_2 statement.

Proposition 5.1. *There is a computable* G_δ *set* G *such that* $\{q \in \mathbb{Q} \mid \mu(G) \geq q\}$ *is not* Σ^0_2.

Proof. Let $\textsc{Tot}$ denote the Π^0_2 complete index set $\{e \in \omega \mid W_e = \omega\}$, where $\{W_e\}_{e\in\omega}$ is the standard enumeration of the c.e. sets. We identify $\textsc{Tot}$ with its characteristic function. Let $r = \sum_{i=0}^\infty \frac{\textsc{Tot}(i)}{2^{i+1}}$, so the binary expansion of r is $\textsc{Tot}$.

Let $\leq_L$ denote lexicographic order on $2^{\leq\omega}$. Define $G = \{X \in 2^\omega \mid X \leq_L \mathrm{Tot}\}$ and note that $r = \mu(G)$. To see that G is a computable G_δ set, notice that

$$X \in G \iff \forall n \exists s (X[n] \leq_L \mathrm{Tot}_{n,s})$$

where $\mathrm{Tot}_{n,s} = \{e < n \mid 0, \dots, n-1 \in W_{e,s}\}$.

Now let $A = \{q \in \mathbb{Q} \mid \mu(G) \geq q\} = \{q \in \mathbb{Q} \mid r \geq q\}$. It is not hard to see that we can recover Tot from A. First, note that $0 \in \mathrm{Tot}$ if and only if $1/2 \in A$ (using the fact that Tot is coinfinite). Next, $1 \in \mathrm{Tot}$ if and only if either $0 \in \mathrm{Tot}$ and $3/4 \in A$ or $0 \notin \mathrm{Tot}$ and $1/4 \in A$. The induction continues in the obvious way, showing that $\mathrm{Tot} \leq_T A$.

As noted above, A is a Π^0_2 set. If A were Σ^0_2, then A would be computable from $\emptyset'$. But this would imply that $\emptyset'' \equiv_T \mathrm{Tot} \leq_T \emptyset'$, which is a contradiction. Therefore, A is not Σ^0_2. □

Corollary 5.2. $\mathsf{REC} \nvDash \mathsf{G}_\delta\text{-}\mathsf{REG}$.

Proof. Consider the computable G_δ set G from Proposition 5.1. Note that $\mu(G)$ is irrational, or else $\mu(G) \geq q$ would clearly be Σ^0_2. Suppose that there is a computable F_σ set F such that $\mu(G) = \mu(F)$, so $\mu(G) \geq q$ if and only if $\mu(F) \geq q$. (Here, we do not even need to assume that $F \subseteq G$.) Recall that $\mu(F) \geq q$ if and only if $\mu(F^c) \leq 1 - q$. Since $\mu(G)$ is irrational, $1 - \mu(G)$ is irrational, so $\mu(F^c) \leq 1 - q$ is a Σ^0_2 predicate. But $\mu(F^c) \leq 1 - q$ is equivalent to $\mu(G) \geq q$, which is a contradiction. □

The following proposition just says that there are Σ^0_1 classes in 2^ω with arbitrarily small measure that contain all computable sets. This is well known: consider the Σ^0_1 classes that make up a universal Martin-Löf test $\{U_n\}_{n\in\omega}$.

Proposition 5.3. *Let $\varepsilon > 0$. There is a computable closed set C such that C contains no computable elements and $\mu(C) \geq 1 - \varepsilon$.*

Proof. We define a computable open set set O such that O contains all of the computable sets and $\mu(O) \leq \varepsilon$. Fix $n \in \omega$ such that $2^{-n} \leq \varepsilon$. We enumerate O in stages. At stage s, we check for every $e \leq s$ if $\varphi_e(x)$ has converged and taken values in $\{0, 1\}$ for all $x \leq n + e$. For those e for which this happens, we enumerate $\langle \varphi_e(0), \dots, \varphi_e(n + e)\rangle$ into O_s.

It is clear that O will contain all of the computable sets. Furthermore, each $e \in \omega$ adds at most $2^{-(n+e+1)}$ to the measure of O. Therefore, $\mu(O) \leq \sum_{e=0}^{\infty} 2^{-(n+e+1)} = 2^{-n} \leq \varepsilon$. □

Corollary 5.4. $\mathsf{REC} \vDash \mathsf{POS}$ *and* $\mathsf{REC} \vDash \mathsf{G}_\delta\text{-}\varepsilon$.

Proof. To see that $\mathsf{REC} \vDash \mathsf{POS}$, fix any computable G_δ set G such that $\mu(G) > 0$. By Proposition 5.3, there is a computable closed set C such that $\mu(C) > 0$ and C contains no computable elements. Therefore, C is a code for a closed set in the ω-model REC and $\mathsf{REC} \vDash C = \emptyset$ (in the sense that $\mathsf{REC} \vDash \neg\exists X(X \in C)$), hence $\mathsf{REC} \vDash C \subseteq G$.

Since C is a computable closed set, we can fix a computable prefix free code O for the complement of C. Because $\mu(C) > 0$, there is a rational $q < 1$ such that $\forall t\, (\mu(O_t) \leq q)$. Since $\mu(O_t) \leq q$ is an arithmetic fact and REC is an ω-model, $\mathsf{REC} \vDash \forall t\, (\mu(O_t) \leq q)$ and hence $\mathsf{REC} \vDash \mu(C) > 0$. Therefore, $\mathsf{REC} \vDash \mathsf{POS}$.

The proof that $\mathsf{REC} \vDash \mathsf{G}_\delta\text{-}\varepsilon$ is the same except that we start with C such that $\mu(C) \geq \mu(G) - \varepsilon$ for the given ε. □

Corollary 5.5. $\mathsf{RCA}_0 \nvdash \mathsf{POS} \to \mathsf{G}_\delta\text{-}\mathsf{REG}$ *and* $\mathsf{RCA}_0 \nvdash \mathsf{G}_\delta\text{-}\varepsilon \to \mathsf{G}_\delta\text{-}\mathsf{REG}$.

Proof. This corollary follows immediately from Corollaries 5.2 and 5.4. □

6. Logarithm Properties

We have now established that although positive measure domination is equivalent to uniform almost everywhere domination, RCA_0 is not strong enough to prove $\mathsf{POS} \to \mathsf{G}_\delta\text{-}\mathsf{REG}$. In the last two sections, we show that WWKL_0 is strong enough to prove this implication. In this section, we sketch the development of the natural logarithm in RCA_0 and give an analogue of Lemma 2.4.

