ON THE STRUCTURE OF THE DEGREES OF RELATIVE PROVABILITY

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ABSTRACT. We investigate the structure of the degrees of provability, which measure the proof-theoretic strength of statements asserting the totality of given computable functions. The degrees of provability can also be seen as an extension of the investigation of relative consistency statements for first-order arithmetic (which can be viewed as Π^0_1-statements, whereas statements of totality of computable functions are Π^0_2-statements); and the structure of the degrees of provability can be viewed as the Lindenbaum algebra of true Π^0_2-statements in first-order arithmetic. Our work continues and greatly expands the second author's paper on this topic by answering a number of open questions from that paper, comparing three different notions of a jump operator and studying jump inversion as well as the corresponding high/low hierarchies, investigating the structure of true Π^0_1-statements as a substructure, and connecting the degrees of provability to escape and domination properties of computable functions.

1. INTRODUCTION

The topic of this paper arises from two different directions in the study of logic. On the one hand, Gödel's Incompleteness Theorems tell us that given any sufficiently strong, consistent, effectively axiomatizable theory T for first-order arithmetic, there are even Π^0_1-statements (stating the consistency of T) that are not provable in T, but that are, of course, true. On the other hand, over the past seventy years, a number of researchers studying witness functions for various combinatorial statements have realized the importance of fast-growing functions and the fact that their totality is often not provable over a given sufficiently strong, consistent, effectively axiomatizable theory T for first-order arithmetic. Two famous examples are the Paris/Harrington Theorem [2] and the Kirby/Paris [4] work on Goodstein's Theorem. Since the totality of a given computable function can be formulated as a Π^0_2-statement, the study of the proof-theoretic strength of statements about the totality of computable functions can be viewed as an extension of Gödel's study of statements about consistency.

This paper continues work of the second author [1], investigating the degree structure of the *provability degrees*, or *p-degrees*. Let TA be true arithmetic, the first order theory of $(\mathbb{N}, +, \cdot, 0, 1)$. Fix any sufficiently strong, effectively axiomatizable

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theory T which is sound, i.e., its arithmetic consequences[1] are in TA (examples of such theories are ZFC and Peano arithmetic). A computable function φ is *p-reducible* to a total computable function ψ if, over T, the totality of φ can be proven from the totality of ψ. (Note, of course, that this really refers to *algorithms* for φ and ψ rather than just the *functions* φ and ψ!) Two computable functions are then *p-equivalent* if they are p-reducible to each other, and all computable functions that are p-equivalent to a given total computable function φ form the *p-degree* $[\varphi]$ of φ. Clearly, the p-degrees can be thought of as true Π^0_2-statements over T (i.e., we think of $[\varphi]$ as "tot(φ)", the statement that "for each x, there is a stage s by which (the algorithm for) φ has converged on x").

Cai's work [1] has already shown that this structure is quite rich. He showed that the p-degrees form a distributive lattice with least element; indeed they form a sublattice of the Lindenbaum algebra of T with the natural interpretations of least element, join, and meet as the (p-degree of the) trivial formula, conjunction, and disjunction, respectively. In addition, it is natural to introduce three so-called "jump" operators to our structure. One corresponds, loosely speaking, to the assertion $\mathrm{Con}_T(P)$ of the consistency of $T + P$ (for a Π^0_2-statement P), which we will call the "skip" of P (relative to T). Formally, the skip of φ will be a computable function $\mathrm{con}(\varphi)$ that "codes" the consistency of $T + \mathrm{tot}(\varphi)$. The skip of φ need not always be p-above φ, so we also introduce the join of φ and its skip as the "hop" of φ, which we write as φ°. Finally, and most importantly, following Cai [1], we define the "jump" of φ, which is a function φ^* whose totality is equivalent to the statement that every function p-reducible to φ is total (i.e., the Π^0_2 soundness of $T + \mathrm{tot}(\varphi)$). Note that in the Paris/Harrington result, the modified version of Ramsey's Theorem is equivalent to the Π^0_2 soundness of Peano Arithmetic (see [2, Theorem 3.1]).

The paper is organized as follows: In Section 2, we give some basic definitions and notation. In Section 3, we formally introduce and compare the three jump operators. In Section 4, we study the jump properties of Π^0_1-degrees and link these degrees to the property of escaping every provably total function. In Section 5, we show the density of the p-degrees; in fact, given any two p-degrees $[\varphi] < [\psi]$, we can find two incomparable p-degrees between them with meet $[\varphi]$ and join $[\psi]$. In Section 6, we study the high/low hierarchy for both the hop and the jump. In Section 7, we study the cappable p-degrees. In Section 8, we show jump inversion for both the hop and the jump. In Section 9, we study the connection between lowness and highness on the one hand, and domination and escape properties of functions on the other. We conclude in Section 10 with some open questions.

