January 15, 2009

DIAGONALLY NON-RECURSIVE FUNCTIONS AND EFFECTIVE HAUSDORFF DIMENSION

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Abstract. We prove that every sufficiently slow growing DNR function computes a real with effective Hausdorff dimension one. Using a proof recently published by Kumabe and Lewis, it follows that there is a real of dimension one and minimal degree. Note that such a real cannot compute a Martin-Löf random real.

1. Introducion

Reimann and Terwijn asked the *dimension extraction problem*: can one effectively increase the information density of a sequence with positive information density? For a formal definition of information density, they used the notion of effective Hausdorff dimension. This effective version of the classical Hausdorff dimension of geometric measure theory was first defined by Lutz [9], using a martingale definition of Hausdorff dimension. Unlike classical dimension, it is possible for singletons to have positive dimension, and so Lutz defined the dimension $\dim(A)$ of a binary sequence $A \in 2^\omega$ to be the effective dimension of the singleton $\{A\}$. Later, Mayordomo [11] (but implicit in Ryabko [14]), gave a characterisation using Kolmogorov complexity: for all $A \in 2^\omega$,

$$\dim(A) = \liminf_{n\to\infty} \frac{K(A \upharpoonright n)}{n} = \liminf_{n\to\infty} \frac{C(A \upharpoonright n)}{n},$$

where C is plain Kolmogorov complexity and K is the prefix-free version.

Given this formal notion, the dimension extraction problem is the following: if $\dim(A) > 0$, is there necessarily a $B \leqslant_T A$ such that $\dim(B) > \dim(A)$? The problem was recently solved by the second author [12], who showed that there is an $A \in 2^\omega$ such that $\dim(A) = 1/2$ and if $B \leqslant_T A$, then $\dim(B) \leqslant 1/2$.

Even while it was still open, the dimension extraction problem spawned variations. The one we consider in the present paper is:

Question 1.1. *Is there an* $A \in 2^\omega$ *such that* $\dim(A) = 1$ *and* A *computes no Martin-Löf random set?*

One motivation for this, and for the dimension extraction problem, is that the obvious ways to obtain nonrandom sets of positive dimension allow us to extract random sets. For example, we could take a random set X (whose dimension is always 1) and water it down by inserting zeros between the bits of X. This would give us a sequence of dimension $1/2$; inserting zeros sparsely would result in a sequence of dimension 1. As long as the insertion positions are computable, we

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2000 *Mathematics Subject Classification.* Primary 68Q30; Secondary 03D80, 28A78.

This work arose from a visit by the first author that was supported by the National Science Foundation Focused Research Group grant DMS-065267. The first author was supported in part by the Marsden Fund of New Zealand. The second author was supported by the National Science Foundation grant DMS-0601021.

can recover the original random set. As another example, a sequence of positive dimension created by flipping a biased coin also computes a random sequence. The dimension extraction question, and Question 1.1, can be understood as asking if there are sequences of positive dimension, and even dimension 1, that behave differently than these examples.

In this paper, we answer Question 1.1 in the negative by showing:

Theorem 1.2. *There is a set of dimension* 1 *that has minimal Turing degree.*

Theorem 1.2 solves Question 1.1 because no Martin-Löf random set can be recursive of have minimal Turing degree.

We remark that a weaker version of Theorem 1.2 was proved by Downey and Greenberg [4], where they show that there is a set of minimal Turing degree whose effective packing dimension is 1. Packing dimension is a variant of Hausdorff dimension that uses inner measure; the complexity characterisation uses $\limsup$ instead of $\liminf$.

To prove Theorem 1.2, we study the computational power of sufficiently slow growing diagonally non-recursive functions. Recall that the *jump function* $J\colon \omega \to \omega$ takes e to the result of the eth program on input e, if it halts. Formally, $J(e) = \varphi_e(e)$, where $\{\varphi_e\}_{e\in\omega}$ is a standard enumeration of the partial recursive functions. A function $f \in \omega^\omega$ is *diagonally non-recursive* (or DNR) if $J(e) \neq f(e)$ for all $e \in \operatorname{dom} J$. Diagonally non-recursive functions were introduced by Jockusch in [6], where he shows that the Turing degrees of DNR functions are exactly those degrees that contain fixed-point free functions (that is, functions f such that for all e, $\varphi_e \neq \varphi_{f(e)}$).

From a computational standpoint, DNR functions, in general, can be quite feeble. For example, it follows from [1] that there is a DNR function that does not compute any set of positive effective Hausdorff dimension, while Kumabe and Lewis [8] showed that there is a DNR function that has minimal Turing degree. On the other hand, DNR *sets*, that is, DNR functions whose range is $\{0, 1\}$, are quite powerful: their degrees are characterized as those that can compute completions of Peano arithmetic [15, 7] and paths through nonempty recursive binary trees. This difference suggests that, possibly, it is the *rate of growth* of a DNR function that may influence its strength as an oracle.

A finer analysis of the computational power of a class of functions is given by considering the issue of uniformity. For any $a \geqslant 2$, let DNR_a be the class of DNR functions whose range is contained in the set $\{0, 1, \dots, a-1\}$ (so DNR_2 is the class of DNR sets). Jockusch [6] has shown that the Turing degrees of the functions in DNR_a coincide, that is, if $a < b$ then every function in DNR_b computes a function in DNR_a; but that this cannot be done uniformly, that is, there is no single Turing functional that given a function in DNR_b produces one in DNR_a. In modern parlance, the Mučnik (weak) degrees of the classes DNR_a coincide for all $a \geqslant 2$, but the Medvedev (strong) degrees of these classes form a strictly decreasing chain (see [16]).

In this paper, we examine what can be computed from an unbounded DNR function that nevertheless does not grow too quickly. For any non-decreasing function $q\colon \omega \to \omega \setminus \{0, 1\}$, let

$$\mathrm{DNR}_q = \{f \in \mathrm{DNR} \,:\, \text{for all } e,\ f(e) < q(e)\}$$

be the class of DNR functions that are bounded by q. A DNR function is called *recursively bounded* if it is bounded by some recursive function q. We shall see (Theorem 4.9) that if q grows slowly, then every DNR_q function computes a set whose effective Hausdorff dimension is 1. This will be a consequence of results on the uniform power of bounded DNR functions, culminating in Proposition 4.8, which says that uniformly in a, functions in DNR_a uniformly compute sets of Hausdorff dimension 1.

