Kolmogorov-Loveland Randomness and Stochasticity

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Abstract

An infinite binary sequence X is Kolmogorov-Loveland (or KL) random if there is no computable non-monotonic betting strategy that succeeds on X in the sense of having an unbounded gain in the limit while betting successively on bits of X. A sequence X is KL-stochastic if there is no computable non-monotonic selection rule that selects from X an infinite, biased sequence.

One of the major open problems in the field of effective randomness is whether Martin-Löf randomness is the same as KL-randomness. Our first main result states that KL-random sequences are close to Martin-Löf random sequences in so far as every KL-random sequence has arbitrarily dense subsequences that are Martin-Löf random. A key lemma in the proof of this result is that for every effective split of a KL-random sequence at least one of the halves is Martin-Löf random. However, this splitting property does not characterize KL-randomness; we construct a sequence that is not even computably random such that every effective split yields two subsequences that are 2-random. Furthermore, we show for any KL-random sequence A that is computable in the halting problem that, first, for any effective split of A both halves are Martin-Löf random and, second, for any computable, nondecreasing, and unbounded function g and almost all n, the prefix of A of length n has prefix-free Kolmogorov complexity at least $n - g(n)$. Again, the latter property does not characterize KL-randomness, even when restricted to left-r.e. sequences; we construct a left-r.e. sequence that has this property but is not KL-stochastic, in fact, is not even Mises-Wald-Church stochastic.

Turning our attention to KL-stochasticity, we construct a non-empty Π^0_1 class of KL-stochastic sequences that are not weakly 1-random; by the usual basis theorems we obtain such sequences that in addition are left-r.e., are low, or are of hyperimmune-free degree.

Our second main result asserts that every KL-stochastic sequence has effective dimension 1, or equivalently, a sequence cannot be KL-stochastic if it has infinitely many prefixes that can be compressed by a factor of $\alpha < 1$. This improves on a result by Muchnik, who has shown that were they to exist, such compressible prefixes could not be found effectively.

1 Introduction

The major criticism brought forward against the notion of Martin-Löf randomness is that, while it captures almost all important probabilistic laws, it is not completely intuitive, since it is not characterized by *computable* martingales but by *recursively enumerable* ones (or by an equivalent r.e. test notion).

This point was issued first by Schnorr [26, 27], who asserted that Martin-Löf randomness was too strong to be regarded as an *effective notion* of randomness. He proposed two alternatives, one defined via coverings with measures which are computable real numbers (not merely left-r.e.), leading to the concept today known as *Schnorr randomness* [27]. The other concept is based on the unpredictability paradigm; it demands that no *computable* betting strategy should win against a random sequence. This notion is commonly referred to as *computable randomness* [27].

If one is interested in obtaining stronger notions of randomness, closer to Martin-Löf randomness, without abandoning Schnorr's paradigm, one might stay with computable betting strategies and think of more general ways those strategies could be allowed to bet. One possibility is to remove the requirement that the betting strategy must bet on a given sequence in an order that is *monotonic on the prefixes of that sequence*, that is, the strategy itself determines which place of the sequence it wants to bet against next. The resulting concept of *non-monotonic betting strategies* is a generalization of the concept of monotonic betting strategies. An infinite binary sequence against which no computable non-monotonic betting strategy succeeds is called *Kolmogorov-Loveland random*, or KL-random, for short. The concept is named after Kolmogorov [9] and Loveland [14], who studied non-monotonic selection rules to define accordant stochasticity concepts, which we will describe later.

The concept of KL-randomness is robust in so far as it remains the same no matter whether one defines it in terms of computable or partial computable non-monotonic betting strategies [18]; in terms of the latter, the concept has been introduced by Muchnik, Semenov, and Uspensky [20] in 1998. They showed that Martin-Löf randomness implies KL-randomness, but it is not known whether the two concepts are different. This question was raised by Muchnik, Semenov, and Uspensky [20] and by Ambos-Spies and Kučera [1]. It is still a major open problem in the area. A proof that both concepts are the same would give a striking argument against Schnorr's criticism of Martin-Löf randomness.

Most researchers conjecture that the notions are different. However, a result of Muchnik [20] indicates that KL-randomness is rather close to Martin-Löf randomness.

Recall that it is possible to characterize Martin-Löf randomness as incompressibility with respect to prefix-free Kolmogorov complexity K: A sequence A is Martin-Löf random if and only if there is a constant c such that for all n the prefix-free Kolmogorov complexity of the length n prefix $A\restriction_n$ of A is at least $n - c$. It follows that a sequence A cannot be Martin-Löf random if there is a function h such that

$$\mathrm{K}(A\restriction_{h(c)}) \leq h(c) - c \qquad \text{for every } c. \tag{1}$$

On the other hand, by the result of Muchnik [20] a sequence A cannot be KL-random if (1) holds for a *computable* function h. So, the difference between Martin-Löf randomness and KL-randomness appears, from this viewpoint, rather small. Not being Martin-Löf random means that for any given constant bound there are infinitely many initial segments for which the compressibility exceeds this bound. If, moreover, we are able to detect such initial segments efficiently (by means of a computable function), then the sequence cannot even be KL-random.

In this paper we continue the investigations by Muchnik, Semenov, and Uspensky, and give additional evidence that KL-randomness is very close to Martin-Löf randomness.

In Section 4 we refine a splitting technique that Muchnik used in order to obtain the result mentioned above. We show the following: if A is KL-random and Z is a computable, infinite and co-infinite set of natural numbers, either the bits of A whose positions are in Z or the remaining bits form a Martin-Löf random sequence. In fact, both do if A is Δ^0_2. Moreover, in that case, for each computable, nondecreasing and unbounded function g it holds that $\mathrm{K}(A\restriction_n) \geq n - g(n)$ for all but finitely many n.

In Section 5 we construct counterexamples that show that two of the implications mentioned in the preceding paragraph cannot be extended to equivalences, i.e. they are not sufficient conditions for KL-randomness (let alone Martin-Löf randomness). First, there is a sequence that is not computably random but all whose "parts" in the sense above (i.e., which can be obtained through a computable splitting) are Martin-Löf random. Second, there is a sequence A that is not even MWC-stochastic such that for all g as above and almost all n, $\mathrm{K}(A\restriction_n) \geq n - g(n)$; moreover, the sequence A can be chosen to be left-r.e. if viewed as the binary expansion of a real.

In the last two sections we consider KL-stochasticity. A sequence is KL-stochastic if there is no computable non-monotonic selection rule that selects from the given sequence a sequence that is biased in the sense that the frequencies of 0's and 1's do not converge to 1/2. First we describe a method for the construction of KL-stochastic sequences, which yields KL-stochastic sequences that are not weakly 1-random with additional properties such as being left-r.e., being low, or being of hyperimmume-free degree. Next we consider effective dimension. Muchnik [20, 31] demonstrates, by an argument similar to the proof that a sequence A cannot be KL-random if there is a computable function that satisfies (1), that a sequence A cannot be KL-stochastic if there is a computable, unbounded function h and a rational $\alpha < 1$ such that

$$\mathrm{K}(A\restriction_{h(i)}) \leq \alpha h(i) \qquad \text{for every } i, \tag{2}$$

i.e., if we can effectively find arbitrarily long prefixes of A that can be compressed by a factor of α. Theorem 41 below states that KL-stochastic sequences have effective dimension 1. This is equivalent to the fact that in the second mentioned result of Muchnik it is not necessary to require that the function h be computable, i.e., it suffices to require the mere existence of arbitrarily long prefixes of A that can be compressed by a factor of α.

In the remainder of the introduction we gather some notation that will be used throughout the text. Unless explicitly stated otherwise, the term *sequence* refers to an infinite binary sequence and a *class* is a set of sequences. A sequence S can be viewed as mapping $i \mapsto S(i)$ from ω to $\{0,1\}$, and accordingly we have $S = S(0)S(1)\ldots$. The term bit i of S refers to $S(i)$, the $(i+1)$st bit of the sequence S. Occasionally we identify a sequence S with the subset $\{i\colon S(i) = 1\}$ of the natural numbers ω.

The *initial segment of length* n, $S\restriction_n$, of a sequence S is the string of length n corresponding to the first n bits of S. Given two binary strings v, w, v is called a *prefix* of w, $v \sqsubseteq w$ for short, if there exists a string x such that $v^\wedge x = w$. If $v \sqsubseteq w$ and $v \neq w$, we will write $v \sqsubset w$. The same relation can be defined between strings and infinite sequences in an obvious way.

The plain and the prefix-free Kolmogorov complexity of a word w are denoted by $\mathrm{C}(w)$ and by $\mathrm{K}(w)$, respectively. For definitions and properties of Kolmogorov complexity we refer to the book by Li and Vitányi [11].

For a word u let $[u]$ denote the class of all sequences that have u as a prefix, and for a set of words U write $[U]$ for the union of the classes $[u]$ over all $u \in U$.

We will often deal with generalized joins and splittings. Assume that Z is an infinite and co-infinite set of natural numbers. The *Z-join* $A_0 \oplus_Z A_1$ of sequences A_0 and A_1 is the result of merging the sequences using Z as a guide. Formally,

$$A_0 \oplus_Z A_1(n) = \begin{cases} A_0(|\overline{Z} \cap \{0, \dots, n-1\}|) & \text{if } Z(n) = 0, \\ A_1(|Z \cap \{0, \dots, n-1\}|) & \text{if } Z(n) = 1. \end{cases}$$

On the other hand, given a sequence A and a set $Z \subseteq \omega$ one can obtain a new sequence (word) $A \restriction_Z$ by picking the positions that are in Z. Let p_Z denote the principal function of Z, i.e. $\mathrm{p}_Z(n)$ is the $(n+1)$st element of Z (where this is undefined if no such element exists). Formally,

$$A \restriction_Z (n) = A(\mathrm{p}_Z(n)), \quad \text{where } \mathrm{p}_Z(n) = \mu x[|Z \cap \{0, \dots, x\}| \geq n+1].$$

If Z is infinite, $A \restriction_Z$ will yield a new infinite sequence, otherwise we define $A \restriction_Z$ to be the word of length $|Z|$ extracted from A via Z. Note that this notation is consistent with the usual notation of initial segments in the sense that $A \restriction_n = A \restriction_{\{0,\dots,n-1\}}$. Observe that $A = A_0 \oplus_Z A_1$ if and only if $A \restriction_Z = A_1$ and $A \restriction_{\overline{Z}} = A_0$.