We wish to define the natural logarithm using the usual integral form

$$\ln(x) = \int_1^x \frac{1}{u}\, du.$$

Because the function $f(u) = 1/u$ does not have a modulus of uniform continuity, we do not automatically obtain a code for $\ln(x)$ as a continuous function in RCA_0. (See Simpson [15], Definition IV.2.1, Lemma IV.2.6, and Theorem IV.2.7 for the relevant background on integrals in subsystems of second order arithmetic.)

Let $q \in \mathbb{Q}^+$. Following the standard procedure for estimating $\int_1^q \frac{1}{u}\, du$ by rectangles, we subdivide the interval $[1, q]$ (or $[q, 1]$ if $q < 1$) into n equal pieces. Because $f(u) = 1/u$ is a decreasing function, we obtain upper and lower estimates of the integral using the left and right endpoints of each interval to define the height of the approximating rectangle. A short calculation shows that

$$\text{Upper Sum} - \text{Lower Sum} = \frac{|q-1|}{n}\left|1 - \frac{1}{q}\right|,$$

which goes to 0 as $n \to \infty$.

In RCA_0, we define the following code for $\ln(x)$. (See Simpson [15], Definition II.6.1, for the formal definition of a code for a continuous function in a subsystem of second order arithmetic.) Let

$$\Phi_{ln} \subseteq \mathbb{N} \times \mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+$$

be given by $(n, a, r, b, s) \in \Phi_{ln}$ if and only if $0 < a - r$, the upper sum for the estimate of $\ln(a+r)$ using n intervals is $< b + s$, and the lower sum for the estimate of $\ln(a - r)$ using n intervals is $> b - s$. Since the difference between the upper and lower sums converges to 0, Φ_{ln} is a code for a continuous function and the function $\ln(x)$ defined by these conditions coincides with $\int_1^x 1/u\, du$. The proof that $1/x$ is the derivative of $\ln(x)$ can be carried out in a straightforward manner within RCA_0.

Lemma 6.1 (RCA_0)**.** *The following results hold.*

1. *The Mean Value Theorem.*
2. *If f is a differentiable function on an open interval in $\mathbb{R}$, then $f' = 0$ on this interval if and only if f is constant. If $f' \geq 0$ on this interval, then f is nondecreasing, and if $f' \leq 0$ on this interval, then f is nonincreasing.*
3. *For all $a, b \in \mathbb{R}^+$, $\ln(ab) = \ln(a) + \ln(b)$.*
4. *For all $k \in \mathbb{N}$ and all sequences of positive rational numbers $a_0, \ldots, a_k$, $\ln(\prod_{i=0}^k a_i) = \sum_{i=0}^k \ln(a_i)$.*

Proof. Part (1) is proved by Hardin and Velleman in [6]. Parts (2) and (3) follow by their classical proofs using the Mean Value Theorem. Part (4) follows by Π^0_1 induction on k since the equality predicate between reals is Π^0_1. □

Lemma 6.2 (RCA_0)**.** *For $0 \le x < 1$, $x \le |\ln(1-x)|$.*

Proof. Consider the function $f(x) = -x - \ln(1-x)$. Since $f(0) = 0$ and

$$f'(x) = -1 + \frac{1}{1-x} \ge 0$$

for $0 \le x < 1$, $f(x)$ is nondecreasing and nonnegative on $[0,1)$. But $-x-\ln(1-x) \ge 0$ implies that $x \le |\ln(1-x)|$. □

Lemma 6.3 (RCA_0)**.** *For $0 \le x \le 1/2$, $|\ln(1-x)| \le 2x$.*

Proof. Consider the function $f(x) = -2x - \ln(1-x)$. Since $f(0) = 0$ and

$$f'(x) = -2 + \frac{1}{1-x} \le 0$$

for $0 \le x \le 1/2$, $f(x)$ is nonincreasing and nonpositive on $[0,1/2]$. But $-2x - \ln(1-x) \le 0$ implies that $|\ln(1-x)| \le 2x$. □

Definition 6.4 (RCA_0)**.** Let a_i, $i \in \mathbb{N}$, be a sequence of real numbers. $\sum_{i=0}^{\infty} a_i$ is *bounded above* if there is a rational q such that for every k, $\sum_{i=0}^{k} a_i \le q$. (We do not assume that the infinite series converges for this definition.) Similarly, $\sum_{i=0}^{\infty} a_i$ is *bounded below* if there is a rational q such that for every k, $\sum_{i=0}^{k} a_i \ge q$.

Definition 6.5 (RCA_0)**.** Let b_i, $i \in \mathbb{N}$, be a sequence of real numbers such that $0 < b_i \le 1$. $\prod_{i=0}^{\infty} b_i$ is *bounded away from* 0 if there is a rational $q > 0$ such that for every k, $\prod_{i=0}^{k} b_i \ge q$.

Finally, we arrive at the version of Lemma 2.4 that we will use in the next section.

Proposition 6.6 (RCA_0)**.** *Let $\langle a_i \mid i \in \mathbb{N}\rangle$ be a sequence of rational numbers such that $0 \le a_i < 1$. $\sum_{i=0}^{\infty} a_i$ is bounded above if and only if $\prod_{i=0}^{\infty}(1-a_i)$ is bounded away from 0.*

Proof. For both expressions, the only way they can be bounded as desired is if a_i converges to 0, in particular for all but finitely many i we have $0 \le a_i \le 1/2$. So by Lemmas 6.2 and 6.3, $\sum_{i=0}^{\infty} a_i$ is bounded above if and only if $\sum_{i=0}^{\infty} |\ln(1-a_i)|$ is bounded above. Because $\ln(1-a_i) = -|\ln(1-a_i)|$, $\sum_{i=0}^{\infty} \ln(1-a_i)$ is bounded below if and only if $\sum_{i=0}^{\infty} |\ln(1-a_i)|$ is bounded above. Therefore, to finish the proof, it suffices to show that $\sum_{i=0}^{\infty} \ln(1-a_i)$ is bounded below if and only if $\prod_{i=0}^{\infty}(1-a_i)$ is bounded away from 0. By Part (3) of Lemma 6.1

$$\sum_{i=0}^{k} \ln(1-a_i) \ge q \Leftrightarrow \ln\left(\prod_{i=0}^{k}(1-a_i)\right) \ge q \Leftrightarrow \prod_{i=0}^{k}(1-a_i) \ge e^q > 0.$$

(We omit the straightforward details of developing the exponential function as the inverse of the natural log.) □

We will also want a more explicit version of one direction of Lemma 2.4.