Theorem 1.2 follows from Theorem 4.9 by showing that the Kumabe–Lewis construction can be modified to show that given an unbounded, non-decreasing recursive function j, there is a DNR function bounded by j and having minimal Turing degree (Theorem 5.1). It follows that in contrast with positive dimension, no matter how slow the rate of growth, unbounded DNR functions cannot always compute random sets. In fact, since the Kumabe–Lewis construction can be altered to produce a hyperimmune-free degree, even Kurtz random sets cannot always be computed from slow-growing DNR functions[1], whereas by Jockusch's result, every bounded DNR function can compute a Martin-Löf random set. This indeed relates again to the uniform computation power of bounded DNR functions: recently, Downey et. al. [5] have shown that DNR_3 functions cannot uniformly compute even Kurtz random sets.

To prove Theorem 4.9, we use variants of Cantor space that we call *h-spaces*. For a recursive, non-decreasing function h, we work in the space

$$\prod_{m<\omega}\{0,1,\dots,h(m)-1\}$$

which for shorthand, in analogy with 2^ω, we name h^ω. In Section 2 we develop the theory of effective dimension for sequences in these spaces. Then, in Section 3, we show that for slow-growing functions h, the notion of dimension in h^ω coincides with dimension in 2^ω or in the Euclidean interval $[0,1]$. This allows us to carry out the plan of stringing together the uniform power of bounded DNR functions to prove Theorem 4.9.

2. Dimension in h-spaces

We generalise effective Hausdorff dimension to spaces that are very similar to Cantor space. Let $h\colon \omega \to \omega \setminus \{0,1\}$ be a recursive function. We let

$$h^\omega = \prod_{m<\omega}\{0,1,\dots,h(m)-1\}.$$

For finite $n<\omega$, we let

$$h^n = \prod_{m<n}\{0,1,\dots,h(m)-1\}$$

and we let

$$h^{<\omega} = \bigcup_{n<\omega} h^n.$$

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[1] This is because every Kurtz random set of hyperimmune-free degree is Martin-Löf random, even weakly 2-random.

The space h^ω is a compact Polish space. An analogue of Lebesgue measure for h^ω is obtained by dividing mass equitably: for $\sigma \in h^{<\omega}$, let

$$\mu(\sigma) = \frac{1}{|h^{|\sigma|}|} = \frac{1}{h(0)\cdots h(|\sigma|-1)}.$$

Then $\mu = \mu^h$ induces a measure $\mu = \mu^h$ on h^ω such that for all $\sigma \in h^{<\omega}$, $\mu([\sigma]) = \mu(\sigma)$. Here as usual $[\sigma] = \{X \in h^\omega : \sigma \subset X\}$. We extend this notation: for sets of strings $A \subseteq h^{<\omega}$, we let $[A] = \bigcup_{\sigma \in A}[\sigma]$.

Definition 2.1. For a real $s \geqslant 0$ and a set $A \subseteq h^{<\omega}$, the *s-weight* of A is

$$\mathtt{wt}_s(A) = \sum_{\sigma \in A} \mu(\sigma)^s.$$

The *weak s-weight* of A is

$$\mathtt{wkwt}_s(A) = \sup \mathtt{wt}_s(C) [\![C \subseteq A \text{ is prefix-free}]\!].$$

Definition 2.2. Let $s \geqslant 0$. An *s-ML-test* is a uniformly r.e. sequence $\langle A_k \rangle$ of $h^{<\omega}$ such that for all $k < \omega$, $\mathtt{wt}_s(A_k) \leqslant 2^{-k}$. A *weak s-ML-test* is a uniformly r.e. sequence $\langle A_k \rangle$ of $h^{<\omega}$ such that for all $k < \omega$, $\mathtt{wkwt}_s(A_k) \leqslant 2^{-k}$.

Definition 2.3. A ML-test $\langle A_k \rangle$ *covers* $X \in h^\omega$ if $X \in \bigcap_k [A_k]$. A sequence $X \in h^\omega$ is *weakly s-null* if it is covered by a weak s-ML-test. It is *s-null* if it is covered by a s-ML-test.

Remark 2.4. In other common parlance, X is called *s-random* if it is not s-null and *strongly s-random* if it is not weakly s-null.

Proposition 2.5 (Reimann)**.** *If $t > s \geqslant 0$ and X is weakly s-null, then X is t-null.*

Proof. We show that there is a constant c such that for all $A \subseteq h^{<\omega}$, $\mathtt{wt}_t(A) \leqslant c \cdot \mathtt{wkwt}_s(A)$. We can then convert a weak s-ML-test to a t-ML-test simply by taking a tail. Let $A \subseteq h^{<\omega}$. For $n < \omega$, let $\gamma_n = 1/|h^n|$ be the measure of any string of length n.

$$\mathtt{wt}_t(A) = \sum_{\sigma \in A} \mu(\sigma)^t = \sum_{n<\omega} \sum_{\sigma \in A \cap h^n} \gamma_n^t = \sum_{n<\omega} \gamma_n^{t-s} \sum_{\sigma \in A \cap h^n} \gamma_n^s \leqslant \mathtt{wkwt}_s(A) \sum_{n<\omega} \gamma_n^{t-s},$$

the last equation following of course from the fact that $A \cap h^n$ is prefix-free. Thus we can let $c = \sum_{n<\omega} \gamma_n^{t-s}$, which is finite because for all m, $h(m) \geqslant 2$ and so $\gamma_n \leqslant 2^{-n}$. □

Thus for $X \in h^\omega$, the infimum of all s for which X is s-null equals the infimum of all s for which X is weakly s-null.