For functions f, g, the notation $f(n) \leq^+ g(n)$ means that there exists a constant c such that for all n, $f(n) \leq g(n) + c$.

2 Non-monotonic Betting Strategies

Intuitively speaking, a non-monotonic betting strategy defines a process that places bets on bits of a given sequence $X \in 2^\omega$. More precisely, the betting strategy determines a sequence of mutually distinct places $n_0, n_1, \dots$ at which it bets a certain portion of the current capital on the value of the respective bit of X being 0 or 1. (Note that, by betting none of the capital, the betting strategy may always choose to “inspect” the next bit only.) The place n_{i+1} and the bet which is to be placed depends solely on the previously scanned bits $X(n_0)$ through $X(n_i)$.

As a formal definition is somewhat tedious, we present it in a sequence of definitions.

1 Definition. *A* finite assignment (f.a.) *is a sequence*

$$x = (r_0, a_0) \dots (r_{n-1}, a_{n-1}) \in (\omega \times \{0,1\})^*$$

of pairs of natural numbers and bits. The set of all finite assignments is denoted by FA*.*

Finite assignments can be thought of as specifying partial values of an infinite binary sequence $X = X(0)\,X(1)\,X(2)\dots$, in the sense that $X(r_i) = a_i$ for $i < n$. If this is the case for some f.a. x, we write $x \sqsubset X$. Given an f.a. $x = (r_0, a_0) \dots (r_{n-1}, a_{n-1})$, the *domain* of x, $\mathrm{dom}(x)$ for short, is the set $\{r_0, \dots, r_{n-1}\}$; note that a f.a. induces a partial function from ω to $\{0,1\}$ with domain $\mathrm{dom}(x)$.

When betting, the player will successively gain more and more information on the sequence he bets against. Depending on his current knowledge of the sequence, he will determine the next place to bet on. We call the function which does this a *scan rule*.

2 Definition. *A* scan rule *is a partial function* $s : \mathrm{FA} \to \omega$ *such that*

$$(\forall w \in \mathrm{FA})\ [s(w) \notin \mathrm{dom}(w)]. \tag{3}$$

Condition (3) ensures that no place is scanned (and bet on) twice. A non-monotonic betting strategy consists of a scan rule and in addition endows each place selected with a bet.

3 Definition. *A* stake function *is a partial function from* FA *to* $[0,2]$*. A* non-monotonic betting strategy *is a pair* $b = (s, q)$ *which consists of a scan rule* s *and a stake function* q*.*

Intuitively, given a f.a. $x \sqsubset X$, the strategy picks $s(x)$ to be the next place to bet on. If $q(x) < 1$ it bets that $X(s(x)) = 1$, if $q(x) > 1$, it bets that $X(s(x)) = 0$, and if $q(x) = 1$, the strategy refrains from making a bet.

Note at this point that it is not really necessary to define a non-monotonic betting strategy on finite assignments. It is sufficient to give a binary word $w \in 2^{<\omega}$ representing the values $a_0, \dots, a_{n-1}$ of an f.a. If the sequence was obtained by a scan rule s, the places selected can be recovered completely from this information. Therefore, it suffices to consider betting strategies $b : 2^{<\omega} \to \omega \times [0,2]$ which satisfy condition (3) for the scan rule that is induced by keeping track of the first component of the given betting strategy.

2.1 Running a betting strategy on a sequence

We now describe the game that takes place when a non-monotonic betting strategy is applied to an infinite binary sequence. Formally, this induces a functional which maps sequences (or even assignments) to assignments (finite or infinite). So, in the following, assume X is a sequence and $b = (s, q)$ is a non-monotonic betting strategy.

The most important partial function o_b^X yields the *f.a. obtained so far*. This only depends on the scan rule s, and, of course, the bits of the sequence X, and is defined as follows: Let $o_b^X(0) = \varepsilon$, and, if $x_n = o_b^X(n)$ is defined, let

$$o_b^X(n+1) = x_n \,^\wedge\, (s(x_n), X(s(x_n))),$$

if $s(x_n)$ is defined, while $o_b^X(n+1)$ is undefined otherwise.

Formally, the payoff described above is then given by a partial function c_b^X where

$$c_b^X(n+1) = \begin{cases} q(o_b^X(n)), & \text{if } X(s(o_b^X(n))) = 0, \\ 2 - q(o_b^X(n)), & \text{if } X(s(o_b^X(n))) = 1. \end{cases}$$

For given initial capital $d_b(\varepsilon)$, the partial *capital function* d_b^X is now easily described:

$$d_b^X(n) = d_b(\varepsilon) \prod_{i=1}^{n} c_b^X(i). \tag{4}$$

Finally, we can define the randomness notion induced by non-monotonic betting strategies.

4 Definition. *(1) A non-monotonic betting strategy b* succeeds on *a sequence A if*

$$\limsup_{n\to\infty} d_b^A(n) = \infty\,.$$

(2) A subclass $\mathcal{C}$ of 2^ω is a KL-nullset *if there is a partial computable non-monotonic betting strategy that succeeds on all $A \in \mathcal{C}$.*

(3) A sequence A is KL-random *if there is no computable non-monotonic betting strategy that succeeds on A.*

The concept of KL-randomness remains the same if one uses in its definition partial computable instead of computable non-monotonic betting strategies [18].

2.2 Further notions of randomness

In a setting of monotonic betting strategies it does matter whether randomness is defined with respect to computable or partial computable betting strategies [18]; accordingly a sequence is called *computably random* and *partial computably random* if there is no computable or partial computable, respectively, monotonic betting strategy that succeeds on the sequence. For monotonic betting strategies, the f.a. obtained so far is always a prefix of the given sequence and often we write for example $d_b(X \upharpoonright_n)$ instead of $d_b^X(n)$. For a monotonic betting strategy b, the function d_b satisfies for all words w the following fairness condition

$$d_b(w) = \frac{d_b(w0) + d_b(w1)}{2} . \quad (5)$$

A function d from words to reals such that (5) with d in place of d_b holds for all words w will be called a *martingale*. A pair of a monotonic betting strategy and an initial capital determines a martingale and, conversely, any martingale determines an initial capital and a monotonic betting strategy. Accordingly, we will extend concepts defined for betting strategies to the corresponding martingales and vice versa and occasionally we will specify betting strategies by giving the corresponding martingale.

Furthermore, we consider Martin-Löf random sequences [15]. Let $W_0, W_1, \ldots$ be a standard enumeration of the recursively enumerable sets. Recall that a sequence $(A_n \colon n \geq 1)$ of sets is called uniformly recursively enumerable if there is a computable function g such that for all $n \geq 1$ we have $A_n = W_{g(n)}$. Recall further that the Lebesgue measure λ on Cantor space is obtained by determining the bits of a sequence by independent tosses of a fair coin, i.e. it is equivalent to the $(1/2, 1/2)$-Bernoulli measure, i.e., the uniform measure on Cantor space.

5 Definition. *A* Martin-Löf test *is a uniformly recursively enumerable sequence* $(A_n \colon n \in \omega)$ *of sets of words such that for every* n*,*

$$\lambda([A_n]) \leq \frac{1}{2^{n+1}} . \quad (6)$$

A sequence X *is* covered *by a sequence* $(A_n \colon n \in \omega)$ *of sets of words if for every* n *the class* $[A_n]$ *contains* X*. A sequence is* Martin-Löf random *if it cannot be covered by any Martin-Löf test.*

A Martin-Löf test is called a *Schnorr test* if the measure on the left-hand side of (6) is computable in n (in the usual sense that the measure can be approximated effectively to any given precision strictly larger than 0); a sequence is called *Schnorr-random* if it cannot be covered by a Schnorr test.

A non-monotonic selection rule is a pair (s, c) of a scan rule s and a partial function $c\colon \mathrm{FA} \to \{0, 1\}$. A selection rule is applied to a given sequence X in the same way as a betting strategy, except that instead of specifying a bet on every next bit to be scanned, the function c simply determines whether the next bit should be selected ($c(x) = 1$) or not ($c(x) = 0$). The sequence selected from X by (s, c) is then the sequence of all bits that are selected, in the order of selection. A sequence X is called stochastic with respect to a given class of admissible selection rules if no selection rule in the class selects from X an infinite sequence that is biased in the sense that the frequencies of 0's and 1's do not converge to 1/2. A sequence is *Kolmogorov-Loveland stochastic* or *KL-stochastic*, for short, if the sequence is stochastic with respect to the class of computable non-monotonic selection rules. Like for KL-randomness, it can be shown that the notion of KL-stochasticity remains the same if we allow partial computable non-monotonic selection rules [18]. A sequence is *Mises-Wald-Church stochastic* or *MWC-stochastic*, for short, if the sequence is stochastic with respect to the class of partial computable monotonic selection rules.

3 Basic results on non-monotonic randomness

This section gives some basic results that illustrate how non-monotonic betting strategies work.

Non-monotonic betting strategies exhibit a behavior quite different from other randomness concepts when studying the combined capabilities of two or more strategies. More precisely, the classes that can be covered by computable or partial computable non-monotonic betting strategies are not closed under union; the latter can be seen by considering the class of all r.e. sets.

6 Proposition. *No partial computable non-monotonic betting strategy can succeed on all r.e. sets.*

Proof. Let $b = (s, q)$ be a partial computable non-monotonic betting strategy. We show that there exists a r.e. set W such that b does not succeed on W. For this purpose, we compute a sequence (x_n) of finite assignments, $x_n = (r_0, a_0) \ldots (r_{n-1}, a_{n-1})$. Start with $x_0 = \varepsilon$, and set $r_{n+1} = s(x_n)$ and

$$a_{n+1} = \begin{cases} 1, & \text{if } q(x_n) \geq 1, \\ 0, & \text{if } q(x_n) < 1. \end{cases}$$

Enumerate r_{n+1} into W if $a_{n+1} = 1$. (If $b(x_n)$ is undefined at some stage, the enumeration process will get stuck here as well and the resulting set W will be finite.) Obviously, W is defined in a way such that b does not win a single bet against it, hence, in particular, does not succeed on W. □

The following proposition contrasts Proposition 6.