2. Preliminaries and Notation

We fix a base theory T. It must be effectively axiomatizable and sound (i.e., arithmetic consequences of T are in TA). It must also be sufficiently strong for our purposes; Peano Arithmetic is more than enough. The axioms of a discretely ordered semiring (PA^-) plus Σ^0_1-induction will always tacitly be assumed, and so T proves every true Σ^0_1-sentence of first-order arithmetic. We use capital Roman letters for sentences and formulas in the language of T. We write $P \vdash Q$ to mean

[1] If the language of T is not arithmetic (for example, the language of set theory), then we fix a standard interpretation of arithmetic in the language of T.

that Q follows from $T + P$ and s: "Q" to mean that s is (the Gödel number of) a proof of Q from T.

We use Greek letters for algorithms. Let $\{\varphi_e\}_{e\in\omega}$ be a standard list of algorithms. For an algorithm φ, we write $\mathrm{tot}(\varphi)$ for the sentence asserting the totality of φ. We define $\varphi \leq_p \psi$ to mean that

$$T \vdash \mathrm{tot}(\psi) \to \mathrm{tot}(\varphi),$$

and $\varphi \equiv_p \psi$ to mean that $\varphi \leq_p \psi$ and $\psi \leq_p \varphi$. We use f, g and h for total functions on ω. It is important to keep a distinction between a total algorithm and the function it *represents*. Write $\varphi \sim \psi$ to mean that φ and ψ represent the same function. It is entirely possible that $\varphi \sim \psi$ but the totality of ψ is much stronger, from a proof-theoretic standpoint, than the totality of φ.

We adopt the convention from [1] that functions converge on initial segments, by simply not considering $\varphi(x)$ to converge unless $\varphi(y)$ has already converged for all $y < x$. This does not affect any of the conclusions, since (under Σ^0_1-induction) being total under this modified notion of convergence is equivalent to being total under the usual notion. However, it simplifies a number of the arguments. There is a natural correspondence between Π^0_2-sentences and algorithms. On the one hand, given an algorithm φ, the sentence $\mathrm{tot}(\varphi)$ is Π^0_2. Conversely, given a Π^0_2-sentence $(\forall x)(\exists y)\, P(x, y)$, there is an algorithm φ that, on input x, outputs the least witness y to $(\exists y)\, P(x, y)$; the totality of this algorithm is provably equivalent to the original Π^0_2-sentence. Moreover, the functions mapping between Gödel numbers of algorithms and Gödel numbers of the corresponding Π^0_2-sentences are primitive recursive.

We enclose a mathematical statement in quotes to indicate a sentence in a formal language equivalent to the given statement. For example, we write "$[\varphi] \leq [\psi]$" for the sentence $\mathrm{Prov}_T(\#(\mathrm{tot}(\psi) \to \mathrm{tot}(\varphi)))$ in the language of arithmetic. (Here # denotes Gödel number and Prov_T is a standard provability predicate for T.) This is much less cumbersome and more readable than carefully writing out the formal sentence, and it is usually obvious how to turn a mathematical idea into the corresponding formal sentence. We sometimes enclose what is already a sentence in a formal language in quotes to separate it from surrounding text. Since every sentence is equivalent to itself, this should not create any ambiguity. We also identify sentences with their Gödel numbers, and will, henceforth, omit any mention of the # function.

If φ is total, then the (*provability*) *degree* of φ is $[\varphi] = \{\psi \colon \varphi \equiv_p \psi\}$. It is clear that $\leq_p$ is degree invariant, so it induces an order on the provability degrees. The provability degrees form a distributive lattice with a least element 0 (the constant 0 function, computed by the obvious algorithm). The join and meet are given by the operations

$$(\varphi \boxplus \psi)(x) = \varphi(x) + \psi(x)$$

(which converges on input x once $\varphi(x)$ and $\psi(x)$ have both converged) and

$$(\varphi \boxtimes \psi)(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \text{ converges at stage } s \text{ and } \psi(x) \text{ did not} \\ & \text{converge at any stage } t < s \\ \psi(x) & \text{if } \psi(x) \text{ converges at stage } t \text{ and } \varphi(x) \text{ did not} \\ & \text{converge at any stage } s \leq t \end{cases}$$

(which converges on input x once either $\varphi(x)$ or $\psi(x)$ has converged). We use the symbols $\vee$ and $\wedge$ for the join and meet. The reader should beware not to confuse these with disjunction and conjunction of Π^0_2 statements.

