THE CONTINUOUS DEGREES: OPEN QUESTIONS

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Introduction. The continuous degrees and representations reducibility are introduced in [3]; the questions that follow refer to the material in that paper and, unless otherwise mentioned, any results described below are proved there.

I. Degrees of d.c.e. functions. We say that $f \in \mathcal{C}[0,1]$ is *weakly computable* if there is a uniformly computable sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $f = \lim_{n\to\infty} f_n$. If $f_0 \leq f_1 \leq f_2 \leq \cdots$, then f is *computably enumerable* (*c.e.*). If $g, h \in \mathcal{C}[0,1]$ are c.e., then we say that $f = g - h$ is *d.c.e.* The following proposition is not proved in [3], but it follows from the facts presented there.

Proposition. *Every c.e. function* $f \in \mathcal{C}[0,1]$ *has total degree. On the other hand, weakly computable functions can have non-total degree.*

Question 1. Must d.c.e. functions have total degree?

II. Diagonally not c.d. sequences. For each $e \in \mathbb{N}$, let γ_e be the eth partial computable function from $[0,1]^{\mathbb{N}}$ to $[0,1]$. We say that $\alpha \in [0,1]^{\mathbb{N}}$ is *diagonally not computably diagonalizable* (*d.n.c.d.*) if $(\forall e)[\gamma_e(\alpha)\downarrow \implies \alpha(e) = \gamma_e(\alpha)]$. We know that fixed points of the multivalued function $\Psi\colon [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ (defined in [3]) are d.n.c.d and also that d.n.c.d. sequences have non-total degree.

Question 2. Do all non-total continuous degrees contain a d.n.c.d. sequence?

Question 3. Do all continuous degrees which contain a d.n.c.d. sequence contain a fixed point of Ψ?

Question 4. If either of the previous questions are answered in the negative, then do the continuous degrees containing a d.n.c.d. sequence play a special role in the continuous degrees? What about those containing a fixed point of Ψ?

III. Definability of substructures.

Question 5. Are the continuous degrees definable within the enumeration degrees?

Question 6. Are the Turing degrees definable within the continuous degrees (i.e., is total a definable property)?

Positive answers to both questions would obviously prove that the Turing degrees are definable in the enumeration degrees (is this still an open question?).

IV. Embedding partial orders. The *Turing ideal below a continuous degree* $\mathbf{v}$ is $\mathcal{I}_T(\mathbf{v}) = \{\mathbf{a} \leq \mathbf{v} \mid \mathbf{a} \text{ is total}\}$. Let $\mathcal{I} = \mathcal{I}_T(\mathbf{v})$ for some non-total continuous degree $\mathbf{v}$; such ideals are characterized in [3]. It is known that there is an antichain of size continuum in the continuous degrees with Turing ideal $\mathcal{I}$. What about other partial orders?

Question 7. Are there continuous degrees $\mathbf{v} < \mathbf{w}$ with $\mathcal{I}_T(\mathbf{v}) = \mathcal{I}_T(\mathbf{w}) = \mathcal{I}$?

Question 8. Do all countable partial orders embed into the continuous degrees with Turing ideal $\mathcal{I}$?

Question 9. Which uncountable partial orders do?

V. Extreme degrees. Other questions can be asked about the collection of continuous degrees which share a given Turing ideal. For example, are there any extreme elements in this collection? Again, let $\mathcal{I} = \mathcal{I}_T(\mathbf{v})$ for some non-total continuous degree $\mathbf{v}$.

Question 10. Is there a minimal continuous degree with Turing ideal $\mathcal{I}$?

Question 11. Is there a maximal continuous degree with Turing ideal $\mathcal{I}$?

It is known that the collection of continuous degree with Turing ideal $\mathcal{I}$ cannot have either a greatest or a least element.

VI. Minimal covers. We know that non-total degrees cannot be minimal covers of total degrees, but other questions about minimal covers remain open.

Question 12.

(i) Do all non-total continuous degrees have minimal covers?
(ii) Can/Must a non-total continuous degree have a total minimal cover?
(iii) Can/Must a non-total continuous degree have a non-total minimal cover?

These problems are related to Questions 10 and 11.

VII. Finite dimensional metric spaces. In [3], it is proved that every continuous degree contains members of $[0,1]^{\mathbb{N}}$ and $\mathcal{C}[0,1]$, both of which are infinite dimensional. Vasco Brattka conjectured a connection between the dimension of a computable metric space and the degrees of its members. Along with the author, he effectivized a result from dimension theory to show the following [1].

Proposition. *Let $\mathcal{M}$ be a finite dimensional computable metric space and $x \in \mathcal{M}$.*

(i) *If $\mathcal{M}$ is co-r.e. compact, then x has total degree.*
(ii) *If $\mathcal{M}$ is compact and $\mathbf{deg}_r(x) \geq \mathbf{0}'$, then x has total degree.*
(iii) *There is a Turing degree $\mathbf{d}$ (independent of $\mathcal{M}$) such that if $\mathbf{deg}_r(x) \geq \mathbf{d}$, then x has total degree.*

Question 13. How large must $\mathbf{d}$ be in the Theorem? Can we take $\mathbf{d} = \mathbf{0}'$?

Every continuous degree below $\mathbf{0}'$ is the degree of a member of a compact zero dimensional computable metric space, but what about non-total degrees incomparable with $\mathbf{0}'$?

Question 14. Can/Must a non-total degree incomparable with $\mathbf{0}'$ be the degree of a member of a compact finite dimensional computable metric space?

The analogous question can be asked for the non-compact case. Finally, we note that not all infinite dimensional computable metric spaces have members of non-total degree. In fact, there is an infinite dimensional co-r.e. compact computable metric space $\mathcal{M}$ such that every member of $\mathcal{M}$ has total degree.

Question 15. Is there a natural condition on a computable metric space which guarantees that it has members of every continuous degree?

References.

[1] Vasco Brattka and Joseph S. Miller, Effective embedding and universal space theorems for finite dimensional computable metric spaces, *Notes*.

[2] Lance Gutteridge, Some results on enumeration reducibility, *Ph.D. Dissertation*, Simon Fraser University, 1971.

[3] Joseph S. Miller, Degrees of unsolvability of continuous functions, *The Journal of Symbolic Logic*, 69(2):555–584, 2004.

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