7 Proposition. *There exist computable non-monotonic betting strategies b_0 and b_1 such that for every r.e. set W, at least one of b_0 and b_1 will succeed on W.*

Proof. Define $b_0 = (s_0, q_0)$ to be the following simple betting strategy which is meant to be applied with initial capital 1. Let the stake function q_0 be constant with value 5/3, i.e., always exactly 2/3 of the current capital are bet on the next bit being 0. Let $s_0(\varepsilon) = 0$ and for all $x_n \neq \varepsilon$ let $s_0(x_n) = 1 + \max \operatorname{dom} x_n$, i.e, in particular, for $x_n = (0, a_0) \dots (n-1, a_{n-1})$ we have $s_0(x_n) = n$. Hence, b_0 is a *monotonic* betting strategy that always bets 2/3 of its current capital on the next bit being 0. An easy calculation shows that this betting strategy succeeds in particular on all sequences A such that there are infinitely many prefixes of A where less than 1/4 of the bits in the prefix are equal to 1.

To define b_1, fix a partition $I_0, I_1, \dots$ of the natural numbers into consecutive, pairwise disjoint intervals I_k such that $|I_{k+1}| \geq 5|I_k|$ and for every natural number e let

$$D_e = \bigcup_{j \in \omega} I_{<e,j>} ,$$

where $< .,. >$ is the usual effective pairing function. By the discussion in the preceding paragraph, b_0 succeeds on any set that has a finite intersection with any of the sets D_e, hence it suffices to construct a non-monotonic betting strategy $b_1 = (s_1, q_1)$ that succeeds on all sets W_e that have an infinite intersection with D_e. Fix an effective enumeration $(e_1, z_1), (e_2, z_2), \dots$ without repetitions of all pairs (e, z) where $z \in W_e \cap D_e$. Divide the capital function $d^X_{b_1}$ into infinitely many parts d^X_e, where it will always hold that the $d^X_e(n)$ sum up to $d^X_{b_1}(n)$. Start with $d^X_e(0) = 2^{-e-1}$, i.e., for every number e reserve a share of 2^{-e-1} of the initial capital 1; then, given a f.a. $x_n = (r_0, a_0) \dots (r_{n-1}, a_{n-1})$, let

$$s_1(x_n) = z_n \quad \text{and} \quad q_1(x_n) = 1 - \frac{d^X_{e_n}(n)}{d^X_{b_1}(n)},$$

hence betting all the capital obtained by $d^X_{e_n}$ so far on the outcome that the z_nth position in the infinite sequence revealed during the application of the strategy is 1. Then b_1 succeeds on all sets W_e where $W_e \cap D_e$ is infinite because for any number in the latter set the capital $d_e^{W_e}$ is doubled. □

We can immediately deduce that KL-nullsets are not closed under finite union.

8 Proposition. *The KL-nullsets are not closed under finite union, that is, if there are partial computable non-monotonic betting strategies b and b' that succeed on classes $\mathcal{X} \subseteq 2^\omega$ and $\mathcal{Y} \subseteq 2^\omega$, respectively, then there is not necessarily a partial computable non-monotonic betting strategy that succeeds on $\mathcal{X} \cup \mathcal{Y}$.*

Proposition 7 also is immediate from the proof of the first mentioned result by Muchnik [20]. This result goes beyond the fact that for any given computable, unbounded function h the sequences A that satisfy (1) cannot be KL-random. Indeed, two partial computable non-monotonic betting strategies are given such that any such sequence is covered by one of them. These non-monotonic betting strategies can be transformed into equivalent total ones by an argument similar to the proof that the concepts of KL-randomness with respect to partial and total computable non-monotonic betting strategies are the same. But for any r.e. set W, the length n prefix of W can be coded by at most $\log n$ bits, hence there is a function h as required that works for all r.e. sets.

9 Remark. Let b be a computable non-monotonic betting strategy that on every sequence scans all places of the sequence. Then there is a monotonic betting strategy that succeeds on every sequence on which b succeeds.

For a proof, observe that by compactness of Cantor space there is a computable function t such that for every sequence X and all n the betting strategy b, when applied to X, uses at most $t(n)$ computation steps before scanning the bit $X(n)$. The assertion then is immediate by a result of Buhrman, van Melkebeek, Regan, Sivakumar and Strauss [4], who show that for any non-monotonic betting strategy b where there is such a computable function t, there is an equivalent computable monotonic betting strategy that succeeds on all the sequences on which b succeeds.

10 Proposition. *The class of computably random sequences is closed under computable permutations of the natural numbers.*

Proof. Assume that there were a computably random sequence X that loses this property after permuting its bits according to a computable permutation, i.e., there is a computable monotonic betting strategy b that succeeds on the permuted sequence. By composing the computable inverse of the given permutation with b we obtain a computable non-monotonic betting strategy b' that succeeds on X, where b' scans all places of every sequence. Hence by Remark 9 there is a computable monotonic strategy that succeeds on X, thus contradicting our assumption on X. □

4 Splitting properties of KL-random sequences

Proposition 8 suggests that KL-nullsets behave very differently from Martin-Löf nullsets, which are all covered by a universal Martin-Löf test (and hence are closed under finite unions). On the other hand, KL-random sequences exhibit some properties which makes them appear quite "close" to Martin-Löf random sequences.

For the splitting property of KL-random sequences stated in Proposition 11 it is essential that the considered martingales are non-monotonic; in Proposition 13 it is shown that a corresponding assertion for computably random sequences is false.

11 Proposition. *Let Z be a computable, infinite and co-infinite set of natural numbers, and let $A = A_0 \oplus_Z A_1$. Then A is KL-random if and only if*

$$A_0 \textit{ is } KL^{A_1}\textit{-random} \quad \textit{and} \quad A_1 \textit{ is } KL^{A_0}\textit{-random}. \tag{7}$$

Proof. First assume that A is not KL-random and let b be a computable nonmonotonic betting strategy that succeeds on A. According to (4) the values of the gained capital $d_b^A(n)$ are given by multiplying the factors $c_b^A(i)$, and when splitting each of these products into two subproducts corresponding to the bets on places in Z and $\overline{Z}$, respectively, then in case d_b^A is unbounded at least one of these two subproducts must be unbounded. So the bets of places in Z or in the complement of Z alone must result in an unbounded gain. Accordingly there is a computable nonmonotonic betting strategy that succeeds on A by scanning exactly the same bits in the same order as b, while betting only on the bits in either the Z part or $\overline{Z}$ part of A, which implies that (7) is false.

Next suppose that a non-monotonic betting strategy b^{A_1} computable in A_1 succeeds on A_0. We devise a new computable non-monotonic betting strategy which succeeds on A. Of course, the idea is as follows: Scan the Z-positions of A (corresponding to A_1) until we find an initial segment of A_1 which allows to compute a new value of b^{A_1}. Consequently, bet on A_0 according to b^{A_1}.

Formally, given an f.a. x_n, split it into two sub-f.a. x_n^0 and x_n^1, where (r_k, a_k) is a part of x_n^i if and only if $Z(r_k) = i$. Now define

$$b(x_n) = \begin{cases} b^{x_n^1}(x_n^0) & \text{if } b^{x_n^1}(x_n^0)\downarrow \text{ in } |x_n^1| \text{ steps,} \\ (\min \overline{\operatorname{dom} x_n^1} \cap Z, 1) & \text{otherwise.} \end{cases}$$

This completes the proof. □

This rather simple observation stated in Proposition 11 has some interesting consequences. One is that splitting a KL-random sequence by a computable set yields at least one part that is Martin-Löf random.

12 Theorem. *Let Z be a computable, infinite and co-infinite set of natural numbers. If the sequence $A = A_0 \oplus_Z A_1$ is KL-random, then at least one of A_0 and A_1 is Martin-Löf random.*

Proof. Suppose neither A_0 nor A_1 is Martin-Löf random. Then there are Martin-Löf tests $(U_n^0 \colon n \in \omega)$ and $(U_n^1 \colon n \in \omega)$ with $U_n^i = \{u_{n,0}^i, u_{n,1}^i, \dots\}$, such that for $i = 0, 1$,

$$A_i \in \bigcap_{n\in\omega} \bigcup_{k\in\omega} [u_{n,k}^i].$$

Define functions f_0, f_1 by $f_i(n) = \min\{k \in \omega \colon u_{n,k}^i \sqsubset A_i\}$. Obviously the following must hold:

$$(\exists i)\ (\overset{\infty}{\exists} m)\ [f_i(m) \geq f_{1-i}(m)].$$

We define a new Martin-Löf test (V_n) by

$$V_n = \bigcup_{m>n} \bigcup_{k=0}^{f_i(m)} [u_{m,k}^{1-i}].$$

Then $\{V_n\}$ is a Schnorr test relative to the oracle A_i (a Schnorr^{A_i}-test) and covers A_{1-i}, so A_{1-i} is not Schnorr^{A_i}-random. Since KL-randomness implies Schnorr-randomness (for relativized versions, too), it follows that A_{1-i} is not KL^{A_i}-random, contradicting Theorem 11. □

We use the same method to give an example of a computably random sequence where relative randomness of parts, in the sense of Proposition 11, fails. Here Z is the set of even numbers, and we write $A \oplus B$ instead of $A \oplus_Z B$. The same example works for Schnorr randomness.

13 Proposition. *There is a computably random (and hence Schnorr random) sequence $A = A_0 \oplus A_1$ such that for some $i \in \{0, 1\}$, A_i is not Schnorr random relative to A_{1-i}.*

Proof. One can construct a computably random sequence $A = A_0 \oplus A_1$ such that, for each n, $\mathrm{K}(A\restriction_n) \leq^+ n/3$ [10, 17]. Then, for $i = 0$ and for $i = 1$, $\mathrm{K}(A_i\restriction_n) \leq^+ 2n/3$, hence by Schnorr's characterization of Martin-Löf randomness [11], neither A_0 nor A_1 are Martin-Löf random. Now the construction in the proof above shows that for some $i \in \{0, 1\}$, A_i is not Schnorr random relative to A_{1-i}. □

14 Remark. Let Z be a computable, infinite and co-infinite set of natural numbers. If the sequence $A = A_0 \oplus_Z A_1$ is KL-random relative to some oracle X, then at least one of A_0 and A_1 is Martin-Löf random relative to X.

For a proof it suffices to observe that the proof of Theorem 12 relativizes.

An interesting consequence of (the relativized form of) Theorem 12 is stated in Theorem 17; in the proof of this theorem we will use Remark 15, due to van Lambalgen [32] (also see Downey et al. [7] for a proof).

15 Remark. Let Z be a computable, infinite and co-infinite set of natural numbers. The sequence $A = A_0 \oplus_Z A_1$ is Martin-Löf random if and only if A_0 is Martin-Löf random and A_1 is Martin-Löf random relative to A_0. (Furthermore, this equivalence remains true if we replace Martin-Löf randomness by Martin-Löf randomness relative to some oracle.)