Any of the results we prove in this paper hold for any theory T sufficient for our purposes (for example, $\mathbf{I\Sigma_1}$). One important consequence is that for any total algorithm φ, the results also hold for the theory $T + \mathrm{tot}(\varphi)$. This gives a natural relativization of any result we prove to any provability degree $[\varphi]$.

3. A hop, skip, and a jump

There are a few natural ways, given a provability degree $[\psi]$, to produce a new provability degree $[\hat{\psi}] > [\psi]$ (or at least $[\hat{\psi}] \not\leq [\psi]$). We call these operations the *hop*, the *skip*, and the *jump*.

Definition 3.1 (Cai [1])**.** Given an algorithm ψ, its *jump* ψ^* is the algorithm

$$\psi^*(x) = \begin{cases} \varphi_e(x) & \text{if } x \text{ is a proof of ``}\varphi_e \leq_p \psi\text{'' for some } e \\ 0 & \text{otherwise.} \end{cases}$$

We define $[\psi]^* = [\psi^*]$.

It is easy to see that $\psi^* \not\leq_p \psi$, because $\psi^* \equiv_p \psi^* + 1$ and $\psi^* + 1$ differs from every $\varphi_e \leq_p \psi$ on any input that is a proof witnessing $\varphi_e \leq_p \psi$. One can also show (Proposition 4.5 of [1]) that the jump operator $\psi \mapsto \psi^*$ is (non-strict) order-preserving for $\leq_p$. One consequence is that the jump is degree-invariant, and so $[\psi]^* = [\psi^*]$ is well-defined.

One helpful way to think about ψ^* is as a single function that is universal for every $\theta \leq_p \psi$. If $\theta \leq_p \psi$, there is a provably total function f such that $\mathrm{tot}(\theta)$ is equivalent to $\mathrm{tot}(\psi^* \circ f)$. Moreover, an index for f can be found uniformly from an index for θ and proof witnessing $\theta \leq_p \psi$. (These results follow from Lemma 4.2 of [1], and the Padding Lemma.) This gives a nice analogy between the jump function ψ^* in this context and the jump function $J^A(e) = \Phi^A_e(e)$ in computability theory, which is similarly universal for all partial A-computable functions. Thinking about ψ^* this way makes it clear that $\psi^* >_p \psi$ for every total algorithm ψ.

Another helpful way to think about ψ^* is that $\mathrm{tot}(0^*)$ is equivalent to soundness of T for Π^0_2-statements (see also Proposition 4.3 of [1]). That is, $T + \mathrm{tot}(0^*)$ proves "for every Π^0_2 sentence P, $(\mathrm{Prov}_T(P) \to P)$." More generally, $\mathrm{tot}(\psi^*)$ is equivalent to soundness of $T + \mathrm{tot}(\psi)$ for Π^0_2-statements. This is because of the correspondence between Π^0_2-statements and algorithms, and the fact that the totality of ψ^* is equivalent to the uniform totality of all $T + \mathrm{tot}(\psi)$-provably total algorithms.

Definition 3.2. Given an algorithm ψ, its *skip* is the algorithm

$$\mathrm{con}(\psi)(x) = \begin{cases} \uparrow & \text{if } x \text{ is a proof of ``}\neg\,\mathrm{tot}(\psi)\text{''} \\ 0 & \text{otherwise.} \end{cases}$$

We define $[\psi]^{\mathrm{con}} = [\mathrm{con}(\psi)]$.

We use the notation $\mathrm{con}(\psi)$ because T proves

$$\mathrm{tot}(\mathrm{con}(\psi)) \iff \text{``}T + \mathrm{tot}(\psi) \text{ is consistent.''}$$

Gödel's Second Incompleteness Theorem implies that $\mathrm{con}(\psi) \not\leq_p \psi$ for every total algorithm ψ. It is easy to see that the con operator is order preserving for $\leq_p$, which implies that con is degree invariant and so $[\psi]^{\mathrm{con}} = [\mathrm{con}(\psi)]$ is well-defined.