Definition 2.6. Let $s \geqslant 0$. A *Solovay s-test* is an r.e. set $A \subseteq h^{<\omega}$ such that $\mathtt{wt}_s(A)$ is finite. We say that $X \in h^\omega$ is *covered* by a Solovay s-test if there are infinitely many initial segments of X in A.

Proposition 2.7 (Reimann, [13])**.** *If X is s-null then X is covered by a Solovay s-test. If X is covered by a Solovay s-test then for all $t > s$, X is t-null.*

Proof. Suppose first that X is s-null, so it is covered by some s-ML-test $\langle A_k\rangle$. Let $A = \bigcup_k A_k$. Then $\mathtt{wt}_s(A) \leqslant 1$. Since every h^n is finite, for all n there is a k such that every string in A_k has length greater than n. Thus A coveres X as a Solovay test.

Suppose that A is a Solovay s-test that covers X. Let $t > s$. For all k we can find a length n_k such that for all $\sigma \in h^{<\omega}$ of length greater than n_k we have $\mu(\sigma)^{t-s} < 2^{-k}/\mathtt{wt}_s(A)$ (again recall that the series $\sum \gamma_n^{s-t}$ from the proof of Proposition 2.5 converges) and so if we let $A_k = \{\sigma \in A : |\sigma| > n_k\}$ then

$$\mathtt{wt}_t(A_k) = \sum_{\sigma\in A_k} \mu(\sigma)^t = \sum_{\sigma\in A_k} \mu(\sigma)^{t-s}\mu(\sigma)^s \leqslant 2^{-k}/\mathtt{wt}_s(A)\sum_{\sigma\in A}\mu(\sigma)^s = 2^{-k}. \qquad \square$$

We also need to characterize dimension by martingales.

Definition 2.8. A *supermartigale* (for h) is a function $d\colon h^{<\omega} \to \mathbb{R}^+$ such that for all $\sigma \in h^{<\omega}$,

$$\sum_{i<h(|\sigma|)} d(\sigma i) \leqslant h(|\sigma|)d(\sigma).^{2}$$

If $s \geqslant 0$ is a real number then we say that a supermartingale d is s-*successful* on $X \in h^\omega$ if the sequence

$$\left\langle d(X\restriction n)\mu(X\restriction n)^{1-s}\right\rangle_{n<\omega}$$

is unbounded.

Proposition 2.9 (Lutz [10], Calude-Staiger-Terwijn [2])**.** *If X is s-null then there is a left-r.e. martingale that s-succeeds on X.*

In fact, the proposition is true even if X is weakly s-null; in light of Proposition 2.5, this does not matter for dimension.

Proof. For any $\sigma \in h^{<\omega}$ there is a recursive matingale d_σ such that $d_\sigma(\sigma) = 1$ and $d_\sigma(\langle\rangle) = \mu(\sigma)$. For any $A \subseteq h^{<\omega}$ let

$$d_A = \sum_{\sigma\in A}\mu(\sigma)^{s-1}d_\sigma.$$

Then

$$d_A(\langle\rangle) = \sum_{\sigma\in A}\mu(\sigma)^{s-1}d_\sigma(\langle\rangle) = \mathtt{wt}_s(A)$$

and so if $\mathtt{wt}_s(A)$ is finite, then d is a martingale (inductively, for all $\tau \in h^{<\omega}$, we get $d_A(\tau) < \infty$). If A is r.e. then d_A is left-r.e.

Thus if $\langle A_k\rangle$ is a s-ML-test that covers X, then we can let

$$d = \sum_{k<\omega} k d_{A_k}.$$

Then $d(\langle\rangle) = \sum k2^{-k} < \infty$ and so d is a martingale. If $\sigma \in A_k$ then

$$d(\sigma)\mu(\sigma)^{1-s} \geqslant \left(k\mu(\sigma)^{s-1}d_\sigma(\sigma)\right)\mu(\sigma)^{1-s} = k.$$

Since there is a $\sigma \subset X$ in A_k we get that d s-succeeds on X. $\square$

Proposition 2.10 (Lutz [10], Calude-Staiger-Terwijn [2])**.** *If there is a left-r.e. supermartingale d that s-succeeds on X, then X is weakly s-null.*

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[2] Note that this is equivalent to $\sum d(\tau)\mu(\tau) \leqslant d(\sigma)\mu(\sigma)$, where the sum is taken over all immediate successors τ of σ.

Proof. The key is that Kolmogorov's inequality holds in the space h^ω as well: for any prefix-free set $C \subset h^{<\omega}$ we have

$$\sum_{\sigma \in C} d(\sigma)\mu(\sigma) \leqslant d(\langle\rangle).$$

The easiest way to see this is to think of $d(\sigma)\mu(\sigma)$ as inducing a submeasure on h^ω.

Thus if d s-succeeds on X, we can let, for $k < \omega$,

$$A_k = \{\sigma \in h^{<\omega} \,:\, d(\sigma)\mu(\sigma)^{1-s} > 2^k\}$$

(by normalizing we assume that $d(\langle\rangle) \leqslant 1$). Then $\langle A_k \rangle$ is uniformly r.e., covers X, and if $C \subseteq A_k$ is prefix-free then

$$\mathtt{wt}_s(C) = \sum_{\sigma \in C} \mu(\sigma)^s \leqslant 2^{-k} \sum_{\sigma \in C} d(\sigma)\mu(\sigma) \leqslant 2^{-k},$$

thus $\mathtt{wkwt}_s(A_k) \leqslant 2^{-k}$. □

Corollary 2.11. *The following four numbers are equal for $X \in h^\omega$. This number $\dim^h(X)$ is the* effective Hausdorff dimension *of X in the space h^ω.*

1. $\inf\{s$: *There is a left-r.e. supermartingale d that s-succeeds on X*$\}$.
2. $\inf\{s$: *X is s-null*$\}$.
3. $\inf\{s$: *X is weakly s-null*$\}$.
4. $\inf\{s$: *X is covered by a Solovay s-test*$\}$.

The proof of the final result in this section is standard.