The closest one can presently come to van Lambalgen's theorem for KL-randomness is Proposition 11. Note the subtle difference: in the case of Martin-Löf randomness, one merely needs A_0 to be random, not random relative to A_1.

16 Definition. *A set Z has* density α *if*

$$\lim_{m\to\infty} \frac{|Z \cap \{0, \dots, m-1\}|}{m} = \alpha \,. \tag{8}$$

17 Theorem. *Let R be a KL-random sequence and let $\alpha < 1$ be a rational. Then there is a computable set Z of density at least α such that $R\restriction_Z$ is Martin-Löf random.*

Proof. For a start, we fix some notation for successive splits of the natural numbers. Let $\{\mathrm{N}_w\}_{w\in\{0,1\}^*}$ be a uniformly computable family of sets of natural numbers such that for all w,

$$(i)\ \mathrm{N}_\varepsilon = \omega \,, \qquad (ii)\ \mathrm{N}_w = \mathrm{N}_{w0} \dot\cup \mathrm{N}_{w1} \,, \qquad (iii)\ \mathrm{N}_w \text{ has density } \frac{1}{2^{|w|}} \,,$$

where $\dot\cup$ denotes disjoint union. For example, we may define

$$\mathrm{N}_{a_0 a_1 \dots a_{m-1}} = \{a_0 + a_1 \cdot 2 + \ldots + a_{m-1} \cdot 2^{m-1} + n \cdot 2^m \colon n \in \omega\} ,$$

which obviously satisfies (i), (ii) and (iii). By (iii), for any word w the complement $\overline{\mathrm{N}_w}$ of N_w has density $1 - 1/2^{|w|}$, thus it suffices to show that there are words $w_1 \sqsubseteq w_2 \sqsubseteq \ldots$ such that for all i,

$$(iv)\ |w_i| = i \quad \text{and} \quad (v)\ R_i = R\restriction_{\overline{\mathrm{N}_{w_i}}} \text{ is Martin-Löf random} .$$

The w_i are defined inductively. For a start, observe that by Theorem 12 for $r_1 = 0$ or for $r_1 = 1$ the sequence $R\restriction_{\mathrm{N}_{r_1}}$ is Martin-Löf random; pick r_1 such that the latter is true and let $w_1 = 1 - r_1$. For $i > 1$, let w_i be defined as follows. By Proposition 11 the sequence $R\restriction_{\mathrm{N}_{w_{i-1}}}$ is KL-random relative to R_{i-1}, hence by (ii) and by the relativized version of Theorem 12 stated in Remark 14, for $r_i = 0$ or for $r_i = 1$ the sequence $R\restriction_{\mathrm{N}_{w_{i-1}r_i}}$ is Martin-Löf random relative to R_{i-1}; pick r_i such the latter is true and let $w_i = w_{i-1}(1 - r_i)$.

Now (iv) follows for all i by an easy induction argument. We already have that R_1 is Martin-Löf random. Assuming that R_i is Martin-Löf random and using van Lambalgen's result from Remark 15, we obtain that the sequence R_{i+1} is Martin-Löf random since r_{i+1} is chosen such that $R\restriction_{\mathrm{N}_{w_i r_{i+1}}}$ is Martin-Löf random relative to R_i and because for some appropriate computable set Z we have

$$R_{i+1} = R\restriction_{\overline{\mathrm{N}_{w_{i+1}}}} = R\restriction_{\overline{\mathrm{N}_{w_i}}} \oplus_Z R\restriction_{\mathrm{N}_{w_i r_{i+1}}} .$$

This completes the proof. □

The functions f_i in the proof of Theorem 12 can be viewed as a modulus for a certain type of approximation to the sequences under consideration. The technique of comparing two given moduli can also be applied to other types of moduli, e.g., to a modulus of convergence of an effectively approximable sequence. Recall that a function g majorizes a function m if $g(n) \geq m(n)$ for all n.

18 Remark. Let A be a sequence in Δ^0_2, i.e., A is the pointwise limit of uniformly computable sequences $A_0, A_1, \ldots$, and let

$$m(n) = \min\{s \colon s > n \text{ and } A_s\restriction_n = A\restriction_n\} .$$

Then A is computable relative to any function g that majorizes m.

For a sketch of proof, let T be the set of all words w such that for all prefixes u of w there is some s where $|u| \leq s \leq g(|u|)$ and u is a prefix of A_s. Then T is a tree that is computable in g and the sequence A is an infinite path of T because g majorizes m. Furthermore, A is the only infinite path on T because any word u that is not a prefix of A has only finitely many extensions on T, hence A is computable in g. For details of this standard argument see for example Odifreddi [21, I, V.5.3 d].

19 Theorem. *Let Z be a computable, infinite and co-infinite set of natural numbers and let $A = A_0 \oplus_Z A_1$ be KL-random where A_1 is in Δ^0_2. Then A_0 is Martin-Löf random.*

Proof. We modify the proof of Theorem 12. For a proof by contradiction, assume that A_0 is not Martin-Löf random and define f_0 as in the proof of Theorem 12. Let f_1 be defined similar to the definition of the modulus m in Remark 18, i.e., $f_1(n)$ is the least $s > n$ such that some fixed effective approximation to A agrees after s steps with A on the first n places. In case f_0 majorized f_1 at all but finitely many places, the sequence A_1 were computable in a finite variant of f_0, hence in f_0, and hence in A_0, contradicting the assumption that A is KL-random. Otherwise we argue as before that A_0 is not Schnorr-random relative to A_1, again contradicting the assumed KL-randomness of A. □

By applying Theorem 19 to the set Z and its complement, the following Corollary is immediate.

20 Corollary. *Let Z be a computable, infinite and co-infinite set of natural numbers and let $A = A_0 \oplus_Z A_1$ be KL-random and Δ^0_2. Then A_0 and A_1 are both Martin-Löf random.*

A function g is an *order* if g is computable, nondecreasing, and unbounded.

21 Corollary. *Suppose A is in Δ^0_2 and is KL-random. Then for each order g and almost all n, $\mathrm{K}(A\restriction_n) \geq n - g(n)$.*

Proof. Let Z be a computable co-infinite set that for all n contains at least $n - g(n)/2$ of the first n natural numbers. Let A_0 and A_1 be the sequences such that $A = A_0 \oplus_Z A_1$. Then

$$\mathrm{K}(A_1\restriction_{(n-g(n)/2)}) \leq^+ \mathrm{K}(A\restriction_n) ,$$

because the first $n - g(n)/2$ bits of A_1 can be effectively recovered from the first n bits of A. So if $\mathrm{K}(A\restriction_n) \leq n - g(n)$ for infinitely many n, for each

such n the prefix of length $n - g(n)/2$ of A_1 would be compressible, up to an additive constant, by at least $g(n)/2$ bits, hence A_1 would not be Martin-Löf random. Since A and hence also A_0 is in Δ^0_2, this contradicts Theorem 19.
□

5 Counterexamples

5.1 Splicing zeroes into Ω

A sequence X can be identified with the real that has the binary expansion $0.X(0)X(1)\ldots$. A real is called *left-r.e.* if it the limit of a nondecreasing computable sequence of rationals. First we give an example of a left-r.e. real A which is not MWC-stochastic, but satisfies $\mathrm{K}(A\upharpoonright_n) \geq^+ n - g(n)$ for each order g and almost all n. Thus even for left-r.e. reals, the conclusion of Corollary 21 is not equivalent to KL-randomness.

The idea is to "splice" into Ω a very sparse Π^0_1 set of zeros. Recall that Ω is the halting probability of some fixed universal prefix-free Turing machine and that the real Ω is left-r.e. and (its binary expansion) is Martin-Löf random [6].

22 Definition. *For sequences X and S, let* $\mathrm{Splice}(X, S)$ *be equal to* $X \oplus_S \emptyset$ (*where* $\emptyset$ *is the sequence consisting of zeroes only*).

23 Lemma. *If X is left-r.e. and B is r.e. then* $\mathrm{Splice}(X, \overline{B})$ *is also left-r.e.*

This is easy to verify for an r.e. set B, as n entering B means that in $\mathrm{Splice}(X, \overline{B})$ one bit 0 is cancelled and certain bits shift to the left.

24 Proposition. *If the r.e. set B is co-infinite, then* $A = \mathrm{Splice}(\Omega, \overline{B})$ *is not MWC-stochastic.*

Proof. We may assume $\mathrm{p}_{\overline{B}}(n) \geq 2n$ for almost all n, else A violates the law of large numbers. When n enters B, then a prefix code of n (of length $\leq^+ 2\log n$) enters the domain of the universal machine. So, for some constant d, if $\Omega\upharpoonright_{2\log n+d}$ has settled by stage s, then $B_s\upharpoonright_n = B\upharpoonright_n$. Fix n_0 such that $2\log n_0 + d \leq n_0/2$, and pick s_0 so that $\tau = A\upharpoonright_{n_0}$ has settled by s_0. Let $m = n_0 - |\overline{B} \cap \{0, \ldots, n_0 - 1\}|$. Note that $\sigma_0 = \Omega\upharpoonright_m$ has settled by s_0.

We define a monotonic partial computable selection rule l which succeeds on A. We also define a sequence $\sigma_0 \sqsubset \sigma_1 \sqsubset \ldots$ and a sequence of stages $s_0 \leq s_1 \leq \ldots$ such that $\sigma_i \sqsubset \Omega_{s_i}$. The selection rule l behaves as follows on reals R extending σ_0. Suppose we have defined σ_i, s_i and so far have scanned $\rho \sqsubset R$, $|\rho| = k$. If $k \notin B_{s_i}$, then select k (note that, if $R = A$, then by the choice of n_0, in fact $k \notin B$). Otherwise, interpret the position k as a further bit of Ω. Thus, scan this bit $h = R(k)$ without selecting, and let $\sigma_{i+1} = \sigma_i h$. Let s_{i+1} be the first stage $s \geq s_i$ (if any) such that $\sigma_{i+1} \sqsubset \Omega_s$. (If s is not found then L is undefined.) Note that l has the desired effect on A; in particular, it selects an infinite sequence of positions on which A is 0. □

25 Proposition. *Let R be Martin-Löf random, let B be dense simple, and let $A = \mathrm{Splice}(R, B)$. Then for each order g and almost all n, we have that*

$$\mathrm{K}(A\restriction_n) \geq^+ n - g(n) \ .$$

Proof. As we may modify g, it suffices to show $(\forall n)\ \mathrm{K}(A\restriction_n) \geq^+ n - g(n) - 2\log g(n)$ for each order g. Let $h(u) = \max\{n : g(n) \leq u\}$. Let $S = \overline{B}$. Since B is dense simple, $\mathrm{p}_S(u) \geq h(u)$ for almost all u. Hence there is n_0 such that, for $n \geq n_0$, $\mathrm{p}_S(g(n)) \geq n$.