Proposition 6.7 (RCA_0)**.** *Let $k \in \mathbb{N}$ and let $\langle a_i \mid 0 \le i \le k\rangle$ be a sequence of rational numbers such that $0 \le a_i \le \frac{1}{2}$. If $\sum_{i=0}^{k} a_i \le 2$, then $\prod_{i=0}^{k}(1-a_i) \ge \frac{1}{81}$.*

Proof. If $0 \leq a_i \leq \frac{1}{2}$, then by Lemma 6.3, $0 \leq -\ln(1-a_i) \leq 2a_i$. Thus

$$\sum_{i=0}^{k} \ln(1-a_i) \geq \sum_{i=0}^{k}(-2a_i) = (-2)\sum_{i=0}^{k} a_i \geq -4$$

so as in Proposition 6.6, $\prod_{i=0}^{k}(1-a_i) \geq e^{-4} \geq 1/81$ (using the fact that $e \leq 3$). □

7. Working in WWKL_0

Throughout this section, we work in WWKL_0 to prove $\mathsf{POS} \to \mathsf{G}_\delta\text{-}\mathsf{REG}$. Our proof will roughly be a formalization of the arguments in Lemma 3.1 and Corollary 3.3 with one important difference. In the proofs leading to Corollary 3.3, we used the fact that every Π^0_2 class contains a $\Sigma^{\emptyset'}_2$ class of the same measure. This fact allowed us to switch from working with a Π^0_2 class to working with closed classes with oracles. Because WWKL_0 cannot prove the existence of $\emptyset'$, we need to work directly with the given G_δ set and approximate its measure within WWKL_0. Throughout this section we work in WWKL_0 (in fact, except for Lemma 7.8, we work in RCA_0), assume POS and prove $\mathsf{G}_\delta\text{-}\mathsf{REG}$.

Let $X = \langle X_i \mid i \in \mathbb{N}\rangle$ be a code for a G_δ set of positive measure. Each X_i is a nonempty prefix-free subset of $2^{<\mathbb{N}}$ and $X_{i,s}$ denotes the set of all strings $\tau \in X_i$ such that $|\tau| \leq s$. We will be notationally sloppy about the distinction between coding sets, such as X and X_i, and the subsets of $2^{\mathbb{N}}$ they code, relying on the context to indicate which is the intended meaning. If the context is not clear, we will use square brackets $[X]$ to denote the coded subset of $2^{\mathbb{N}}$.

For each pair $i, n \in \mathbb{N}$, we define a function $m_{i,n}(t)$ by primitive recursion (uniformly in i and n) to approximate $\mu(X_i)$. Set $m_{i,n}(0) = 0$ and

$$m_{i,n}(t+1) = \begin{cases} m_{i,n}(t) & \text{if } \mu(X_{i,t+1} - X_{i,m_{i,n}(t)}) < 2^{-n-i-1}, \\ t+1 & \text{otherwise.} \end{cases}$$

Lemma 7.1. *The following properties hold for each $i, n \in \mathbb{N}$.*

(1) $\forall t, u\,(t < u \to m_{i,n}(t) \leq m_{i,n}(u))$.
(2) $\forall t, u\,(m_{i,n}(t) < m_{i,n}(u) \to (t < u \wedge \mu(X_{i,m_{i,n}(u)} - X_{i,m_{i,n}(t)}) \geq 2^{-i-n-1}))$.
(3) $\exists t \forall u \geq t\,(m_{i,n}(u) = m_{i,n}(t))$.

Proof. Properties (1) and (2) follow directly from the definitions. To prove Property (3), we proceed by contradiction. If Property (3) fails for a particular i and n, then by Property (1), for all t, there is a $u > t$ such that $m_{i,n}(u) > m_{i,n}(t)$. We define a function f such that $f(0) = 0$ and $f(j+1) =$ the least $u > f(j)$ such that $m_{i,n}(u) > m_{i,n}(f(j))$. By Property (2), we have that $\mu(X_{i,m_{i,n}(f(j))}) \geq j \cdot 2^{-i-n-1}$, which for $j > 2^{i+n+1}$ gives the desired contradiction. □

We let $m^\infty_{i,n} = \lim_s m_{i,n}(s)$. (So in a sense $m^\infty_{i,n}$ is the last stage that is significant for the pair (i,n).) As we are working in WWKL_0, we cannot form a function taking each pair $\langle i, n\rangle$ to $m^\infty_{i,n}$, so we understand each statement $m^\infty_{i,n} = k$ to be an abbreviation for the Δ^0_2 formula given by the equivalent formulations $\exists t \forall u \geq t(m_{i,n}(u) = k)$ and $\forall t \exists u \geq t(m_{i,n}(u) = k)$.

We say that $\langle \sigma, n\rangle \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}$ is *correct at s* if $|\sigma| \leq s$, $n \leq s$, and $\sigma(i) = m_{i,n}(s)$ for all $i < |\sigma|$. (The collection of triples $\langle \sigma, n, s\rangle$ such that $\langle \sigma, n\rangle$ is correct at s is a set.) We say that $\langle \sigma, n\rangle$ is *correct* if $\sigma(i) = m^\infty_{i,n}$ for all $i < |\sigma|$ and we let $\mathbf{C}^\infty_\mathbf{n}$ denote the Δ^0_2 class of all strings σ such that $\langle \sigma, n\rangle$ is correct. (To help maintain the distinction between sets of strings and classes of strings, we use boldface letters for classes. Any statement involving a class is to be regarded as shorthand for the statement given by substituting in the defining formula for the class.) Notice that in addition to being a Δ^0_2 class, $\mathbf{C}^\infty_\mathbf{n}$ is also d.c.e. (a difference of two computably enumerable sets) in the sense that if $\langle \sigma, n\rangle$ becomes correct at s, then either $\langle \sigma, n\rangle$ remains correct at all future stages (and $\sigma \in \mathbf{C}^\infty_\mathbf{n}$) or $\langle \sigma, n\rangle$ ceases to be correct at some $t > s$ and is never correct at any stage $\geq t$.

We need to define the appropriate version of the set I from Lemma 3.1 for our argument. Consider an arbitrary n, a stage s, and a value $k \leq s$. The string $\sigma = \langle m_{0,n}(s), m_{1,n}(s), \dots, m_{k-1,n}(s)\rangle$ is the unique string of length k such that $\langle \sigma, n\rangle$ is correct at s. It gives rise to the following sequence of clopen sets

$$(X_{0,\sigma(0)})^c \subseteq (X_{0,\sigma(0)} \cap X_{1,\sigma(1)})^c \subseteq \cdots \subseteq \left(\bigcap_{j<|\sigma|} X_{j,\sigma(j)}\right)^c.$$

The difference $(\bigcap_{j<|\sigma|} X_{j,\sigma(j)})^c - (\bigcap_{j<|\sigma|-1} X_{j,\sigma(j)})^c$ is a clopen set generated by a finite set of minimal length strings (so these strings form an antichain). We define the set $I \subseteq \mathbb{N}^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$ by $\langle \sigma, \tau, n, s\rangle \in I$ if and only if $\langle \sigma, n\rangle$ is correct at s and τ is a minimum length string used to cover $(\bigcap_{j<|\sigma|} X_{j,\sigma(j)})^c - (\bigcap_{j<|\sigma|-1} X_{j,\sigma(j)})^c$.