Proposition 2.12. *There is an optimal left-r.e. supermartingale d^h for h. That is, for any left-r.e. supermartigale d for h, there is a constant c such that $d(\sigma) \leqslant cd^h(\sigma)$, for all $\sigma \in h^{<\omega}$.*

3. TRANSLATING BETWEEN SPACES

There is a natural measure-preserving surjection of h^ω onto the Euclidean interval $[0,1]$. First we map strings to closed intervals: we let $\pi^h(\langle\rangle) = [0,1]$, and if $\pi^h(\sigma) = I$ is defined for some $\sigma \in h^{<\omega}$ then we divide I into $h(|\sigma|)$ many intervals I_k of equal length and let $\pi^h(\sigma k) = I_k$. Note that indeed $\mu^h(\sigma) = \lambda(\pi^h(\sigma))$, where λ is Lebesgue measure on $[0,1]$. Finally we extend π^h continuously to h^ω by letting $\{\pi^h(X)\} = \bigcap_n \pi^h(X \upharpoonright n)$. The fact that $h(n) \geqslant 2$ ensures that this intersection is indeed a singleton. The mapping π^h is not quite 1-1; it is, if we ignore the (countably many) sequences that are eventually constant.

Note that for all $X \in h^\omega$, $X \equiv_T \pi^h(X)$.

Some parts of the theory of effective dimension can also be developed for the space $[0,1]$. We do not have martingales, but we can still, for example, define Solovay tests: for $s \geqslant 0$, a *Solovay s-test* for $[0,1]$ is an r.e. collection A of (rational) closed intervals in $[0,1]$ whose s-weight $\mathtt{wt}_s(A) = \sum_{I \in A} \lambda(I)^s$ is finite. Such a test A *covers* a point $x \in [0,1]$ if for infinitely many $I \in A$ we have $x \in I$. We let $\dim^{[0,1]}(x)$ be the infimum of all s such that there is a Solovay s-test (for $[0,1]$) that covers x.

Proposition 3.1. *For any h and all $X \in h^\omega$, $\dim^{[0,1]}(\pi^h(X)) \leqslant \dim^h(X)$.*

Proof. Let $s > \dim^h(X)$ and let A be a Solovay s-test for h that covers X. Since π^h is measure-preserving, the s-weight of $\pi^h[A]$ (in $[0,1]$) is also finite and covers $\pi^h(X)$, so $\dim^{[0,1]}(\pi^h(X)) \leqslant s$. $\square$

Equality does not hold in general. We show below that it holds if h does not grow too quickly or irregularly. Suppose that A is a Solovay s-test for $[0,1]$ that covers $\pi^h(X)$. We would like to cover X by something like $(\pi^h)^{-1}A$. The problem is that the basic sets (rational closed intervals) in $[0,1]$ are finer than the basic sets in h^ω; not every closed rational interval is in the range of π^h. Thus we need to replace A by a coarser collection: replace every $I \in A$ by finitely many intervals in the range of π^h. We can control the Lebesgue measure of such a collection, but if $s < 1$, then the process of replacing large intervals by a collection of smaller ones may increase the s-weight significantly. We show that if h does not grow too irregularly and we increase the exponent, the weight remains finite.

Let $\mathcal{I}_n = \pi^h[h^n]$ and let $\mathcal{I} = \bigcup_n \mathcal{I}_n = \pi^h[h^{<\omega}]$. Recall that $\gamma_n = 1/|h^n|$ is the μ^h-measure of any string in h^n. Thus $\mathcal{I}_n$ consists of $|h^n|$ many closed intervals, each of length γ_n.

For a closed interval $I \subsetneq [0,1]$, let n_I be the unique n such that $\gamma_n \geqslant \lambda(I) > \gamma_{n+1}$. Let k_I be the greatest integer k such that $k\gamma_{n_I+1} \leqslant \lambda(I)$; so $k_I \leqslant h(n_I)$ and there is a set $\hat{I} \subseteq \mathcal{I}_{n_I+1}$ of size $k_I + 1$ such that $I \subseteq \bigcup \hat{I}$.

For a set A of closed rational subintervals of $[0,1]$, let $\hat{A} = \bigcup_{I\in A} \hat{I}$. Thus $\hat{A} \subseteq \mathcal{I}$, and if $x = \pi^h(X)$ and x is covered by A, then X is covered by $(\pi^h)^{-1}\hat{A}$. If A is r.e. then so is $\hat{A}$ (and $(\pi^h)^{-1}\hat{A}$). Note that the t-weight of $\hat{A}$ in $[0,1]$ equals the t-weight of $(\pi^h)^{-1}\hat{A}$ in h^ω.

We express the regularity of h in terms of the following conditions, where $t > s \geqslant 0$:

$$(*)(t,s) \qquad \sum_{n<\omega} \frac{h(n)^{1-s}}{\left(h(0)\cdots h(n)\right)^{t-s}} < \infty.$$

Lemma 3.2. *Suppose that $(*)(t,s)$ holds for h and that A is a set of closed intervals in $[0,1]$ such that $\mathtt{wt}_s(A)$ is finite. Then $\mathtt{wt}_t(\hat{A})$ is also finite.*

Thus for all $X \in h^\omega$, if $(*)(t,s)$ holds for h and $\dim^{[0,1]}(\pi^h(X)) < s$, then $\dim^h(X) \leqslant t$. Thus if $(*)(t,s)$ holds for all $t > s \geqslant 0$ then for all $X \in h^\omega$, $\dim^{[0,1]}(\pi^h(X)) = \dim^h(X)$.

Proof. Let I be any closed interval in $[0,1]$. Let $n = n_I$ and $k = k_I$. Recalling that $\lambda(I)/\gamma_{n+1} \geqslant k$, we have

$$\begin{aligned}\mathtt{wt}_t(\hat{I}) = (k+1)\gamma_{n+1}^t \leqslant 2k\gamma_{n+1}^{t-s}\gamma_{n+1}^s = 2k\frac{\gamma_{n+1}^s}{\lambda(I)^s}\gamma_{n+1}^{t-s}\lambda(I)^s \\ < 2k^{1-s}\gamma_{n+1}^{t-s}\lambda(I)^s \leqslant \frac{2h(n)^{1-s}}{\left(h(0)\dots h(n)\right)^{t-s}}\lambda(I)^s.\end{aligned}$$

because $k \leqslant h(n)$.