Given $n \geq n_0$, let $v = |S \cap \{0, \ldots, n-1\}|$. So $v \leq g(n)$. Given $A\restriction_n$ and v, we may enumerate B till v elements $< n$ are left, and then recover $R\restriction_{n-v}$. Since R is random and $v \leq g(n)$,

$$n - g(n) \leq n - v \leq^+ \mathrm{K}(R\restriction_{n-v}) \leq^+ \mathrm{K}(A\restriction_n) + 2\log g(n).$$

This completes the proof. □

26 Theorem. *There is a left-r.e. sequence A that is not MWC-stochastic such that for each order g and almost all n,*

$$\mathrm{K}(A\restriction_n) \geq^+ n - g(n) \ .$$

Proof. By the results of this section, it is immediate that it suffices to let $A = \mathrm{Splice}(\Omega, \overline{B})$ for some dense simple (and hence r.e.) set B. □

27 Remark. Miller and Yu [19] have shown that A is Martin-Löf random iff $(\forall n)\ \mathrm{C}(A\restriction_n) \geq^+ n - \mathrm{K}(n)$. A similar argument to the previous two proposition shows that this bound is quite sharp. It is not hard to construct an r.e. co-infinite set B such that $|B \cap \{0, \ldots, n-1\}| \geq n - K(n)$. (This can be done, for example, by using a standard *movable markers* construction from computability theory.) Now it can be proved that if $A = \mathrm{Splice}(\Omega, \overline{B})$, then A is a non-stochastic left-r.e. real such that $(\forall n)\ \mathrm{C}(A\restriction_n) \geq^+ n - 2K(n)$.

5.2 A non-random sequence all of whose parts are random

Our next example shows that splitting properties like the one considered in Corollary 20 do not necessarily imply Martin-Löf randomness. Recall from the introduction that $\mathrm{p}_G(i)$ is the $(i+1)$st element of G and that $A\restriction_G$ is defined by $A\restriction_G(i) = A(\mathrm{p}_G(i))$.

28 Theorem. *There is a sequence A which is not computably random such that for each computable infinite and co-infinite set V, $A\restriction_V$ is 2-random.*

Proof. We build an r.e. equivalence relation on ω where the equivalence classes are finite intervals. The idea underlying the construction of A is to make the last bit of each such interval a parity bit, while A is 2-random on the other bits. The approximations to the equivalence classes are not changed too often, and accordingly a computable martingale can succeed on A by betting on all the places that are the maximum element of some approximation of an equivalence class. Furthermore, during the construction it is ensured that for any computable infinite and co-infinite set V, the complement of V meets almost every equivalence class, hence the sequence $A\restriction_V$ is 2-random because, intuitively speaking, the parity bits do not help a martingale that is left-computable relative to $\emptyset'$ when betting on $A\restriction_V$.

The equivalence class $[x]$ of each place x will be a finite interval. Then, in order to define A, fix any 2-random sequence Z and let

$$A\restriction_S = Z \quad \text{where} \quad S = \{x : x \neq \max[x]\},$$

while for x that are maximum in the interval $[x]$, let $A(x)$ be equal to the parity bit of the restriction of A to the remainder of the interval, i.e., for x not in S, let

$$A(x) = u_x \quad \text{where} \quad u_x = \begin{cases} 0 & \text{if } |A \cap ([x] - \{x\})| \text{ is even;} \\ 1 & \text{otherwise.} \end{cases}$$

During the construction, it will be ensured that each infinite r.e. set intersects almost all equivalence classes and, furthermore, that with oracle access to the halting problem $\emptyset'$, a canonical index for $[x]$ can be computed from x. The properties of the equivalence relation stated so far already suffice to verify the condition on 2-randomness.

Claim 1 *Let G be a set that is computable in $\emptyset'$ and contains for each x the whole interval $[x]$ except for exactly one place. Then the sequence $A\restriction_G$ is 2-random.*

Proof. Schnorr's characterization [27] of Martin-Löf randomness in terms of incompressibility can be relativized to $\emptyset'$, i.e., for any sequence R we have

$$R \text{ is 2-random} \quad \Leftrightarrow \quad (\exists c \in \omega)(\forall n \in \omega)[\mathrm{K}^{\emptyset'}(R\restriction_n) \geq n - c],$$

where $\mathrm{K}^{\emptyset'}(.)$ denotes prefix-free Kolmogorov complexity relativized to the halting problem. By this characterization it suffices to show that with respect to the latter type of Kolmogorov complexity and up to an additive constant c_G, the complexity of the prefixes of $A\restriction_G$ is as least as large as the complexity of the prefixes of the 2-random sequence Z, i.e., we have for all n,

$$\mathrm{K}^{\emptyset'}(Z\restriction_n) \leq^+ \mathrm{K}^{\emptyset'}((A\restriction_G)\restriction_n). \tag{9}$$

In order to demonstrate (9), we show that

$$\mathrm{K}^{\emptyset'}(Z\restriction_n) \leq^+ \mathrm{K}^{\emptyset'}(A\restriction_{\mathrm{p}_S(n)}) \leq^+ \mathrm{K}^{\emptyset'}(A\restriction_{\mathrm{p}_G(n)}) \leq^+ \mathrm{K}^{\emptyset'}((A\restriction_G)\restriction_n). \tag{10}$$

Every equivalence class contains exactly one place that is not in S and exactly one place that is not in G, hence the functions p_S and p_G differ at most by one and the middle inequality in (9) follows. Concerning the first inequality observe that by definition of A, with access to the halting problem it is possible to compute $Z\restriction_n$ from $A \restriction \mathrm{p}_S(n)$ by simply cancelling all bits of the latter word that correspond to places not in S. Concerning the last inequality, observe that with access to the halting problem one can compute $A\restriction_{\mathrm{p}_G(n)}$ from $(A\restriction_G)\restriction_n$ because for any y, due to the parity bit $A(\max[y])$, the bit $A(y)$ is determined by the restriction of A to $[y] \setminus \{y\}$, hence for any place y not in G, which is then the only place in $[y]$ that is not in G, the bit $A(y)$ is determined by $A\restriction_G$ □

Claim 2 *For any computable infinite and co-infinite set V, the sequence $A\restriction_V$ is 2-random.*

Proof. Given V as in the claim, let $W_e = \overline{V}$. Then by construction, almost all equivalence classes have a nonempty intersection with W_e. Let H contain the least element of each such intersection plus the least elements of the finitely many equivalence classes that do not intersect W_e and let G be the complement of H. The set H intersects each equivalence class in exactly one place and is computable in $\emptyset'$, hence by Claim 1 the sequence $A\restriction_G$ is 2-random. Recalling the characterization of Martin-Löf randomness in terms of left-computable martingales, also $A\restriction_V$ is 2-random because if there were a $\emptyset'$-left-computable martingale d_V that succeeded on $A\restriction_V$, there would be a $\emptyset'$-left-computable martingale d_G that succeeds on $A\restriction_G$ by simply simulating the bets of d_V on all the bits of $A\restriction_G$ that correspond to bits of $A\restriction_V$. Observe in this connection that V is a subset of G up to finitely many places, hence for almost all the bets of d_V on $A\restriction_V$ there are corresponding bets of d_G on $A\restriction_G$. □

The equivalence classes $[x]$ are constructed in stages $s = 0, 1, \ldots$, where $[x]_s$ denotes the approximation of $[x]$ at the beginning of stage s. Let $g(x)$ denote the number of equivalence classes $[y]$ where $y \leq x$, and let $g_s(x)$ be defined likewise with $[y]$ replaced by $[y]_s$. Note that g is a nondecreasing unbounded function and that the $g_s(x)$ converge nonincreasingly to $g(x)$.

In order to ensure that every infinite r.e. set intersects almost all equivalence classes, the requirement

$$P_e\colon e < \log(g(x)) - 1 \Rightarrow W_e \cap [x] \neq \emptyset$$

is met for every e such that W_e is infinite. For all x, let $[x]_0 = \{x\}$. During stage $s \geq 0$, the finite intervals $[x]_{s+1}$ are defined as follows. Requirement P_e *requires attention* at stage s via x if $x \leq s$, $x = \min([x]_s)$, and there is some $e < \log(g_s(x)) - 1$ where

$$[x]_s \cap W_{e,s} = \emptyset \qquad \text{and} \qquad \{x, x+1, \ldots, s\} \cap W_{e,s} \neq \emptyset.$$

If some P_e needs attention at stage s, take the least such e and choose the least x such that P_e needs attention via x, and let

$$[y]_{s+1} = \begin{cases} \{x, x+1, \ldots, s\} & \text{in case } y \in \{x, x+1, \ldots, s\}, \\ [y]_s & \text{otherwise.} \end{cases}$$

Otherwise, in case no P_e requires attention, let $[y]_{s+1} = [y]_s$ for all y. By a standard finite injury argument, which is left to the reader, all equivalence classes are finite intervals and requirement P_e is met for all e such that W_e is infinite. Since the construction is effective, a canonical index for $[x]$ can be computed from x with oracle access to the halting problem.

It remains to show that A is not computably random, i.e., that there is a computable martingale d that succeeds on A. The martingale d exploits the redundancy given by the parity bits $A(\max[x])$, and the idea underlying the construction of d is to work with a small number of candidates for the maximum place of the current equivalence class, where d plays on these candidates a doubling strategy until the first win occurs. In order to define the candidates, observe that for every x the maximum number z in $[x]$ enters this equivalence class during stage z, hence z is in the set

$$D_x = \{y\colon y = \max[x]_{y+1}\}\ .$$

It is instructive to observe that the set D_x contains exactly the $y \geq x$ such that $y = \max[x]_s$ for some s and that $\min D_x = x$ and $\max D_x = \max[x]$.