Unlike the jump of ψ, which is always strictly above ψ, the skip of ψ may be either above ψ or off to the side (hence the name "skip"). For this reason we also define the "hop" as a strictly increasing version of skip.

Definition 3.3. Given an algorithm ψ, its *hop* is the algorithm $\psi^\circ = \psi \boxplus \mathrm{con}(\psi)$.

It is easy to see that $\psi <_p \psi^\circ$ for every total algorithm ψ, and that the operator $\psi \mapsto \psi^\circ$ is (non-strict) order-preserving and hence degree-invariant. Define $[\psi]^\circ = [\psi^\circ]$. A hop is a small jump, and indeed for every total ψ, we have $\psi <_p \psi^\circ <_p \psi^*$. We prove that $\psi^\circ \leq_p \psi^*$ in this section; in the next section, we will show that the inequality is strict.

Lemma 3.4. *For every total algorithm ψ, we have $\psi^\circ \leq_p \psi^*$.*

Proof. Since $\psi \leq_p \psi^*$, it suffices to show that $\mathrm{con}(\psi) \leq_p \psi^*$. Fix an algorithm φ_e that T-provably never halts (on any input). We argue within T, by contrapositive:

> Suppose that $\mathrm{con}(\psi)$ is not total. Since $\mathrm{tot}(\mathrm{con}(\psi))$ is equivalent to the consistency of $T + \mathrm{tot}(\psi)$, there is a T-proof p_1: "$\mathrm{tot}(\psi) \to 0 = 1$." Furthermore, since T proves $0 \neq 1$ and all propositional validities, there is a T-proof p_2 : "$0 = 1 \to \mathrm{tot}(\varphi_e)$." By adjoining p_1 and p_2, there is a T-proof x : "$\mathrm{tot}(\psi) \to \mathrm{tot}(\varphi_e)$," so $\psi^*(x) = \varphi_e(x)\uparrow$, and hence ψ^* is non-total.

Therefore $T \vdash \mathrm{tot}(\psi^*) \to \mathrm{tot}(\psi^\circ)$. □

As we did in the proof of the above lemma, we sometimes need to argue within the theory T. (When we are arguing within some formal theory, we will indent the internal argument.) For that reason, we will need to show that some of the inequalities between various provability degrees mentioned in this section are provable in T or extensions of T, so that we may use this fact when we argue internally in T in proofs in later sections. We will not worry about what base theory is necessary to prove the results of this paper, except when this is necessary so that a result may be used internally inside a later argument.

Lemma 3.5. *For every total algorithm ψ, the theory $T + \mathrm{tot}(\mathrm{con}(\psi))$ suffices to prove "$\psi^* \not\leq_p \psi$."*

Proof. Let $\varphi_e = \psi^* + 1$. We argue within T:

> Suppose there is a T-proof of $\mathrm{tot}(\psi) \to \mathrm{tot}(\psi^*)$. Then there is a T-proof of $\mathrm{tot}(\psi) \to \mathrm{tot}(\varphi_e)$, and hence T proves
>
> $$\text{"}T \vdash \mathrm{tot}(\psi) \to \mathrm{tot}(\varphi_e)\text{."}$$
>
> Therefore, T proves that if ψ is total, then $(\exists x)\, \psi^*(x) = \varphi_e(x) = \psi^*(x) + 1$. Hence $T + \mathrm{tot}(\psi) \vdash \neg(\mathrm{tot}(\psi^*))$. Putting this together with the T-proof of $\mathrm{tot}(\psi) \to \mathrm{tot}(\psi^*)$, we have a $T + \mathrm{tot}(\psi)$-proof of an inconsistency. □

4. Π^0_1 and Non-escaping degrees

Definition 4.1. A degree $[\varphi]$ is Π^0_1 if $\mathrm{tot}(\varphi)$ is provably equivalent to a (true) Π^0_1-sentence.

Definition 4.2. A degree $[\varphi]$ is *escaping* if there is $\psi \in [\varphi]$ such that ψ escapes every provably total function. Otherwise, $[\varphi]$ is *non-escaping.*

Theorem 4.3. *A degree is non-escaping if and only if it is bounded by a Π^0_1-degree. (There are degrees that are non-escaping but not Π^0_1. See Corollary 6.7.)*

Proof. First we show that every degree bounded by a Π^0_1-degree is non-escaping. Let $(\forall s)\, P(s)$ be a true Π^0_1-statement such that T proves

$$(\forall s)\, P(s) \to \mathrm{tot}(\varphi).$$

Consider the following algorithm: $\bar{\varphi}$, on input x, searches for the least s such that

(1) $\varphi(x)$ converges in at most s steps, or
(2) $\neg P(s)$.