We will be interested in the following projections and restrictions of I.

$$\begin{aligned} I_{\sigma,n,s} &= \{\tau \mid \langle \sigma, \tau, n, s\rangle \in I\} \\ I_s &= \{\langle \sigma, \tau, n\rangle \mid \langle \sigma, \tau, n, s\rangle \in I\} \\ I^{\exists\sigma}_{n,s} &= \{\tau \mid \exists\sigma(\langle \sigma, \tau, n, s\rangle \in I)\} \\ \mathbf{I}^\infty &= \{\langle \sigma, \tau, n\rangle \mid \exists t \forall s \geq t(\langle \sigma, \tau, n, s\rangle \in I\} \\ \mathbf{I}^\infty_{\sigma,\mathbf{n}} &= \{\tau \mid \exists s(\langle \sigma, \tau, n, s\rangle \in I\} \end{aligned}$$

$I_{\sigma,n,s}$, I_s and $I^{\exists\sigma}_{n,s}$ are all finite sets, while $\mathbf{I}^\infty$ is a Δ^0_2 class of strings (via the equivalent condition $\forall t \exists s \geq t(\langle \sigma, \tau, n, s\rangle \in I)$) and $\mathbf{I}^\infty_{\sigma,\mathbf{n}}$ is a Σ^0_1 class of strings. (To see that $I^{\exists\sigma}_{n,s}$ is a finite set, notice that $I^{\exists\sigma}_{n,s}$ is the union of the finite sets $I_{\mu,n,s}$ over the finitely many μ such that $\langle \mu, n\rangle$ is correct at s.) The following properties are easily verified from the definitions. In the current argument, Property (7) plays the role of the Kraft inequality in Lemma 3.1.

Lemma 7.2. *The following properties hold for all σ, τ, n and s.*

(1) *If $\langle \sigma, n\rangle$ is not correct at s, then $I_{\sigma,n,s} = \emptyset$.*
(2) *If $\langle \sigma, n\rangle$ is correct at s, then $I_{\sigma,n,s} \subseteq I^{\exists\sigma}_{n,s}$.*
(3) *$\langle \sigma, \tau, n\rangle \in \mathbf{I}^\infty$ if and only if $\langle \sigma, n\rangle$ is correct and $\tau \in \mathbf{I}^\infty_{\sigma,\mathbf{n}}$. Furthermore, if $\langle \sigma, n\rangle$ is correct and is correct at s, then $\mathbf{I}^\infty_{\sigma,\mathbf{n}} = I_{\sigma,n,s}$.*
(4) *For each n and k, there is a unique string σ such that $|\sigma| = k$ and $\langle \sigma, n\rangle$ is correct (that is, $\sigma \in \mathbf{C}^\infty_\mathbf{n}$). For each $i < k$, $\langle \sigma \upharpoonright i, n\rangle$ is correct,*

$$\left(\bigcap_{i<k} X_i\right)^c \subseteq \left(\bigcap_{i<k} X_{i,\sigma(i)}\right)^c = \left(\bigcap_{i<k} X_{i,m^\infty_{i,n}}\right)^c = \bigcup_{i<k} [\mathbf{I}^\infty_{\sigma\upharpoonright\mathbf{i},\mathbf{n}}]$$

and

$$\mu\left(\bigcup_{i<k} [\mathbf{I}^\infty_{\sigma\upharpoonright\mathbf{i},\mathbf{n}}] - \left(\bigcap_{i<k} X_i\right)^c\right) \leq \sum_{i<k} 2^{-n-i-1}.$$

(5) *Extending Property (3), for each fixed n,*

$$\mu\left(\bigcup_{\sigma\in\mathbf{C}_{\mathbf{n}}^{\infty}}[\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}]-X^c\right)=\mu\left(\bigcup_{\sigma\in\mathbf{C}_{\mathbf{n}}^{\infty}}[\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}]-\left(\bigcap_{i\in\mathbb{N}}X_i\right)^c\right)\leq\sum_{i=0}^{\infty}2^{-n-i-1}=2^{-n}.$$

(6) *$I_{\sigma,n,s}$ and $I_{n,s}^{\exists\sigma}$ are finite antichains and therefore*

$$\sum_{\tau\in I_{\sigma,n,s}}2^{-|\tau|-n}\leq\sum_{\tau\in I_{n,s}^{\exists\sigma}}2^{-|\tau|-n}\leq 2^{-n}\cdot\sum_{\tau\in I_{n,s}^{\exists\sigma}}2^{-|\tau|}\leq 2^{-n}.$$

(7) *For any fixed s,*

$$\sum_{n\in\mathbb{N}}\sum_{\tau\in I_{n,s}^{\exists\sigma}}2^{-|\tau|-n}\leq\sum_{n\in\mathbb{N}}2^{-n}\leq 2$$

and therefore

$$\sum_{\langle\sigma,\tau,n\rangle\in I_s}2^{-|\tau|-n}\leq 2.$$

Using these ideas, we define the following Π_3^0 class $\mathbf{Z}$. (We use boldface type for $\mathbf{Z}$ since it is introduced via a formula rather than a set code.)

$$\mathbf{Z}=\bigcap_{n\in\mathbb{N}}\bigcup_{\substack{s\in\mathbb{N}\\ \sigma\in\mathbb{N}^{<\mathbb{N}}}}\bigcap_{t\geq s}[I_{\sigma,n,t}]=\bigcap_{n\in\mathbb{N}}\bigcup_{\sigma\in\mathbf{C}_{\mathbf{n}}^{\infty}}[\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}]$$

To be clear, since this definition involves a class predicate, it is to be read in terms of the defining formulas. That is

$$\begin{aligned}A\in\mathbf{Z}&\Leftrightarrow\forall n\exists\sigma,s\forall t\geq s\exists\tau\in I_{\sigma,n,t}(A\in[\tau])\\&\Leftrightarrow\forall n\exists\sigma(\langle\sigma,n\rangle\text{ is correct }\wedge\exists\tau\in\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}(A\in[\tau])).\end{aligned}$$

Since $\exists\tau\in I_{\sigma,n,t}$ is a bounded quantifier, $\langle\sigma,n\rangle$ is correct is a Σ_2^0 statement, and $\exists\tau\in\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}$ is a Σ_1^0 statement, each of these equivalent definitions is Π_3^0.