Claim 3 *For any x, the set D_x has at most $\log(g_x(x))$ elements.*

Proof. The maxima of the equivalence classes $[x]_s$ and $[x]_{s+1}$ can only differ for stages $s \geq x$ where the minimal pair (e, x') such that e requires attention via x satisfies $x' \leq x$. In this situation the index e satisfies

$$e < \log(g_s(x')) - 1 \leq \log(g_s(x)) - 1 \leq \log(g_x(x)) - 1$$

by definition of requiring attention and by the monotonicity properties of the functions g and log. Furthermore, for all $x' \leq x$, the set W_e intersects $[x']_s$ and thus intersects $[x']_{s'}$ for all $s' \geq s$, hence e will not require attention via some $x' \leq x$ at any later stage. In summary, the approximation to the maximum of $[x]$ will be changed at most $\log(g_x(x)) - 1$ times and Claim 3 follows. □

Claim 4 *The sequence A is not computably random.*

Proof. For given x, consider the monotonic martingale d_x that bets only on places in D_x, starting with a bet of stake $q_x = 1/g_x(x)$ at place x, then doubling the stake at each consecutive bet; the martingale stops betting after the first win occurred (or if there are no more places in D_x left or the current capital is too small). For each place $y \in D_x$, the strategy d_x bets in favor of the assumption that the parity of the bits in

$$[y]_{y+1} = [x] \cap \{x, \ldots, y\}$$

is 0, which corresponds to the assumption that y is indeed maximum in $[x]$. The martingale d_x doubles its initial stake q_x at most $|D_x| - 1$ times, hence by Claim 3 at most $|\log(q_x)| - 1$ times, and consequently d_x does not run out of capital in case its initial capital is at least 1. When d_x bets against the sequence A, at the latest the bet on $y = \max[x]$ will be a win, hence by betting on the places in $|D_x|$ the martingale d_x eventually wins, thereby realizing an overall win of q_x according to the doubling strategy.

Now let d be the martingale that bets successively according to martingales $d_{x_1}, d_{x_2}, \ldots$ where x_1 is 0 and in case the first win of d_{x_i} is on z, the starting location x_{i+1} of the subsequent submartingale is set to $z + 1$. By the discussion in the preceding paragraph, an easy inductive argument shows that when betting on the sequence A, all submartingales d_{x_i} of d start with capital of at least 1 and increase this capital by their initial stake q_x.

It remains to show that the capital of d is unbounded when betting on A. Fix any place z where $z = \min[z]$. Then $[y]_z = [y]$ for all $y < z$, and accordingly we have

$$g_z(z) = g(z) , \qquad \text{hence} \qquad q_z = \frac{1}{g(z)} .$$

In particular, in case the minimum z of the kth equivalence class appears among the x_i, the corresponding martingale d_z will add an amount of $q_z = 1/k$ to the gain of d. As a consequence, the gain of d on A will be at least $1 + 1/2 + 1/3 + \ldots$, i.e., d will succeed on A, in case for all equivalence classes the minimum of the class is equal to some x_i. The latter holds because a straightforward inductive argument shows that for every equivalence class $[x]$ some d_{x_i} wins when betting on the last bit of $[x]$ and that accordingly x_{i+1} is set equal to the minimum of the subsequent equivalence class. □

This ends the proof of Theorem 28. □

6 Kolmogorov-Loveland Stochasticity

There are two standard techniques for constructing KL-random sequences. The first one is a probabilistic construction due to van Lambalgen [13, 18, 28, 32] and is based on the observation that if one chooses a sequence at random according to a quasi-uniform Bernoulli-distribution, i.e., determines the bits of the sequence by independent flips of biased coins where the probabilities for 0 and 1 converge to 1/2, then the sequence will be KL-random with probability 1. The second one is to construct directly a Martin-Löf random sequence, e.g., by diagonalizing against a universal left-computable martingale. In the proof of Theorem 30, we present another technique that allows us to construct KL-stochastic sequences with certain additional properties that could not be achieved by the standard methods mentioned above. For example, we obtain a KL-stochastic sequence that is not weakly 1-random and left-r.e. whereas the probabilistic construction of van Lambalgen cannot be used to obtain a KL-stochastic sequence in the countable class of left-r.e. sequences, and by constructing a Martin-Löf random sequence the sequence obtained will in particular be weakly 1-random.

The *Bernoulli measure* specified by a sequence (β_i) of rational numbers with $0 \le \beta_i \le 1$ is the distribution on Cantor space that is obtained by determining the bits of a sequence according to independent tosses of biased coins where β_i is the probability that the outcome of toss i is 1. Such a Bernoulli measure is called computable if the function $i \mapsto \beta_i$ is computable, and is called *quasi-uniform* if $\lim_i \beta_i = 1/2$.

29 Remark. If a sequence R is Martin-Löf random with respect to a quasi-uniform Bernoulli measure, then R is KL-stochastic.

The assertion is immediate from results of Muchnik, Semenov, and Uspensky [20], who introduce a notion of KL-stochasticity with respect to a given Bernoulli measure and then show that, first, with respect to any Bernoulli measure Martin-Löf randomness implies KL-stochasticity and, second, KL-stochasticity with respect to any quasi-uniform measure is the same as KL-stochasticity.

A sequence X is *weakly 1-random* (also called *Kurtz-random*) if X is contained in every r.e. open class of uniform measure 1. Note that Schnorr randomness implies weak 1-randomness, but not conversely, as one can construct a weakly 1-random sequence that does not satisfy the law of large numbers by a non-effective finite extension construction where one alternates between appending long runs of 0's and hitting the next r.e. open class of uniform measure 1.

30 Theorem. *There is a non-empty Π^0_1 class $\mathcal{P}$ of KL-stochastic sequences such that no $X \in \mathcal{P}$ is weakly 1-random.*

Proof. We use some standard techniques for the construction of stochastic sequences [18, 20]. We will define an appropriate quasi-uniform computable Bernoulli measure β. Then, for a universal Martin-Löf test $(R_n : n \in \omega)$ with respect to β, we let $\mathcal{P}$ be the complement of R_1. Since β is computable, the set R_1 is recursively enumerable, hence $\mathcal{P}$ is a Π^0_1 class. Furthermore, every sequence in $\mathcal{P}$ is Martin-Löf random with respect to β and hence is KL-stochastic according to Remark 29.

It remains to choose a quasi-uniform computable Bernoulli measure β such that no $X \in P$ is weakly 1-random. By elementary probability theory, given a rational $\varepsilon > 0$ and $k \in \mathbb{N}$, one can compute $m = m(k, \varepsilon)$ such that in m independent tosses of a 0/1 "coin" with bias toward 1 of at least ε, with probability at least $1 - 2^{-k}$ the majority of the outcomes reflect the bias—in other words, there are more 1's than 0's [18, Remark 8]. (It turns out that $m(k, \varepsilon) = \lceil 6k\varepsilon^{-2} \rceil$ is sufficient.) Now let $\varepsilon_r = 1/r$ and partition the natural numbers into consecutive intervals $I_0, I_1, \ldots$ where I_r has length $m(r, \varepsilon_r)$. For all $i \in I_r$, let $\beta_i = 1/2 + \varepsilon_r$ and let β be the Bernoulli measure determined by the β_i. By construction, for the closed-open class

$$\mathcal{D}_r = \{Z : Z \text{ has at least as many 0's as 1's on positions in } I_r\} ,$$

we have $\beta[\mathcal{D}_r] \leq 2^{-r}$ and thus the classes $\widehat{\mathcal{D}}_n = \bigcup_{r \geq n} \mathcal{D}_r$ form a Martin-Löf test with respect to β. The sequences in $\mathcal{P}$ are Martin-Löf random with respect to β, hence any sequence in $\mathcal{P}$ is not contained in some class $\widehat{D}_n$. But each class $\widehat{\mathcal{D}}_n$ is an r.e. open set of uniform measure 1 and consequently no sequence in $\mathcal{P}$ is weakly 1-random. Observe that the Lebesgue measure of $\widehat{\mathcal{D}}_n$ is 1 because the complement of each class $\mathcal{D}_r$ has uniform measure of at most 1/2 and since the $\mathcal{D}_r$ are stochastically independent. □

In Theorem 30, it can be arranged for any given $\varepsilon > 0$ that the class $\mathcal{P}$ satisfies in addition $\beta[\mathcal{P}] \geq 1 - \varepsilon$; here it suffices to replace in the proof the first component of a universal Martin-Löf test with respect to β by a component that has measure less than ε.

By the usual basis theorems [21], the following corollary is immediate from Theorem 30.

31 Corollary. *There is a left-c.e., not weakly* 1*-random, KL-stochastic sequence. There is a low, not weakly* 1*-random, KL-stochastic sequence. There is a not weakly* 1*-random, KL-stochastic sequence that is of hyperimmune-free degree.*

7 The dimension of KL-stochastic sequences

There exists an interesting connection between the asymptotic complexity of sequences and Hausdorff dimension. Hausdorff dimension can be seen as a generalization of Lebesgue measure, in the sense that it allows us to distinguish the size of Lebesgue nullsets. It is defined via Hausdorff measures, and similar to Lebesgue measure, one can define effective versions of them. This leads to the concept of *effective dimension*, first introduced by Lutz [12].

32 Definition. *Let* $0 \leq s \leq 1$ *be a rational number. A class* $\mathcal{C} \subseteq 2^\omega$ *has* effective s-dimensional Hausdorff measure 0 *if there is a sequence* $\{C_n\}_{n\in\omega}$ *of uniformly r.e. sets of words such that for every* $n \in \omega$,

$$\mathcal{C} \subseteq [C_n] \quad \text{and} \quad \sum_{w\in C_n} 2^{-s|w|} \leq 2^{-n}. \tag{11}$$

Note that for a sequence A to have effective 1-dimensional Hausdorff measure zero (as a singleton class) is equivalent to it not being Martin-Löf random. So being a effective s-dimensional Hausdorff null sequence for smaller s means being "less" random. The effective Hausdorff dimension captures this "degree" of randomness.

33 Definition. *The* effective (Hausdorff) dimension *of a class* $\mathcal{C} \subseteq 2^\omega$ *is defined as*

$$\dim_1(\mathcal{C}) = \inf\{s \geq 0 : \mathcal{C} \text{ has effective } s\text{-dimensional Hausdorff measure } 0.\}.$$

The effective dimension of a sequence is the effective dimension of the corresponding singleton class.

It turns out that, just as Martin-Löf randomness corresponds to incompressibility with respect to prefix-free Kolmogorov complexity, effective dimension corresponds to linear lower bounds on compressibility.