If (1) holds, then $\bar{\varphi}(x) = \varphi(x)$; if (2) holds, $\bar{\varphi}(x) = 0$.

Then $\bar{\varphi} \sim \varphi$ because $P(s)$ holds for all s, and T proves that $\bar{\varphi}$ is total. Hence T proves the totality of the function $\bar{\varphi} + 1$, which dominates φ.

Next we show that if $[\varphi]$ is non-escaping, then $[\varphi]$ is bounded by a Π^0_1-degree. Suppose $[\varphi]$ is non-escaping, and let ψ be a provably total algorithm for a function that bounds the computing time function of φ. Now consider the sentence:

$$(\forall x)\ \varphi(x)\downarrow \text{ before stage } \psi(x).$$

This is a true Π^0_1-sentence and clearly implies the totality of φ. □

Corollary 4.4. *The join of two non-escaping degrees is non-escaping.* □

The next corollary follows from the *proof* of Theorem 4.3.

Corollary 4.5. *A degree $[\varphi]$ is non-escaping if and only if all $\psi \leq_p \varphi$ compute the same function as some provably total algorithm.* □

Corollary 4.6. *The inequality $[\varphi]^\circ \leq [\varphi]^*$ proved as Lemma 3.4 is strict.*

Proof. The totality of 0° is equivalent to the Π^0_1 sentence asserting the consistency of T, and hence $[0]^\circ$ is non-escaping. In contrast, $[0]^*$ is escaping by the previous corollary. Thus we have $[0]^\circ < [0]^*$, and hence $[\varphi]^\circ < [\varphi]^*$ by relativization. □

Definition 4.7. A degree $[\varphi]$ is Δ^0_2 if $\mathrm{tot}(\varphi)$ is provably equivalent to a (true) Σ^0_2-sentence. (Note that it is already in the form of a Π^0_2-sentence.)

Clearly, every Π^0_1-degree is Δ^0_2. It is also easy to see:

Proposition 4.8. *Every Δ^0_2-degree is bounded by a Π^0_1-degree.*

Proof. Assume that if $\mathrm{tot}(\varphi)$ is true and is provably equivalent to $(\exists y)(\forall x)\, P(y,x)$ over T. Fix y such that $(\forall x)\, P(y,x)$; this is a true Π^0_1-statement and T proves $(\forall x)\, P(y,x) \Longrightarrow \mathrm{tot}(\varphi)$. □

We show in Proposition 10.3 that there is a Δ^0_2-degree that is not Π^0_1. On the other hand, we will see in Corollary 6.7 that not every degree bounded by a Π^0_1-degree is Δ^0_2. Key to the proof will be the following result, which puts an important limitation on the behavior of Δ^0_2-degrees.

Theorem 4.9. *Every Δ^0_2-degree ψ is $\mathbf{GL}_1$, i.e., $[\psi]^* = [0]^* \vee [\psi]$.*

Proof. Arguing in $T + \mathrm{tot}(0^*) + \mathrm{tot}(\psi)$, we will show that ψ^* is total. That is, we must show that for every e, if $T \vdash \mathrm{tot}(\psi) \to \mathrm{tot}(\varphi_e)$ then φ_e is total. Let $Q = (\exists y)(\forall s)\, P(y,s)$ be a true Σ^0_2-statement that is provably equivalent to $\mathrm{tot}(\psi)$. We begin the internal argument in $T + \mathrm{tot}(0^*) + \mathrm{tot}(\psi)$:

Fix e, s where s : "$\mathrm{tot}(\psi) \to \mathrm{tot}(\varphi_e)$." Define $\bar{\varphi_e}$ as follows. On input x, search for the least t such that
(1) $\varphi_e(x)$ converges in at most t steps, or
(2) $(\forall y \le x)(\exists s \le t)\neg P(y, s)$.
Let $\bar{\varphi_e}(x)$ be either $\varphi_e(x)$ or 0, respectively.

Because T proves $\mathrm{tot}(\varphi_e)$ or $(\forall y)(\exists s)\neg P(y, s)$, it also proves that $\bar{\varphi_e}$ is total. This implies (by $\mathrm{tot}(0^*)$, which is equivalent to soundness of T for Π^0_2-sentences) that $\bar{\varphi_e}$ is total. But we are assuming $\mathrm{tot}(\psi)$, which is equivalent to Q, so there is a y such that $(\forall s)\, P(y, s)$. For all $x \ge y$, $\bar{\varphi_e}(x)$ converges if and only if $\varphi_e(x)$ does, so φ_e is total.