Lemma 7.3. $\mathbf{Z}$ *has the following properties.*

(1) $X^c\subseteq\mathbf{Z}$.
(2) $\mu(\mathbf{Z}-X^c)=0$.

Proof. To establish (1), for any fixed $n\in\mathbb{N}$, we have

$$X^c=\left(\bigcap_{i\in\mathbb{N}}X_i\right)^c\subseteq\left(\bigcap_{i\in\mathbb{N}}X_{i,m_{i,n}^{\infty}}\right)^c=\bigcup_{\sigma\in\mathbf{C}_{\mathbf{n}}^{\infty}}[\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}]$$

and therefore

$$X^c\subseteq\bigcap_{n\in\mathbb{N}}\bigcup_{\sigma\in\mathbf{C}_{\mathbf{n}}^{\infty}}[\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}]=\mathbf{Z}.$$

To establish (2), for any fixed $n\in\mathbb{N}$, we have by Property (5) of Lemma 7.2,

$$\mu\left(\bigcup_{\sigma\in\mathbf{C}_{\mathbf{n}}^{\infty}}[\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}]-X^c\right)\leq 2^{-n}$$

and therefore

$$\mu\left(\bigcap_{n\in\mathbb{N}}\bigcup_{\sigma\in\mathbf{C}_{\mathbf{n}}^{\infty}}[\mathbf{I}_{\sigma,\mathbf{n}}^{\infty}]-X^c\right)=0.$$

□

Now that we have a nicely approximated Π^0_3 superset $\mathbf{Z}$ of X^c such that $\mu(\mathbf{Z}) = \mu(X^c)$, it remains to find a Π^0_2 superset $\mathbf{Y}$ of $\mathbf{Z}$ such that $\mu(\mathbf{Y}) = \mu(\mathbf{Z})$. $\mathbf{Y}^c$ will be our desired F_σ subset of X of the same measure.

Fix a bijection between $\mathbb{N}$ and $\mathbb{N}^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times \mathbb{N}$ and let $\langle \sigma_j, \tau_j, n_j\rangle$ denote the triple coded by j. Let V_s, $s \in \mathbb{N}$, be as in Lemma 2.5 for the function $f(\langle \sigma_j, \tau_j, n_j\rangle) = |\tau_j| + n_j$ and note that V_s, $s \in \mathbb{N}$, are defined by primitive recursion on j. By Lemma 2.5, for each s, $\mu([V_s]) = 2^{-|\tau_s|-n_s}$ and $\mu([V_s]^c) = 1 - 2^{-|\tau_s|-n_s}$. Furthermore, because the V_s sets are independent, if $K \subseteq \mathbb{N}$ is finite, then $\mu(\bigcap_{s\in K}[V_s]^c) = \prod_{s\in K}(1 - 2^{-|\tau_s|-n_s})$.

Next, we define the G_δ set (i.e., a Π^0_2 class) $P = \bigcap_{i\in\mathbb{N}} P_i$. Fix a bijection between $\mathbb{N}$ and $\mathbb{N}^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$. Let $\langle \sigma_i, \tau_i, n_i, s_i\rangle$ denote the tuple coded by i. Define $P_i \subseteq 2^{<\mathbb{N}}$ as a union $P_i = \bigcup_{s\geq s_i} P_{i,s}$ of nested finite sets of strings as follows. If $\langle \sigma_i, \tau_i, n_i\rangle \notin I_{s_i}$, then $P_{i,s} = \{\lambda\}$ for all $s \geq s_i$, where λ denotes the empty string. So $P_i = \{\lambda\}$ and $[P_i] = 2^{\mathbb{N}}$. If $\langle \sigma_i, \tau_i, n_i\rangle \in I_{s_i}$, then set P_{i,s_i} = a finite set of strings so that $[P_{i,s_i}] = [V_{\sigma_i,\tau_i,n_i}]^c$. For $t > s_i$, check to see if $\langle \sigma_i, \tau_i, n_i\rangle \in I_t$. If so, then $P_{i,t} = P_{i,t-1} = P_{i,s_i}$. If not, then at the first $t > s_i$ at which $\langle \sigma_i, \tau_i, n_i\rangle \notin I_t$, we extend $P_{i,t}$ (using strings of length $> t$) to a finite set of strings such that $[P_{i,t}] = 2^{\mathbb{N}}$, and for all $u > t$, we set $P_{i,u} = P_{i,t} = P_i$. Note that for each i, either $P_{i,s} = P_{i,t}$ for all $s, t \geq s_i$ or there is a unique $t > s_i$ such that $P_{i,t} \neq P_{i,t-1}$.

Lemma 7.4. $\forall j \exists u \forall i \leq j (P_{i,u} = P_i)$.

Proof. Suppose the lemma is false and fix j such that for all stages u, there is an $i \leq j$ such that $P_{i,u} \neq P_i$. In other words, for all u, there is an $i \leq j$ and a stage $t > u$ such that $P_{i,t} \neq P_{i,t-1}$. Let $m = \max\{s_i \mid i \leq j\}$. Define a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ by $f(0)$ = the least t such that $t > m$ and $\exists i \leq j(P_{i,t} \neq P_{i,t-1})$ and $f(n+1)$ = the least t such that $t > f(n)$ and $\exists i \leq j(P_{i,t} \neq P_{i,t-1})$. By Bounded Σ^0_1 Comprehension, let $A = \{t \mid \exists n \leq j+1(f(n) = t)\}$. Since $|A| = j+2$, there must be a value $i \leq j$ and stages $t_1, t_2 \in A$ with $t_1 \neq t_2$, $P_{i,t_1} \neq P_{i,t_1-1}$ and $P_{i,t_2} \neq P_{i,t_2-1}$. These stages t_1, t_2 contradict the fact that there is at most one stage $t > s_i$ for which $P_{i,t} \neq P_{i,t-1}$, completing the proof of this lemma. (Note that despite this proof, we cannot assume the existence of a function g such that for all i, $P_{i,g(i)} = P_i$.) □

Lemma 7.5. $P = \displaystyle\bigcap_{\langle \sigma,\tau,n\rangle \in \mathbf{I}^\infty} [V_{\sigma,\tau,n}]^c$.