34 Theorem. *For any sequence A it holds that*

$$\dim_1 A = \liminf_{n\to\infty} \frac{\mathrm{K}(A\restriction_n)}{n}. \tag{12}$$

Theorem 34 was proven in the presented form by Mayordomo [16], but much of it was already implicit in earlier works by Ryabko [24, 25], Staiger [29, 30], and Cai and Hartmanis [5]. Note that, since plain Kolmogorov complexity C and prefix-free complexity K behave asymptotically equal, or, more precisely, since for every string w it holds that

$$\mathrm{C}(w) \leq \mathrm{K}(w) \leq \mathrm{C}(w) + 2\log|w|,$$

one could replace K by C in Theorem 34. For more on effective dimension see also Reimann [22].

Muchnik [20] refuted a conjecture by Kolmogorov — who asserted that there exists a KL-stochastic sequence A such that $\mathrm{K}(A\restriction_n) = \mathrm{O}(\log n)$ — by showing that, if A is KL-stochastic, then $\limsup_{n\to\infty} \mathrm{K}(A\restriction_n)/n = 1$. In the following, we are going to strengthen this result by showing that $\dim_1 A = 1$ for any KL-stochastic sequence A.

Ryabko [23] observed that van Lambalgen's probabilistic argument for the construction of KL-stochastic sequences yields with probability 1 a sequence that has effective dimension 1. It is also not hard to see that also the construction in Theorem 30 creates sequences of dimension 1. Theorem 41, which is the main result of this section, states that in fact any KL-stochastic sequence has effective dimension 1.

Reimann [22] has posed the question whether each sequence of effective dimension 1 computes a Martin-Löf random. By our result, one could try to find a counterexample using KL-stochasticity. However, the construction in Theorem 30 will not do it: by a result of Levin [33], the Turing degree of any sequence which is Martin-Löf random relative to some computable measure contains in fact a Martin-Löf random sequence relative to the uniform measure.

The proof of Theorem 41 bears some similarities to the proof of Theorem 17, where it has been shown that any KL-random sequence has arbitrarily dense subsequences that are Martin-Löf random. The proof of the latter theorem worked by successively splitting the given sequence into subsequences, where then the join of all the Martin-Löf random subsequences obtained this way was again Martin-Löf random. The proof used Theorem 12 and a result of van Lambalgen stated in Remark 15; in the proof of Theorem 41, the latter two results are replaced by Lemmas 38 and 40, respectively. The proof of Lemma 38, in turn uses Proposition 35 and Remark 37. Proposition 35 is a slightly generalized version of a corresponding result by Muchnik et al. [20].

35 Proposition. *For any rational $\alpha < 1$ there is a natural number k_α and a rational $\varepsilon_\alpha > 0$ such that the following holds. Given an index for a computable monotonic martingale d with initial capital 1, we can effectively find indices for computable monotonic selection rules $s_1, \dots, s_{2k_\alpha}$ such that for all words w which satisfy*

$$d(w) \geq 2^{(1-\alpha)|w|} \tag{13}$$

there is an index i such that the selection rule s_i selects from w a finite sequence of length at least $\varepsilon_\alpha |w|$ such that the ratio of 0's and the ratio of 1's in this finite sequence differ by at least ε_α.

Proof. In a nutshell, the proof works as follows. First we use an observation due to Schnorr [3, 27] in order to transform the monotonic martingale d into a monotonic martingale d' where the fraction of the current capital that is bet is always in a fixed finite set of weights, while we have $d'(w) \geq 2^{\frac{1}{2}(1-\alpha)|w|}$ for all w that satisfy (13); furthermore, an averaging argument shows that for some rational $\beta > 0$ and and for all such w, for one of these finitely many weights the bets with this weight alone yield a gain of size $2^{\beta|w|}$. Finally, we argue along the lines of Ambos-Spies et al. [2] that if a monotonic martingale wins that much on a word w while being restricted to a single weight, then the martingale must bet on a nonzero constant fraction of all places of w, the correct predictions must outnumber the incorrect ones by a constant nonzero fraction of all predictions, and consequently these martingales can be converted into selection rules s_i as required.

Let d' be the variant of d where for some appropriate rational $\delta > 0$, the fraction γ of the capital d would have bet is rounded down to the next multiple of δ less than or equal to γ, i.e., for any bet of d where d bets a fraction γ of its current capital in the half-open interval $[i\delta, (i+1)\delta)$, the new martingale bets a fraction of $i\delta$ of its current capital.

In case a bet of d is lost, the new martingale d' loses at most as much as d, while in case the bet is won, the new martingale increases its capital by a factor of $1 + i\delta$ compared to an increase by a factor of at most $1 + (i+1)\delta$ for d'. We have that

$$\frac{1+i\delta}{1+(i+1)\delta} = 1 - \frac{\delta}{1+(i+1)\delta} \geq 1 - \frac{\delta}{1+\delta} = \frac{1}{1+\delta} \, . \tag{14}$$

Let $\delta = 1/k_\alpha$ where k_α is chosen so large that

$$\delta \leq 2^{(1-\alpha)/2} - 1\,, \quad \text{hence} \quad \frac{1}{1+\delta} \geq \frac{1}{2^{(1-\alpha)/2}}\,.$$

With this choice of δ we have for all words w that satisfy (13) that

$$d'(w) \geq \left(\frac{1}{1+\delta}\right)^{|w|} d(w) \geq \left(\frac{1}{2^{(1-\alpha)/2}}\right)^{|w|} 2^{(1-\alpha)|w|} = 2^{\frac{(1-\alpha)}{2}|w|}\,, \tag{15}$$

where the two inequalities hold by the discussion preceding (14) and by choice of δ and by (13), respectively.

Now fix any word w that satisfies (13) and hence satisfies (15). Consider the bets of d' on w, i.e., consider the first $|w|$ bets of d' while betting against any sequence that has w as a prefix. For $r \in \{0,1\}$, let $n_{i,r}$ be the number of such bets where a fraction of $i\delta$ of the current capital is bet on the next bit being equal to r; similarly, let $n^+_{i,r}$ and $n^-_{i,r}$ be the number of bets of the latter type that are won and lost, respectively, i.e., $n_{i,r}$ is the sum of $n^+_{i,r}$ and $n^-_{i,r}$. Then we have

$$d'(w) = d'(\varepsilon) \prod_{r\in\{0,1\}} \prod_{i=0,\ldots,k_\alpha-1} [(1+i\delta)^{n^+_{i,r}}(1-i\delta)^{n^-_{i,r}}]\,, \tag{16}$$

where $d'(\varepsilon)$ is 1. Then for some $r \in \{0,1\}$ and some $i \leq k_\alpha - 1$ we have

$$d'(w)^{-2k_\alpha} \leq [(1+i\delta)^{n^+_{i,r}}(1-i\delta)^{n^-_{i,r}}] \leq (1+i\delta)^{n^+_{i,r}-n^-_{i,r}} \leq 2^{n^+_{i,r}-n^-_{i,r}}\,. \tag{17}$$

The first inequality in (17) follows because one of the factors of the product on the right-hand side of (16) must be at least as large as the geometric mean $d'(w)^{-2k_\alpha}$ of this product. The second inequality holds because of $(1+i\delta)(1-i\delta) < 1$, while the latter, together with (17) and the assumption $d'(w) > 1$, implies $n^+_{i,r} > n^-_{i,r}$, from which the third inequality in (17) is immediate. Putting together (15) and (17) and taking logarithms yields

$$\varepsilon_\alpha |w| \leq n^+_{i,r} - n^-_{i,r} \qquad \text{where } \varepsilon_\alpha = \frac{(1-\alpha)}{2}\frac{1}{2k_\alpha} = \frac{(1-\alpha)}{4k_\alpha} > 0\,. \tag{18}$$

For $j = 0, \ldots, k_\alpha - 1$, let s_{2j+1} and s_{2j+2} be the computable monotonic selection rules that on input w selects the next bit if and only if d' on input w bets a fraction of $j\delta$ of its current capital on the next bit being equal to 0 and 1, respectively. Observe that an index for s_j can be computed from α and an index for d. By construction, selection rule s_{2i+r+1} selects from w a sequence of length $n_{i,r}$ where $n^+_{i,r}$ of these bits are equal to r and the remaining $n_{i,r}^{-}$ bits are equal to $1-r$. So by (18), the selected sequence has length at least $\varepsilon_\alpha|w|$ and the ratios of 0's and of 1's differ by at least ε_α. Since w was chosen as an arbitrary word that satisfies (13), this finishes the proof of Proposition 35. □

Remark 37 shows our intended application of Proposition 35

36 Definition. *Let α be a rational. A word w is called* α-compressible *if* $\mathrm{K}(w) \le \alpha|w|$.

37 Remark. Given a rational $\alpha < 1$ and a finite set D of α-compressible words, we can effectively find an index for a computable monotonic martingale d with initial capital 1 such that for all $w \in D$ we have $d(w) \ge 2^{(1-\alpha)|w|}$.

For a proof, let d_w be the monotonic martingale that starts with initial capital $2^{-\alpha|w|}$ and plays a doubling strategy along w, i.e., always bets all its capital on the next bit being the same as the corresponding bit of w; then we have in particular $d_w(w) = 2^{(1-\alpha)|w|}$.

Let d be the sum of the martingales d_w over all words $w \in D$, i.e., betting according to d amounts to playing in parallel all martingales d_w where $w \in D$. Obviously $d(v) \ge d_w(v)$ for all words v and all $w \in D$, so it remains to show that the initial capital of d does not exceed 1. The latter follows because every $w \in D$ is α-compressible, i.e., can be coded by a prefix-free code of length at most $\alpha|w|$, hence the sum of $2^{-\alpha|w|}$ over all $w \in D$ is at most 1.

38 Lemma. *Let Z be a computable, infinite and co-infinite set of natural numbers and let $A = A_1 \oplus_Z A_2$ be KL-stochastic. Then one of the sequences A_1 and A_2 has effective dimension* 1.

Proof. For a proof by contradiction, assume that the consequence of the lemma is false, i.e., that there is some rational number $\alpha_0 < 1$ such that A_1 and A_2 both have effective dimension of at most α_0. Pick rational numbers α_1 and α such that $\alpha_0 < \alpha_1 < \alpha < 1$. By the characterization of effective dimension in terms of the prefix-free variant K of Kolmogorov complexity according to Theorem 34, for $r = 1, 2$ there are arbitrarily large prefixes w of A_r that are α_1-compressible, i.e., $\mathrm{K}(w) \le \alpha_1|w|$. We argue next that for any m there are arbitrarily large intervals I with $\min I = m$ such that the restriction of A_r to I is α-compressible.