The above shows $[\psi]^* \le [0]^* \vee [\psi]$. The reverse inequality is obvious. $\square$

Note that the proof actually shows that $T + \mathrm{tot}(0^*)$ proves that for every Δ^0_2-degree $[\psi]$, $\mathrm{tot}(\psi) \to \mathrm{tot}(\psi^*)$.

Corollary 4.10. *If ψ is Δ^0_2, then $([\varphi] \vee [\psi])^* = [\varphi]^* \vee [\psi]$.*

Proof. This is because $[\varphi] \vee [\psi]$ is relatively Δ^0_2 over $[\varphi]$ and so $([\varphi] \vee [\psi])^* = [\varphi]^* \vee ([\varphi] \vee [\psi]) = [\varphi]^* \vee [\psi]$. $\square$

Corollary 4.11. *For all φ, we have $([\varphi]^\circ)^* = [\varphi]^*$.*

Proof. Since $[\varphi]^{\mathrm{con}}$ is Π^0_1, we have $([\varphi]^\circ)^* = ([\varphi] \vee [\varphi]^{\mathrm{con}})^* = [\varphi]^* \vee [\varphi]^{\mathrm{con}} = [\varphi]^*$. $\square$

The following lemma is well known.

Lemma 4.12. *If P is a Π^0_1-sentence, then $T \vdash \mathrm{Con}_T(P) \to P$.*

Proof. The sentence $\neg P$ is Σ^0_1, and so $T \vdash \neg P \to \mathrm{Prov}_T(\neg P)$, or, equivalently, $T \vdash \neg P \to \neg\,\mathrm{Con}_T(P)$. Now take the contrapositive. $\square$

Restating this result in the language of provability degrees, we have:

Corollary 4.13. *If $[\varphi]$ is Π^0_1, then $[\varphi] < [\varphi]^{\mathrm{con}}$, and hence $[\varphi]^{\mathrm{con}} = [\varphi]^\circ$.* $\square$

5. Density theorems

The Skull Action. There is a recurring idea in the proofs in this section and later on, when we want to ensure $[\psi] \not\le [\varphi]$ for some ψ we build. If $[\psi] \le [\varphi]$, then there is a proof s: "$\mathrm{tot}(\varphi) \to \mathrm{tot}(\psi)$." Using the Recursion Theorem, the algorithm for ψ can know its own index, so can identify a proof of this form. When it sees such a proof, it can replace ψ with some algorithm θ that we know is not below φ. We threaten *action* $\skull_\theta$: For all inputs x on which the algorithm $\psi(x)$ has not already converged, we simply copy θ, i.e., we set $\psi(x) = \theta(x)$ for these x. If we take action $\skull_\theta$, it creates a contradiction, because then $[\psi] = [\theta]$. So merely by threatening to perform action $\skull_\theta$ in the construction of ψ, without carrying it out, we ensure that $[\psi] \not\le [\varphi]$.

Similarly, if we want to ensure $[\psi] \not\ge [\varphi]$ for some ψ we build, we can threaten action $\skull_\theta$ for some $[\theta] \not\ge [\varphi]$. In this latter case, we often choose θ to be the constant 0 function (computed with the obvious algorithm). For this reason, action $\skull_0$ occurs frequently. We also refer to action $\skull_0$ as *annihilating* ψ.

The first two theorems where this idea occurs are about the density of the p-degrees. We prove both theorems (though the first is a corollary of the second) since they are interesting for other reasons.

Theorem 5.1. *Given any nonzero* $[\varphi]$*, there is another degree* $[\psi]$ *strictly between* $[0]$ *and* $[\varphi]$*.*

Proof. Construction of ψ:

At stage s, if s: "$\mathrm{tot}(\psi) \to \mathrm{tot}(\varphi)$," then annihilate ψ. If s: "$\mathrm{tot}(\psi)$," then perform action ☠$_\varphi$. Otherwise, define $\psi(s) = 1$.

Verification:

At every stage s, depending on some primitive recursive condition (whether there is some $p \leq s$ that is a proof of one of two fixed sentences), $\psi(s)$ outputs either 0, 1, or copies $\varphi(s)$. Hence T proves $\mathrm{tot}(\varphi) \to \mathrm{tot}(\psi)$, i.e., $[\psi] \leq [\varphi]$.