Proof. This lemma follows from two calculations. Consider a triple $\langle \sigma, \tau, n\rangle \in \mathbf{I}^\infty$. By Property (3) of Lemma 7.2, $\langle \sigma, n\rangle$ is correct and $\tau \in \mathbf{I}^\infty_{\sigma,\mathbf{n}}$. Fix the least s such that $\langle \sigma, n\rangle$ is correct at s, and hence $\langle \sigma, n\rangle$ is correct at every $t \geq s$. Because s is chosen least, for all $u < s$, $\langle \sigma, n\rangle$ is not correct at u and hence for all i of the form $\langle \sigma, \tau, n, u\rangle$ for $u < s$, we have $[P_i] = 2^{\mathbb{N}}$. On the other hand, because $\tau \in \mathbf{I}^\infty_{\sigma,\mathbf{n}}$, $\langle \sigma, \tau, n\rangle \in I_t$ for all $t \geq s$. Therefore, for all i of the form $\langle \sigma, \tau, n, t\rangle$ for $t \geq s$, we have $[P_i] = [V_{\sigma,\tau,n}]^c$.

Consider a triple $\langle \sigma, \tau, n\rangle \notin \mathbf{I}^\infty$. Fix any i of the form $\langle \sigma, \tau, n, s\rangle$. First, suppose that $\langle \sigma, n\rangle$ is not correct. Then there is a $t \geq s$ such that $\langle \sigma, n\rangle$ is not correct at t. By Property (1) of Lemma 7.2, $I_{\sigma,n,t} = \emptyset$, so $\langle \sigma, \tau, n, t\rangle \notin I$ and $[P_i] = 2^{\mathbb{N}}$. On the other hand, suppose that $\langle \sigma, n\rangle$ is correct and fix $t \geq s$ such that $\langle \sigma, n\rangle$ is correct at t. By Property (3) of Lemma 7.2, $\tau \notin \mathbf{I}^\infty_{\sigma,\mathbf{n}}$ and hence $\tau \notin I_{\sigma,n,t}$ and $\langle \sigma, \tau, n\rangle \notin I_t$. Therefore, $[P_i] = 2^{\mathbb{N}}$. □

Lemma 7.6. $\mu(P) > 0$.

Proof. We need to show that there is an $\varepsilon \in \mathbb{Q}^+$ such that

$$\forall j \left(\mu \left(\bigcap_{i \leq j} P_i \right) \geq \varepsilon \right).$$

We proceed by contradiction. Suppose that for every $\varepsilon > 0$, there is a j such that $\mu(\bigcap_{i \leq j} P_i) < \varepsilon$. Fix an arbitrary ε and the corresponding j. Fix u such that $P_{i,u} = P_i$ for all $i \leq j$. As above, we assume $i = \langle \sigma_i, \tau_i, n_i, t_i \rangle$.

For each $i \leq j$, $P_{i,u} = [V_{\sigma_i,\tau_i,n_i}]^c$ implies $\langle \sigma_i, \tau_i, n_i \rangle \in I_u \cap \mathbf{I}^\infty$, and $P_{i,u} \neq [V_{\sigma_i,\tau_i,n_i}]^c$ implies $P_{i,u} = 2^{\mathbb{N}}$. Furthermore, because each $P_{i,u}$ is a finite set of strings, we can tell which of these cases applies. Form the finite set

$$K = \{ \langle \sigma_i, \tau_i, n_i \rangle \mid i \leq j \wedge P_{i,u} = [V_{\sigma_i,\tau_i,n_i}]^c \} \subseteq I_u.$$

Calculating measures, we have

$$\prod_{\langle \sigma_i,\tau_i,n_i \rangle \in K} (1 - 2^{-|\tau_i| - n_i}) = \mu \left(\bigcap_{i \leq j} P_{i,u} \right) = \mu \left(\bigcap_{i \leq j} P_i \right) < \varepsilon.$$

Furthermore, we have

$$\sum_{\langle \sigma_i,\tau_i,n_i \rangle \in K} 2^{-|\tau_i| - n_i} \leq \sum_{\langle \sigma,\tau,n \rangle \in I_u} 2^{-|\tau| - n} \leq 2.$$

(The first inequality follows because $K \subseteq I_u$ and the second inequality follows from Property (7) of Lemma 7.2.) For a small enough value of ε, the fact that $\prod_{\langle \sigma_i,\tau_i,n_i \rangle \in K} (1 - 2^{-|\tau_i| - n_i}) < \varepsilon$ and $\sum_{\langle \sigma_i,\tau_i,n_i \rangle \in K} 2^{-|\tau_i| - n_i} \leq 2$ contradicts Proposition 6.7. □

Lemma 7.7. *For all σ, τ and n, $[V_{\sigma,\tau,n}] \cap P = \emptyset$ if and only if $\langle \sigma, n \rangle$ is correct and $\tau \in \mathbf{I}^\infty_{\sigma,\mathbf{n}}$.*

Proof. Suppose that $\langle \sigma, n \rangle$ is correct and $\tau \in \mathbf{I}^\infty_{\sigma,\mathbf{n}}$. By Property (3) of Lemma 7.2, $\langle \sigma, \tau, n \rangle \in \mathbf{I}^\infty$. By Lemma 7.5, $[V_{\sigma,\tau,n}]^c$ is one of the intersected sets forming P and therefore $[V_{\sigma,\tau,n}] \cap P = \emptyset$.

Now assume that it is not the case that $\langle \sigma, n \rangle$ is correct and $\tau \in \mathbf{I}^\infty_{\sigma,\mathbf{n}}$. Again by Property (3) of Lemma 7.2, we have $\langle \sigma, \tau, n \rangle \notin \mathbf{I}^\infty$. So $[V_{\sigma,\tau,n}]^c$ does not occur in the intersection forming P. Let $s = \langle \sigma, \tau, n \rangle$. Recall how the sets V_t were formed in Lemma 2.5. Let k be the length of the longest string in $\bigcup_{t<s} V_t$. Consider the sequence $X = 1^k 0^{f(s)} 1^{\mathbb{N}}$. It follows from the construction of the sets V_t, $t \in \mathbb{N}$, that $X \in [V_s]$ but $X \in [V_t]^c$ for every $t \neq s$. Therefore, $X \in [V_{\sigma,\tau,n}] \cap P$, so $[V_{\sigma,\tau,n}] \cap P \neq \emptyset$. □

By Lemma 7.7, we can write $\mathbf{Z}$ as

$$A \in \mathbf{Z} \Leftrightarrow \forall n \exists \sigma, \tau ([V_{\sigma,\tau,n}] \cap P = \emptyset \wedge A \in [\tau]).$$

By POS, we can fix a closed set $Q \subseteq P$ such that $\mu(Q) > 0$. Following the proof of Lemma 3.1, it would make sense to define $\mathbf{J}$ to be the class containing all triples $\langle \sigma, \tau, n \rangle$ such that $[V_{\sigma,\tau,n}] \cap Q = \emptyset$. The problem is that without WKL_0, this would not necessarily be a Σ^0_1 condition. Since we want to work in WWKL_0, we need a slightly different definition of $\mathbf{J}$. Take $k \in \mathbb{N}$ such that $\mu(Q) > 2^{-k}$. Let

$$\mathbf{J} = \{\langle \sigma, \tau, n\rangle \mid \mu([V_{\sigma,\tau,n}] \cap Q) < 2^{-\langle \sigma,\tau,n\rangle - k - 2}\}.$$

In Section 4 we saw that if O is an open set and $q \in \mathbb{Q}$, then $\mu(O) > q$ is a Σ^0_1 statement. Thus, $\mathbf{J}$ is a Σ^0_1 class.