Thus if we let $A_n = \{I \in A : n_I = n\}$ then

$$\mathtt{wt}_t(\hat{A}) \leqslant \sum_{n<\omega} \sum_{I \in A_n} \mathtt{wt}_t(\hat{I}) \leqslant \sum_{n<\omega} \sum_{I \in A_n} \frac{2h(n)^{1-s}}{(h(0)\dots h(n))^{t-s}} \lambda(I)^s$$

$$\leqslant 2\mathtt{wt}_s(A) \sum_{n<\omega} \frac{h(n)^{1-s}}{(h(0)\dots h(n))^{t-s}},$$

which by assumption is finite. □

Note, for example, that the condition $(*)(t,s)$ holds (for all $t > s \geqslant 0$) for the constant function $h(n) = 2$. Thus dimension in $[0,1]$ is the same as dimension in Cantor space 2^ω. However, the condition holds for some unbounded functions h as well (for example $h(n) = 2^n$).

The reader may note that we do not appear to use the full strength of $(*)(t,s)$ in the proof of Lemma 3.2. It would be enough to assume that the sequence $\left\langle h(n)^{1-s} (h(0) \cdots h(n))^{s-t} \right\rangle_{n<\omega}$ is bounded. On the other hand, if this holds for all $t > s \geqslant 0$, then $(*)(t,s)$ holds as well, so we have no need to make a distinction.

Lemma 3.3. *Suppose that*

$$\lim_{n\to\infty} \frac{\log h(n)}{\sum_{m\leqslant n} \log h(m)} = 0.$$

Then for all $t > s \geqslant 0$, the condition $(*)(t,s)$ holds for h, and so for all $X \in h^\omega$, $\dim^h(X) = \dim^{[0,1]}(\pi^h(X))$.

Proof. Let $f(n) = \log h(n)$. Let $t > s \geqslant 0$ and let $\varepsilon < (t-s)/(1-s)$. There is some $N < \omega$ such that for all $n \geqslant N$, $f(n) < \varepsilon \sum_{m\leqslant n} f(m)$. Let $g(n) = h(0) \cdots h(n)$ and let $\delta = \varepsilon(1-s) - (t-s) < 0$. For $n \geqslant N$, $h(n) < g(n)^\varepsilon$ and so

$$\frac{h(n)^{1-s}}{g(n)^{t-s}} < g(n)^\delta.$$

Thus

$$\sum_{n\geqslant N} \frac{h(n)^{1-s}}{(h(0)\dots h(n))^{t-s}} < \sum_{n<\omega} \left(2^\delta\right)^{\log_2 g(n)},$$

which is finite becuase $2^\delta < 1$ and $\log_2 g(n) \geqslant n$ (as $h(n) \geqslant 2$). □

The regularity condition of Lemma 3.3 is not, strictly speaking, a slowness condition, because, for example, $h(n) = 2^{n^2}$ satisfies the condition yet there is a monotone function that is dominated by h and does not satisfy the condition. However, the condition holds for all sufficiently slow monotone functions:

Lemma 3.4. *Suppose that h is non-decreasing and dominated by 2^{kn} (for some constant k). Then*

$$\lim_{n\to\infty} \frac{\log h(n)}{\sum_{m\leqslant n} \log h(m)} = 0,$$

and so for all $X \in h^\omega$, $\dim^h(X) = \dim^{[0,1]}(\pi^h(X))$.

Proof. There are two cases. If h is bounded then it is eventually constant, and the condition is easily verified. Assume then that h is unbounded. Fix $c < \omega$. There is an $N_c < \omega$ such that for all $n \geqslant N_c$, $\log h(n) > c$. For $n > N_c$,

$$\frac{\log h(n)}{\sum_{m\leqslant n} \log h(m)} \leqslant \frac{kn}{c(n-N_c) + \sum_{m<N_c} \log h(m)} \leqslant \frac{k}{c} + \frac{kN_c}{c(n-N_c)}.$$

As n grows, $kN_c/c(n-N_c)$ tends to 0 and so

$$\lim_{n\to\infty} \frac{\log h(n)}{\sum_{m\leqslant n} \log h(m)} \leqslant \frac{k}{c}.$$

As this is true for all $c < \omega$, we get that the limit is 0. □

4. Using slow DNR functions

Our goal is to prove that if q grows sufficiently slowly, then every DNR_q function computes a set of effective Hausdorff dimension 1. The power of slow growing DNR functions is related to the uniform power of bounded DNR functions, so we begin by investigating what can be computed uniformly from functions in DNR_a (even uniformly in a). Along the way we investigate $\mathcal{P}^c_a$, the class of functions $f \in a^\omega$ such that $f(n)$ avoids $J(n,0)$ through $J(n,c-1)$. This class was previously investigated by Cenzer and Hinman [3] and Corollary 4.6 follows from their work.

Definition 4.1. For positive natural numbers $a > b$, let $\mathcal{Q}^b_a$ be the collection of functions $f\colon \omega \to V_\omega$ such that for all $n < \omega$:

- $f(n) \subseteq \{0, \dots, a-1\}$;
- $|f(n)| = b$;
- If $n \in \operatorname{dom} J$ then $J(n) \notin f(n)$.

Via standard coding, $\mathcal{Q}^b_a$ can be seen as a recursively bounded Π^0_1 subclass of ω^ω. $\mathcal{Q}^1_a$ is essentially the same as DNR_a.

Recall that a class $\mathcal{P} \subseteq \omega^\omega$ is *strongly reducible* (or *Medvedev reducible*) to another class $\mathcal{Q}$ (we write $\mathcal{P} \leqslant_s \mathcal{Q}$) if there is a Turing functional Ψ such that for all $f \in \mathcal{Q}$, $\Psi(f) \in \mathcal{P}$. If $\mathcal{Q}$ is a Π^0_1 class, then without loss of generality, we can assume that Ψ is total and so we get a truth-table reduction.