If we annihilate ψ, then $[\varphi] \leq [\psi] = [0]$, contradicting $[\varphi]$ being nonzero. If we perform action ☠$_\varphi$, then $[\varphi] = [\psi] \leq [0]$, again contradicting $[\varphi]$ nonzero. Hence neither action is performed, which ensures $[\psi] \neq [0]$ and $[\psi] \neq [\varphi]$. □

Relativizing this theorem, we obtain full density as a corollary.

Corollary 5.2. *Given* $[\theta] < [\varphi]$*, there is some* $[\psi]$ *such that* $[\theta] < [\psi] < [\varphi]$*.* □

By Theorem 4.3, the function ψ constructed in the proof of Theorem 5.1 is non-escaping; the true Π^0_1-sentence that implies that ψ is total is $P =$ "action ☠$_\varphi$ is never taken." In fact, $\mathrm{tot}(\psi)$ is equivalent to (P or $\mathrm{tot}(\varphi)$). Therefore, if φ has Π^0_1 or Δ^0_2-degree, then so does ψ. This proves that there is no minimal nonzero Π^0_1-degree, and no minimal nonzero Δ^0_2-degree. Relativizing, we obtain:

Corollary 5.3. *The* Π^0_1*-degrees are dense as a substructure: given* Π^0_1*-degrees* $[\theta] < [\varphi]$*, there is a* Π^0_1 *degree* $[\psi]$ *such that* $[\theta] < [\psi] < [\varphi]$*. Similarly, the* Δ^0_2*-degrees are dense as a substructure.* □

On the other hand, if we replaced 1 with $\varphi(s)$ in the definition of ψ in Theorem 5.1, then it would still be the case that $[0] < [\psi] < [\varphi]$, but now φ would be non-escaping relative to ψ. This is because for this modified construction, $\mathrm{tot}(\varphi)$ follows from $\mathrm{tot}(\psi)$ and the true Π^0_1-statement "ψ is never annihilated."

Theorem 5.4. *Given any nonzero* $[\varphi]$*, there is a pair of nonzero degrees* $[\psi_0]$ *and* $[\psi_1]$ *below* $[\varphi]$ *such that* $[\psi_0] \vee [\psi_1] = [\varphi]$ *and* $[\psi_0] \wedge [\psi_1] = [0]$*, i.e.,* $[\varphi]$ *is a join of a minimal pair.*

Proof. Construction:

At stage s we have defined $\psi_0(x)$ and $\psi_1(y)$ for $x < s$ and $y < y_s$, respectively. There are two basic steps taken at stage s:

1. If s: "$\mathrm{tot}(\psi_0) \to \mathrm{tot}(\psi_1)$," then annihilate ψ_0 and perform action ☠$_\varphi$ on ψ_1. If s: "$\mathrm{tot}(\psi_1) \to \mathrm{tot}(\psi_0)$," then perform action ☠$_\varphi$ on ψ_0 and annihilate ψ_1.
2. Wait for $\varphi(s)$ to converge. While this waiting is taking place, define $\psi_1(y_s) = 0$, and $\psi_1(r) = 0$ for each $r > y_s$ such that $\varphi(s)$ does not converge in r steps. If such t is found, define $\psi_0(s) = \varphi(s)$, and $y_{s+1} = t$.[2]

Verification:

If we ever perform a ☠ action, it is because we see $[\psi_i] \leq [\psi_{1-i}]$, in which case we make $[\psi_i] = [\varphi]$ and $[\psi_{1-i}] = [0]$, contradicting $[\varphi]$ nonzero. So no ☠ action is ever performed, which means that $[\psi_0]$ and $[\psi_1]$ are incomparable.

[2]We could make the construction symmetric by alternating the roles of ψ_0 and ψ_1 in the construction at alternate stages, but this is not required.

By inspecting the algorithm used to define ψ_0 and ψ_1, it is clear that so long as φ is total, ψ_0 and ψ_1 are both total. In fact, this can be proven in T, so $[\psi_i] \leq [\varphi]$ for $i < 2$.

It is clear that if both algorithms are total, then we cannot permanently stay in the step (2) at any stage of the construction. Then by an easy case analysis, φ must also be total, which means $[\psi_0] \vee [\psi_1] = [\varphi]$.