Lemma 7.8. *If* $[V_{\sigma,\tau,n}] \cap Q = \emptyset$*, then* $\langle \sigma, \tau, n\rangle \in \mathbf{J}$.

Proof. This follows from WWKL_0 and it is our only use of the principle. If $\langle \sigma, \tau, n\rangle \notin \mathbf{J}$, then $\mu([V_{\sigma,\tau,n}] \cap Q) > 0$. But then WWKL_0 implies that $[V_{\sigma,\tau,n}] \cap Q \neq \emptyset$. □

Lemma 7.9. *The sum* $\sum_{\langle \sigma,\tau,n\rangle \in \mathbf{J}} 2^{-|\tau|-n}$ *is bounded above.*

Proof. Because $\mathbf{J}$ is a Σ^0_1 class, this sum can be expressed as $\sum a_i$ where the sequence $a_i \in \mathbb{Q}$ is determined by the enumeration of $\mathbf{J}$. That is, $a_i = 2^{-|\tau|-n}$ if the i-th element enumerated into $\mathbf{J}$ is $\langle \sigma, \tau, n\rangle$. (Recall that we think of a Σ^0_1 class such as $\mathbf{J}$ enumerated in stages with $\mathbf{J}_s$ equal to the finite set of tuples $\langle \sigma, \tau, n\rangle < s$ which are in $\mathbf{J}$ with an existential witness $< s$.)

We define an open set R as follows. At the stage s when $\langle \sigma, \tau, n\rangle$ goes into $\mathbf{J}$, we have $\mu([V_{\sigma,\tau,n}] \cap Q_s) < 2^{-\langle \sigma,\tau,n\rangle - k-2}$. Enumerate the clopen set $[V_{\sigma,\tau,n}] \cap Q_s$ into R. Note that $\mu(R) \leq \sum_{\langle \sigma,\tau,n\rangle \in \mathbf{J}} 2^{-\langle \sigma,\tau,n\rangle - k - 2} \leq 2^{-k-1}$. Also note that if $\langle \sigma, \tau, n\rangle \in \mathbf{J}$, then $[V_{\sigma,\tau,n}] \subseteq R \cup Q^c$. Therefore, $Q - R \subseteq \bigcap_{\langle \sigma,\tau,n\rangle \in \mathbf{J}_s} [V_{\sigma,\tau,n}]^c$.

For any $s \in \mathbb{N}$, we have

$$\prod_{\langle \sigma,\tau,n\rangle \in \mathbf{J}_s} (1 - 2^{-|\tau|-n}) = \mu\left(\bigcap_{\langle \sigma,\tau,n\rangle \in \mathbf{J}_s} [V_{\sigma,\tau,n}]^c\right) \geq \mu(Q - R)$$
$$\geq \mu(Q) - \mu(R) > 2^{-k} - 2^{-k-1} = 2^{-k-1} > 0.$$

and therefore the product $\prod_{\langle \sigma,\tau,n\rangle \in \mathbf{J}} (1 - 2^{-|\tau|-n})$ is bounded away from 0. Hence, by Proposition 6.6, $\sum_{\langle \sigma,\tau,n\rangle \in \mathbf{J}} 2^{-|\tau|-n}$ is bounded above. □

To approximate the defining condition for $\mathbf{Z}$ given immediately after Lemma 7.7, we look at the Σ^0_1 predicate

$$\langle \sigma, \tau, n\rangle \in \mathbf{J} \wedge \exists t \geq s(\langle \sigma, \tau, n\rangle \in I_t).$$

Define

$$\mathbf{T_{n,s}} = \{\langle \sigma, \tau, n\rangle \mid \langle \sigma, \tau, n\rangle \in \mathbf{J} \wedge \exists t \geq s(\langle \sigma, \tau, n\rangle \in I_t)\},$$
$$\mathbf{U_{n,s}} = \{\tau \mid \exists \sigma(\langle \sigma, \tau, n\rangle \in T_{n,s})\}.$$

Note that $\mathbf{T_{n,s}}$ and $\mathbf{U_{n,s}}$ are Σ^0_1 classes and for any fixed n, we have

$$\mathbf{T_{n,0}} \supseteq \mathbf{T_{n,1}} \supseteq \mathbf{T_{n,2}} \supseteq \cdots,$$
$$\text{and } \mathbf{U_{n,0}} \supseteq \mathbf{U_{n,1}} \supseteq \mathbf{U_{n,2}} \supseteq \cdots.$$

We finally define our desired Π^0_2 class $\mathbf{Y}$

$$\mathbf{Y} = \bigcap_{n \in \mathbb{N}} \bigcap_{s \in \mathbb{N}} [\mathbf{U_{n,s}}].$$

Lemma 7.10. $\mathbf{Z} \subseteq \mathbf{Y}$.

Proof. Let $A \in \mathbf{Z}$ and fix any n. We show that $A \in \bigcap_{s\in\mathbb{N}}[\mathbf{U_{n,s}}]$. Since $A \in \mathbf{Z}$, there are strings σ and τ such that $[V_{\sigma,\tau,n}] \cap P = \emptyset$ and $A \in [\tau]$. Since $Q \subseteq P$, we have $[V_{\sigma,\tau,n}] \cap Q = \emptyset$, so $\langle\sigma,\tau,n\rangle \in \mathbf{J}$. By Lemma 7.7, we have that $\langle\sigma, n\rangle$ is correct and $\tau \in \mathbf{I}^{\infty}_{\sigma,\mathbf{n}}$. Therefore, for all s, there is $t \geq s$ such that $\langle\sigma,\tau,n\rangle \in I_t$. (In fact, this is true for almost all $t \geq s$.) It follows that for all s,

$$\langle\sigma,\tau,n\rangle \in \mathbf{J} \wedge \exists t \geq s(\langle\sigma,\tau,n\rangle \in I_t),$$

and hence that $\langle\sigma,\tau,n\rangle \in \mathbf{T_{n,s}}$ and $\tau \in \mathbf{U_{n,s}}$ for all s. Since $A \in [\tau]$, we have that $A \in \bigcap_{s\in\mathbb{N}}[\mathbf{U_{n,s}}]$ as required. □

Lemma 7.11. $\mu(\mathbf{Y} - \mathbf{Z}) = 0$.