Let $w_0, w_1, \ldots$ be an effective enumeration of all α-compressible words w. For the scope of this proof, say a word w is a *subword of X at m* if

$$w = X(m)X(m+1)\ldots X(m+|w|-1)\ .$$

Let ε_α be the constant from Proposition 35.

Claim 1 *For $r = 1, 2$, the function g_r defined by*

$$g_r(m) = \min\{i \colon w_i \textit{ is a subword of } A_r \textit{ at } m \textit{ and } |w_i| > \frac{2}{\varepsilon_\alpha^2} m\}$$

is total.

Proof. There are infinitely many α_1-compressible prefixes of A_r. Given any such prefix v of length at least m, let u and w be the words such that $v = uw$ and $|u| = m$. Then we have

$$\mathrm{K}(w) \leq^+ \mathrm{K}(v) + 2\log m \leq \alpha_1|v| + 2\log m = \alpha|w| \left(\frac{\alpha_1}{\alpha} \frac{|v|}{|w|} + \frac{2\log m}{\alpha|w|} \right) ,$$

where the expression in brackets goes to $\alpha_1/\alpha < 1$ when the length of w goes to infinity. As a consequence, we have $\mathrm{K}(w) \leq \alpha|w|$ for all such words w that are long enough, hence by assumption on A for any m and t there is a word w_i and an index i as required in the definition of $g_r(m)$. □

Let $m_0 = 0$ and for all $t > 0$, let

$$m_{t+1} = m_t + \max\{|w_i| \colon i \leq \max\{g_1(m_t), g_2(m_t)\}\} .$$

In the following, we assume that there are infinitely many t where

$$g_1(m_t) \leq g_2(m_t) \; ; \tag{19}$$

we omit the essentially identical considerations for the symmetric case where there are infinitely many t such that $g_1(m_t) \geq g_2(m_t)$. Let

$$D_t = \{w_0, w_1, \ldots, w_{g_2(m_t)}\}$$

Claim 2 *There are infinitely many t such that some word in D_t is a subword of A_1 at m_t.*

Proof. By definition of $g_1(m_t)$, the word $w_{g_1(m_t)}$ is a subword of A_1 at m_t, where this word is in D_t for each of the infinitely many t such that $g_1(m_t)$ is less than or equal to $g_2(m_t)$. □

Claim 3 *Given D_t and m_t, we can compute an index for a monotonic computable selection rules $s(t)$ that scans only bits of the form*

$$A_1(m_t), A_1(m_t + 1), \ldots, A_1(m_{t+1} - 1)$$

of A such that for infinitely many t the selection rule $s(t)$ selects from these bits a finite sequence of length at least $2m_t/\varepsilon_\alpha$ where the ratios of 0's and of 1's in this finite sequence differ by at least ε_α.

Proof. By Proposition 35 and Remark 37, from the set D_t we can compute indices for monotonic computable selection rules $s_1, \dots, s_{2k_\alpha}$ such that for each $w \in D_t$ there is an index i such that the selection rule s_i selects from w a finite sequence of length at least $\varepsilon_\alpha |w|$ such that the ratio of 0's and 1's in this finite sequence differ by at least ε_α. Any word $w \in D_t$ has length of at least $2m_t/\varepsilon_\alpha^2$, hence the selected finite sequence has length of at least $2m_t/\varepsilon_\alpha$. Furthermore, by Claim 2, there are infinitely many t such that some $w \in D_t$ is a subword of A_1 at m_t, and among the corresponding indices i some index i_0 between 1 and $2k_\alpha$ must appear infinitely often. So it suffices to let for any t the selection rule $s(t)$ be equal to the i_0th selection rule from the list of selection rules computed from D_t. □

Now we construct an non-monotonic computable selection rule s that witnesses that A is not KL-stochastic. The selection rule s works in stages $t = 0, 1, \dots$ and scans during stage t the bits of A that correspond to bits of the form

$$A_1(y) \quad \text{and} \quad A_2(y) \text{ where } m_t \leq y < m_{t+1} \; .$$

At the beginning of stage t, the value of $g_2(m_t)$ and the set D_t is computed as follows. Successively for $i = 0, 1, \dots,$ check whether w_i is a subword of A_2 at m_t by scanning all the bits

$$A_2(m_t), \dots, A_2(m_t + |w_i| - 1)$$

of A that have not been scanned so far, until eventually the index i equal to $g_2(m_t)$ is found, i.e., until we find some minimum i such that w_i is a subword of A_2 at m_t. Observe that by definition of m_{t+1}, the index i is found while scanning only bits of the form $A_2(y)$ where $y < m_{t+1}$. Next the selection rule s scans and selects the bits $A_1(m_t), A_1(m_t + 1), \dots$ according to the selection rule s_{i_0} as in Claim 3; recall that this selection rule can be computed from D_t. Finally, stage t is concluded by computing m_{t+1} from $g_1(t)$ and $g_2(t)$, where $g_1(t)$ is obtained like $g_2(t)$, i.e., in particular, the computation of m_{t+1} only requires to scan bits of the form $A_r(y)$ where $y < m_{t+1}$.

By Claim 2 there are infinitely many t such that some $w \in D_t$ is a subword of A_1 at m_t. By choice of $s(t)$ and definition of s, for each such t the selection rule s selects during stage t a finite sequence of length at least $2m_t/\varepsilon_\alpha$ where the ratios of 0's and 1's in this finite sequence differ by at least ε_α. Consequently, the at most m_t bits of A that might have been selected by s before stage t are at most a fraction of $\varepsilon_\alpha/2$ of the bits selected during stage t, hence with respect to all the bits selected up to stage t the ratios of 0's and 1's differ by at least $\varepsilon_\alpha/2$. This contradicts the fact that A is KL-stochastic, hence our assumption that A_1 and A_2 both have effective dimension strictly less than 1 is wrong. □

Lemma 38 can be relativized to any oracle Z by essentially the same proof, we leave the necessary minor adjustments to the reader.

39 Lemma. *Let Z be a computable, infinite and co-infinite set of natural numbers, let X be any sequence, and let $A = A_1 \oplus_Z A_2$ be KL-stochastic relative to the oracle X. Then one of the sequences A_1 and A_2 has effective dimension 1 relative to the oracle X.*

40 Lemma. *Let Z be a computable, infinite and co-infinite subset of the natural numbers with density δ. Then it holds for any sequences U and V that*

$$\dim_1 U \oplus_Z V \geq (1-\delta)\dim_1 U + \delta \dim_1^U V. \tag{20}$$

Proof. For any n, let w_n be the prefix of $U \oplus_Z V$ of length n and let u_n and v_n be the prefixes of U and V of length $|\overline{Z} \cap \{0,\dots,n-1\}|$ and $|Z \cap \{0,\dots,n-1\}|$, respectively; i.e., intuitively speaking, the word w_n is the Z-join of u_n and v_n. Fix any $\varepsilon > 0$. Then for almost all n, by definition of density we have that

$$|u_n| \geq (1-\delta-\varepsilon)n \quad \text{and} \quad |v_n| \geq (\delta-\varepsilon)n \,, \tag{21}$$

and by the characterization of effective dimension of a sequence X as the limit inferior of $\mathrm{K}(X\restriction_n)/n$, we have that

$$\mathrm{K}(u_n) \geq |u_n|(\dim_1 U - \varepsilon) \quad \text{and} \quad \mathrm{K}^U(v_n) \geq |v_n|(\dim_1^U(V) - \varepsilon)\,. \tag{22}$$

Furthermore, we have for all n

$$\mathrm{K}(w_n) \geq \mathrm{K}(u_n) + \mathrm{K}(v_n|u_n, \mathrm{K}(u_n)) \geq \mathrm{K}(u_n) + \mathrm{K}^U(v_n) - 4\log n\,, \tag{23}$$

where the first inequality holds by a property of K-complexity related to symmetry of information [11] and the second inequality holds because with oracle U the prefix u_n of U can be recovered from its length, hence v_n has a prefix-free code that consists of a code witnessing the size of $\mathrm{K}(v_n|u_n, \mathrm{K}(u_n))$ plus codes for n and $\mathrm{K}(u_n)$. Using (21) and (22) for substituting in (23) and dividing by n yields

$$\frac{\mathrm{K}(w_n)}{n} \geq (1-\delta)\dim_1 U + \delta \dim_1^U V + \frac{g(\varepsilon) - 4\log n}{n}\,, \tag{24}$$

for some function g such that for all $\varepsilon \leq 1$ the value of $g(\varepsilon)$ is bounded by a constant that does not depend on ε. The lemma follows since $\varepsilon > 0$ has been chosen arbitrarily and because the effective dimension of $U \oplus_Z V$ is equal to the limit inferior of the left-hand side of (24). □

41 Theorem. *If R is KL-stochastic, then* $\dim_1 R = 1$.

Proof. The proof is rather similar to the proof of Theorem 17, in particular, we use the notation N_w from there. It suffices to show that there are words $w_1 \sqsubseteq w_2 \sqsubseteq \ldots$ such that for all i, we have $|w_i| = i$ and

$$\dim_1 R_i = 1 , \qquad \text{where } R_i = R\restriction_{\overline{\mathrm{N}_{w_i}}} ;$$

the theorem then follows by Lemma 40 and because for any word w, the set $\overline{\mathrm{N}_w}$ has density $1 - 1/2^{|w|}$.

The w_i are defined inductively. For a start, observe that by Lemma 38 for $r_1 = 0$ or for $r_1 = 1$ the sequence $R\restriction_{\mathrm{N}_{r_1}}$ has effective dimension 1; pick r_1 such that the latter is true and let $w_1 = 1 - r_1$. For $i > 1$, let w_i be defined as follows. By an argument similar to the proof of Proposition 11, the sequence $R\restriction_{\mathrm{N}_w}$ is KL-stochastic relative to R_{i-1}, hence by the relativized version of Lemma 38, for $r_i = 0$ or for $r_i = 1$ the sequence $R\restriction_{\mathrm{N}_{wr_i}}$ has effective dimension 1 relative to R_w; pick r_i such the latter is true and let $w_i = w(1 - r_i)$.

It remains to show by induction on i that all the sequences R_i have effective dimension 1. For $i = 1$, this is true by construction, while the induction step follows according to the choice of the w_i and due to Lemma 40 by an argument similar to the corresponding part of the proof of Theorem 17; details are left to the reader. □

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