Lemma 4.2. *For all $a > b > 0$, $\mathcal{Q}^{b+1}_{a+1} \leqslant_s \mathcal{Q}^b_a$, uniformly in a and b.*

The uniformity means that an index for the reduction functional Ψ can be obtained effectively from a and b.

Proof. First, for all $n < \omega$ and $y < a$ we can compute an input $m_{n,y}$ such that:

- For all $x < a$ such that $x \neq y$, $J(m_{n,y})\!\downarrow = x \Leftrightarrow J(n)\!\downarrow = x$;
- $J(m_{n,y})\!\downarrow = y$ iff either $J(n)\!\downarrow = y$ or $J(n)\!\downarrow = a$.

Let $f \in \mathcal{Q}^b_a$ and let $n < \omega$. Note that for all $y < a$, if $n \in \operatorname{dom} J$ then $J(n) \notin f(m_{n,y})$ and so $J(n) \notin g(n) = \bigcup_{y<a} f(m_{n,y})$. Further, if there is some $y < a$ such that $y \in f(m_{n,y})$ then we also know that $J(n) \neq a$ and so $J(n) \notin g(n) \cup \{a\}$. Finally, if $|g(n)| = b$ then $f(m_{n,y})$ is in fact constant for all $y < a$ and so in that case $y \in f(m_{n,y})$ for all $y \in g(n)$. Thus we can define Ψ as follows:

$$\Psi(f,n) = \begin{cases} \text{some subset of } g(n) \text{ of size } b+1, & \text{if } |g(n)| > b; \\ g(n) \cup \{a\}, & \text{if } |g(n)| = b. \end{cases}$$ □

Corollary 4.3. *For all* $a \geqslant 2$ *and* $b \geqslant 0$, $\mathcal{Q}^{b+1}_{a+b} \leqslant_s \mathrm{DNR}_a$, *uniformly in* a *and* b.

Definition 4.4. For $a \geqslant 2$ and $c > 0$, let $\mathcal{P}^c_a$ be the collection of functions $f \in a^\omega$ such that for all n and all $x < c$, if $(n, x) \in \operatorname{dom} J$ then $J(n, x) \neq f(n)$.

Again $\mathcal{P}^1_a \equiv_s \mathrm{DNR}_a$.

Lemma 4.5. *For all* $a > b > 0$ *and* $c \geqslant 1$, *if* $c(a-b) < a$ *then* $\mathcal{P}^c_a \leqslant_s \mathcal{Q}^b_a$, *uniformly in* a, b *and* c.

Proof. Given $f \in \mathcal{Q}^b_a$ and $n < \omega$, for all $x < c$, if $(n, x) \in \operatorname{dom} J$ then

$$J(n, x) \in \bigcup_{x<c} \{0, 1, \dots, a-1\} \smallsetminus f(n, x).$$

The set on the right has size at most $c(a - b)$ and so if $c(a - b) < a$ we can choose an $x < a$ not in that set and define $\Psi(f, n) = x$. ☐

Corollary 4.6 (See Cenzer and Hinman [3])**.** *For all* $a \geqslant 2$ *and* $c > 0$, $\mathcal{P}^c_{ca} \leqslant_s \mathrm{DNR}_a$, *uniformly in* a *and* c.

Proof. Let $b = c(a - 1) - a + 1$. Then $\mathcal{Q}^{b+1}_{a+b} \leqslant_s \mathrm{DNR}_a$ and

$$c((a + b) - (b + 1)) = c(a - 1) < a + b$$

so $\mathcal{P}^c_{c(a-1)+1} = \mathcal{P}^c_{a+b} \leqslant_s \mathcal{Q}^{b+1}_{a+b}$. Since $c > 0$ we have $c(a - 1) + 1 \leqslant ca$ and so $\mathcal{P}^c_{c(a-1)+1} \subseteq \mathcal{P}^c_{ca}$ and so $\mathcal{P}^c_{ca} \leqslant_s \mathcal{P}^c_{c(a-1)+1}$. All reductions are uniform. ☐

We now use the classes $\mathcal{P}^c_{ca}$ to construct sequences of positive dimension. We begin lightly.

Proposition 4.7. *Let* $a \geqslant 2$. *For every* $\varepsilon > 0$, *every* $f \in \mathrm{DNR}_a$ *computes a real* X *of dimension* $\dim(X) > 1 - \varepsilon$. *This is uniform in* a *and* ε.

That each $f \in \mathrm{DNR}_a$ computes such reals is of course not new, since the Turing degree of a bounded DNR function is PA and so computes a random real. The extra information is the uniformity.

Proof. Fix $c > 1$. We work in the space $(ca)^\omega$. Let d be the universal left-r.e. supermartingale for this space; by scaling we may assume that $d(\langle\rangle) < 1$.

For every $\sigma \in (ca)^{<\omega}$ we compute a number $m_\sigma < \omega$ such that for all $x < c$, $J(m_\sigma, x)\downarrow = k$ if σk is the x^{th} immediate successor of σ in $(ca)^{<\omega}$ that is discovered (via some left-r.e. approximation for d) such that $d(\sigma k) \geqslant a^{|\sigma|+1}$.

The point is that if $d(\sigma) \leqslant a^{|\sigma|}$, then at most c immediate successors τ of σ can have $d(\tau) \geqslant a^{|\tau|}$ (by the supermartingale condition that the sum of $d(\tau)$ for all immediate successors τ of σ is at most $ca \cdot d(\sigma)$), thus all "heavy" extensions of σ are "captured" by J. We can thus use a function $g \in \mathcal{P}^c_{ca}$ to avoid all such extensions: given such g, inductively define $X \in (ca)^{<\omega}$ by letting the $n + 1^{\text{st}}$ bit of X be $g(m_{X\restriction n})$. Then by induction we prove that for all n, $d(X \restriction n) \leqslant a^n$.