Proof. For $k \in \mathbb{N}$ we let

$$\mathbf{Z_k} = \bigcup_{\sigma\in\mathbf{C}^{\infty}_{\mathbf{k}}}[\mathbf{I}^{\infty}_{\sigma,\mathbf{k}}],$$

$$\mathbf{Y_k} = \bigcap_{s\in\mathbb{N}}[\mathbf{U_{k,s}}].$$

The proof of Lemma 7.10 shows that $\mathbf{Z_k} \subseteq \mathbf{Y_k}$. Since $\mathbf{Z} = \bigcap_k \mathbf{Z_k}$ and $\mathbf{Y} = \bigcap_k \mathbf{Y_k}$, it suffices to show that $\mu(\mathbf{Y_k} - \mathbf{Z_k}) = 0$. To prove this measure statement, we need to prove that for every $\varepsilon \in \mathbb{Q}^+$, there is a c such that $\mu(\mathbf{U_{k,c}} - \mathbf{Z_k}) < \varepsilon$.

Fix $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{Q}^+$. By Lemma 7.9, fix m such that

$$\sum_{\substack{\langle\sigma,\tau,n\rangle\in\mathbf{J}\\ \langle\sigma,\tau,n\rangle\geq m}} 2^{-|\tau|-n} < \varepsilon \cdot 2^{-k}.$$

(In this sum, σ, τ and n vary.) Fixing $n = k$ in this summation and multiplying by 2^k, we have (now letting only σ and τ vary)

$$\sum_{\substack{\langle\sigma,\tau,k\rangle\in\mathbf{J}\\ \langle\sigma,\tau,k\rangle\geq m}} 2^{-|\tau|} < \varepsilon.$$

For each tuple $\langle\sigma,\tau,k\rangle \in \mathbf{T_{k,0}}$ such that $\langle\sigma,\tau,k\rangle \notin \mathbf{I}^{\infty}$, there must be an c such that for all $u \geq c$, $\langle\sigma,\tau,k\rangle \notin I_u$, and hence $\langle\sigma,\tau,k\rangle \notin \mathbf{T_{k,c}}$. We would like to obtain a single witness c which works for all such $\langle\sigma,\tau,k\rangle < m$.

Consider the bounded quantifier statement $\varphi(\sigma,\tau,k,u)$ which says that u is a witness for $\langle\sigma,\tau,n\rangle \in \mathbf{J}$, that $\exists t \leq u(\langle\sigma,\tau,k\rangle \in I_t)$, and that $\langle\sigma,\tau,k\rangle \notin I_u$. Fix any $\langle\sigma,\tau,k\rangle$ such that $\exists u\, \varphi(\sigma,\tau,k,u)$, fix the witness u for this statement and fix $t \leq u$ that witnesses the second conjunct of φ. Because $\langle\sigma,\tau,n\rangle \in \mathbf{J}$ and $\langle\sigma,\tau,k\rangle \in I_t$, we have that $\langle\sigma,\tau,k\rangle \in \mathbf{T_{k,0}}$. Because $\langle\sigma,\tau,k\rangle \notin I_u$ and $t < u$, we have that $\forall v \geq u(\langle\sigma,\tau,k\rangle \notin I_v)$ and hence $\langle\sigma,\tau,k\rangle \notin \mathbf{T_{k,u}}$. Furthermore, by the previous paragraph, if $\langle\sigma,\tau,k\rangle \in \mathbf{T_{k,0}}$ and $\langle\sigma,\tau,k\rangle \notin \mathbf{I}^{\infty}$, then $\exists u\, \varphi(\sigma,\tau,k,u)$.

The strong Σ^0_1 bounding scheme (which holds in RCA_0, see Simpson [15] Exercise II.3.14) implies that

$$\exists c\, \forall\langle\sigma,\tau,k\rangle \leq m\, (\exists u\, \varphi(\sigma,\tau,k,u) \rightarrow \exists u \leq c\, \varphi(\sigma,\tau,k,u)).$$

Fix such a c. For any $\langle\sigma,\tau,k\rangle < m$, if $\langle\sigma,\tau,k\rangle \in \mathbf{T_{k,0}}$ and $\langle\sigma,\tau,k\rangle \notin \mathbf{I}^{\infty}$, then $\langle\sigma,\tau,k\rangle \notin \mathbf{T_{k,c}}$.

To finish the proof, it suffices to show that

$$\mu(\mathbf{U_{k,c}} - \mathbf{Z_k}) \leq \sum_{\substack{\langle\sigma,\tau,k\rangle\in\mathbf{J} \\ \langle\sigma,\tau,k\rangle\geq m}} 2^{-|\tau|} < \varepsilon.$$

Suppose that $\tau \in \mathbf{U_{k,c}}$ but $\tau \notin \mathbf{Z_k}$ (that is, $\tau \notin \mathbf{I}^\infty_{\sigma,\mathbf{k}}$ for any $\sigma \in \mathbf{C}^\infty_\mathbf{k}$). Fix σ such that $\langle\sigma,\tau,k\rangle \in \mathbf{T_{k,c}}$. We need to show that $\langle\sigma,\tau,k\rangle \geq m$. $\langle\sigma,\tau,k\rangle \in \mathbf{T_{k,c}}$ implies that $\exists t \geq c(\langle\sigma,\tau,k\rangle \in I_t)$ and hence $\tau \in \mathbf{I}^\infty_{\sigma,\mathbf{k}}$. Since $\tau \notin \mathbf{Z}$, $\langle\sigma,k\rangle$ must not be correct and hence $\langle\sigma,\tau,k\rangle \notin \mathbf{I}^\infty$ by Property (3) of Lemma 7.2. Suppose for a contradiction that $\langle\sigma,\tau,k\rangle < m$. Since $\langle\sigma,\tau,k\rangle \in \mathbf{T_{k,c}} \subseteq \mathbf{T_{k,0}}$ and $\langle\sigma,\tau,k\rangle \notin \mathbf{I}^\infty$, we have (by our choice of c) that $\langle\sigma,\tau,k\rangle \notin \mathbf{T_{k,c}}$, which is the desired contradiction. □

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