Now let $s \geqslant 0$ and suppose that d s-succeeds on X, that is, $\left\langle d(X \restriction n)\mu^{ca}(X \restriction n)^{1-s}\right\rangle$ is unbounded. Now $\mu^{ca}(X \restriction n) = (ca)^{-n}$ and so

$$d(X \restriction n)\mu^{ca}(X \restriction n)^{1-s} \leqslant a^n (ca)^{-n(1-s)} = \left(c^{s-1}a^s\right)^n.$$

Thus if d s-suceeds on X then $c^{s-1}a^s > 1$, so $a^s > c^{1-s}$. Taking a logarithm on both sides, we get

$$\frac{s}{\log_a c} > 1 - s$$

so

$$s > 1 - \frac{1}{1 + \log_a c}.$$

Thus $\dim^{ca}(X) \geqslant 1 - 1/(1 + \log_a c)$. Now as c grows, $\log_a c \to \infty$ so given ε we can find a c such that the corresponding X has dimension $\dim^{ca}(X) > 1 - \varepsilon$. Of course the constant function ac satisfies the regularity property of the last section, so X computes a $Y \in 2^\omega$ such that $\dim Y > 1 - \varepsilon$. □

Proposition 4.8. *For any $a \geqslant 2$, each $f \in \mathrm{DNR}_a$ computes a real X of dimension 1. This is uniform in a.*

Proof. We combine the constructions of reals of dimensions closer and closer to 1 into one construction. Let $h(n) = (n + 1)a$; let d be the universal left-r.e. supermartingale for h^ω. Given $f \in \mathrm{DNR}_a$, for $n > 0$ obtain (uniformly) $g_n \in \mathcal{P}^n_{na}$. For $\sigma \in h^n$ find some $m_\sigma < \omega$ such that for $x < n + 1$, $J(m_\sigma, x)\downarrow = k$ if σk is the x^{th} immediate successor of σ in $h^{<\omega}$ that is discovered (via some left-r.e. approximation for d) such that $d(\sigma k) \geqslant a^{|\sigma|+1}$. Again the point is that if $\sigma \in h^n$ and $d(\sigma) \leqslant a^{|\sigma|}$ then since $\sum d(\tau) \leqslant (n + 1)a \cdot d(\sigma)$ where we sum over immediate successors τ of σ, so there can be at most $n + 1$ many such τ's such that $d(\tau) \geqslant a^{|\tau|}$. We can then define $X(n) = g_{n+1}(m_{X\restriction n})$ and inductively prove that $d(X \restriction n) \leqslant a^n$ for all n.

Now $\mu^h(X \restriction n) = a^{-n}/n!$ so for $s \geqslant 0$,

$$d(X \restriction n)\mu^h(X \restriction n)^{1-s} \leq \frac{a^{sn}}{(n!)^{1-s}}.$$

If $s < 1$, then for almost all n we have $n! > a^{\frac{sn}{1-s}}$, so for almost all n we have

$$\frac{a^{sn}}{(n!)^{1-s}} < 1$$

and d cannot s-succeed on X. Thus $\dim^h(X) = 1$. Since h is dominated by 2^n, it satisfies the regularity condition and X computes a $Y \in 2^\omega$ of dimension 1. □

Finally we can paste together these constructions for all $a \geqslant 2$ and get the desired theorem.

Theorem 4.9. *There is a recursive, nondecreasing, unbounded function $j\colon \omega \to \omega \setminus \{0, 1\}$ such that every $f \in \mathrm{DNR}_j$ computes a real $X \in 2^\omega$ of dimension 1.*

Proof. Let $h(n) = (n+1)2^n$, and let d be the universal left-r.e. supermartingale for h^ω. For every $\sigma \in h^n$ find an input m_σ such that for all $x < 2^n$, $J(m_\sigma, x)\downarrow = k$ if σk is the x^{th} immediate successor of σ in $h^{<\omega}$ that is discovered such that $d(\sigma k) \geqslant (n + 1)!$.

We know that for all $n \geqslant 1$, $\mathcal{P}^{2^n}_{h(n)} \leqslant \mathrm{DNR}_{n+1}$ uniformly in n, so there is an effective list of truth-table functionals Ψ_n such that for all $f \in \mathrm{DNR}_{n+1}$, $\Psi_n(f) \in \mathcal{P}^{2^n}_{h(n)}$. Let ψ_n be a recursive bound on the use function of Ψ_n. Let

$$m^*_n = 1 + \sup\{\langle m_\sigma, x\rangle \,:\, \sigma \in h^n \;\&\; x < 2^n\}$$

and let

$$u_n = \psi_n(m^*_n).$$

Thus for all n, if ρ is a sequence of length u_n that is a DNR_{n+1}-string (that is, $\rho \in (n + 1)^{u_n}$ and for all $y < u_n$ in $\operatorname{dom} J$, $J(y) \neq \rho(y)$; equivalently, ρ is an initial segment of sequences in DNR_{n+1}) then $\Psi_n(\rho)$ is a $\mathcal{P}^{2^n}_{h(n)}$-string (an initial segment of a sequence in $\mathcal{P}^{2^n}_{h(n)}$) of length at least m^*_n. By increasing ψ_n we may assume that for all n, $u_n < u_{n+1}$. We define $j(k) = n+1$ iff $u_{n-1} \leqslant k < u_n$. So if $f \in \mathrm{DNR}_j$ then for all n, $f \upharpoonright u_n$ is a DNR_{n+1}-string and so gluing the reductions Ψ_n there is a $g \leqslant_T f$ such that for all $n < \omega$ and all $\sigma \in h^n$,

1. $g(m_\sigma) < h(|\sigma|)$;
2. For all $x < 2^n$, if $(m_\sigma, x) \in \operatorname{dom} J$ then $g(m_\sigma) \neq J(m_\sigma, x)$.

We can now use g to construct $X \in h^\omega$ as in the last two constructions: $X(n) = g(m_{X\upharpoonright n})$. By induction on n we prove that $d(X \upharpoonright n) \leqslant n!$; again the point is that if $\sigma \in h^n$ and $d(\sigma) \leqslant n!$ then there are at most 2^n many immediate successors τ of σ such that $d(\tau) \geqslant (n+1)!$ and so they are all captured by J and avoided by g.

Finally we calculate dimension to show that $\dim^h(X) = 1$. We then note that h satisfies the regularity condition of Lemma 3.4 and so X computes a $Y \in 2^\omega$ of dimension 1.

Let $s < 1$ and let $\varepsilon = 1 - s > 0$. For any $\sigma \in h^n$,

$$\mu^h(\sigma) = \frac{1}{2^0 2^1 \cdots 2^{(n-1)} n!} = \frac{1}{2^{\binom{n}{2}} n!}.$$

Thus for all $n < \omega$,

$$d(X \upharpoonright n)\mu^h(X \upharpoonright n)^\varepsilon \leqslant \frac{(n!)^s}{2^{\varepsilon\binom{n}{2}}} \leqslant \frac{n!}{2^{\varepsilon\binom{n}{2}}},$$

which is bounded—indeed, it tends to 0, as $2^{\varepsilon\binom{n}{2}}$ grows faster than $2^{\delta n^2}$, for any $\delta < \varepsilon/2$, and $n!$ grows slower than $2^{n \log n}$. Thus d does not s-succeed on X. ☐

5. Fixed-point-free minimal degrees

In [8], Kumabe and Lewis construct a DNR function whose Turing degree is minimal. In fact, an investigation of their construction shows the following:

Theorem 5.1. *For any non-decreasing, unbounded, recursive function $j \colon \omega \to \omega \setminus \{0, 1\}$, there is an $f \in \mathrm{DNR}_j$ whose Turing degree is minimal.*

Let us elaborate. Kumabe and Lewis define the following notion of forcing $\mathbb{P}$. Conditions are recursive triples (T, g, w) such that:

1. T is an infinite partial function tree from $\omega^{<\omega}$ to $\omega^{<\omega}$. That is, T preserves extension and incomparability, and the domain of T is an initial segment of $\omega^{<\omega}$.
2. The domain of T is recursive; for all $\sigma \in \operatorname{dom} T$, $\{k < \omega : \sigma k \in \operatorname{dom} T\}$ is a finite initial segment of ω.
3. T takes levels to levels: for all $\sigma, \sigma' \in \operatorname{dom} T$, if $|\sigma| = |\sigma'|$ then $|T(\sigma)| = |T(\sigma')|$.
4. $T(\langle\rangle)$ is a DNR string.
5. $g \colon \omega \to \omega$ is strictly increasing (and for all n, $g(n)$ is a power of 2).
6. For $\sigma \in \operatorname{dom} T$, we say that σ is *terminal* in T if there is no proper extension of σ in $\operatorname{dom} T$.

   For all $n < \omega$ and all non-terminal $\sigma \in \operatorname{dom} T$ of length n, the set of $T(\tau)$ where τ is an immediate successor of σ in $\operatorname{dom} T$ is $g(n)$-*bushy* above $T(\sigma)$. That is, if S is the collection of all initial segments of such $T(\tau)$, then for all $\rho \in S$ such that $\rho \supseteq T(\sigma)$ and ρ is not a leaf of S, there are at least $g(n)$ many immediate successors of ρ in S.

(7) $w < \omega$, indeed $0 < w < g(0)$ and there is no finite set R of terminal strings such that $T[R]$ is w-bushy above $T(\langle\rangle)$. Again, this means that for the finite tree S obtained by taking all initial segments of strings in $T[R]$, if $\rho \in S$, $\rho \supseteq T(\langle\rangle)$ and $\rho \notin T[R]$ then ρ has at least w many immediate successors in S.

Kumabe and Lewis show that there is a partial ordering $\leqslant_{\mathbb{P}}$ on $\mathbb{P}$ such that if $G \subset \mathbb{P}$ is sufficiently generic, then

$$f_G = \bigcup T(\langle\rangle)\,[\![\text{For some } g \text{ and } w,\ (T, g, w) \in G]\!]$$

is a DNR function that has minimal Turing degree.[3] Thus to prove Theorem 5.1 it is sufficient to show that given an order j as in the theorem, there is a condition $(T, g, w) \in \mathbb{P}$ such that the range of T is contained in $j^{<\omega}$.

We build the condition. In fact, we build T so that $[T] = j^{\omega}$; we just need to pick sufficiently fast-growing levels so that the bushiness given by j is sufficient. Let $g(n) = 2^{n+1}$ and let k_n be the least k such that $j(k) \geqslant g(k)$. By slowing j down we may assume that $k_n < k_{n+1}$ for all n. Note that $k_0 = 0$. We then define T so that $T[\omega^n] = j^{k_n}$. Namely, define $T(\langle\rangle) = \langle\rangle$; if $\sigma \in \omega^n$ and we already defined $T(\sigma) \in j^{k_n}$, let $a_n = j(k_n)j(k_n + 1) \cdots j(k_{n+1} - 1)$ and let $\{T(\sigma x)\}$, for $x < a_n$, consist of all extensions of $T(\sigma)$ in $j^{k_{n+1}}$. Finally let $w = 1$.

We verify that the conditions above hold. (1), (2) and (3) are immediate. (4) holds because the empty string is a DNR string. (5) holds by definition of g. (6) holds by definition of k_n: for all l, $k_n \leqslant l$, we have $j(l) \geqslant g(n)$ and so we put at least $g(n)$ splitting on T above level n of T. Finally, (7) holds because there are no terminal strings on T.

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[3] For completeness, the partial ordering is defined as follows: (T', g', w') extends (T, g, w) if there is a function tree R such that $T' = T \circ R$ and the following holds. For all $k < \omega$ there is an $l(k) < \omega$ such that for all strings $\sigma \in \operatorname{dom} R$ of length k, $|R(\sigma)| = l(k)$ (Thus strings of level k on T' are on level $l(k)$ of T). We require that for all $k \geqslant 1$, $g'(k) \leqslant g(l(k))/2^{5+3a_k}$, where a_k is the number of strings that are either on level k of T' or on a level below k on T' and terminal on T'.

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