Similarly, one of the two functions is always copying the 0 function, so T can prove that on every input at least one of the two algorithms converges. So $[\psi_0] \wedge [\psi_1] = [0]$. □

By Theorem 4.3, the function ψ_1 constructed in this proof is non-escaping, and φ is non-escaping relative to ψ_0. This is because the true Π^0_1-sentence that implies that ψ_1 is total says that "no 💀 action is ever taken"; and the same sentence implies that ψ_0 is total if and only if φ is total.

6. The high/low hierarchy

Definition 6.1. We denote by $\varphi^{(n)}$ (or $[\varphi]^{(n)}$) the nth iterate of the jump operator applied to φ (or $[\varphi]$, respectively). A degree $[\varphi]$ is *low*$_n$ if $[\varphi]^{(n)} = [0]^{(n)}$. A degree $[\varphi]$ is *high*$_n$ if $[\varphi]^{(n)} \geq [0]^{(n+1)}$. It is *intermediate* if it is between $[0]$ and $[0^*]$ but neither high$_n$ nor low$_n$ for every n. We write low and high rather than low$_1$ and high$_1$. (We can make sense of low$_0$ and high$_0$ as meaning provably-total and $\geq_p 0^*$, respectively.)

Theorem 6.2. *The high/low hierarchy is strict, and there are intermediate degrees. (All results follow by direct construction, and all degrees constructed are below* $[0]^*$*.)*

Proof. Fix $n \geq 0$. We first define an algorithm $\psi \in \text{high}_{n+1} \smallsetminus \text{high}_n$:

$$\psi(x) = \begin{cases} 0 & (\exists s < x)\ s : \text{“}\operatorname{tot}(\psi^{(n)}) \to \operatorname{tot}(0^{(n+1)})\text{”} \\ 0^*(x) & \text{otherwise.} \end{cases}$$

Clearly $[\psi] \leq [0]^*$. If ψ were high$_n$, there would be a proof s: "$\operatorname{tot}(\psi^{(n)}) \to \operatorname{tot}(0^{(n+1)})$," that implies that ψ is provably total, and therefore not high$_n$.

By Lemma 3.5, $T + \operatorname{tot}(\operatorname{con}(0^{(n)}))$ is sufficient to show that $[0]^{(n+1)} \not\leq [0]^{(n)}$. Therefore, $T + \operatorname{tot}(\operatorname{con}(0^{(n)}))$ suffices to prove that annihilation never occurs. This implies that $[0]^* \leq [\psi] \vee [\operatorname{con}(0^{(n)})]$, and hence $[0]^{(n+2)} \leq ([\psi] \vee [\operatorname{con}(0^{(n)})])^{(n+1)}$. The degree $[\operatorname{con}(0^{(n)})]$ is Π^0_1, so by Corollary 4.10, we have

$$[0]^{(n+2)} \leq ([\psi] \vee [\operatorname{con}(0^{(n)})])^{(n+1)} = [\psi]^{(n+1)} \vee [\operatorname{con}(0^{(n)})] = [\psi]^{(n+1)}.$$

(The last equality holds because $[\psi]^{(n+1)} \geq [0]^{n+1} > [\operatorname{con}(0^{(n)})]$.) This shows that $[\psi]$ is high$_{n+1}$.

The construction for low$_{n+1} \smallsetminus$ low$_n$ is a dual construction switching 0^* and 0:

$$\psi(x) = \begin{cases} 0^*(x) & (\exists s < x)\ s : \text{“}\operatorname{tot}(0^{(n)}) \to \operatorname{tot}(\psi^{(n)})\text{”} \\ 0 & \text{otherwise.} \end{cases}$$

Again, it is clear that $[\psi] \leq [0^*]$ and $[\psi]$ is not low$_n$. To see that $[\psi]$ is low$_{n+1}$, again we use the fact that $T + \operatorname{tot}(\operatorname{con}(0^{(n)}))$ proves $[0]^{(n+1)} \not\leq [0]^{(n)}$. This implies that $[\psi] \le [\mathrm{con}(0^{(n)})]$, because $[\mathrm{con}(0^{(n)})]$ proves that the first case in the definition of ψ can never occur. Thus we have $[\psi] \le [0^*] \wedge [\mathrm{con}(0^{(n)})]$, and hence

$$\begin{aligned}[\psi]^{(n+1)} &\le ([0^*] \wedge [\mathrm{con}(0^{(n)})])^{(n+1)} \le [\mathrm{con}(0^{(n)})]^{(n+1)} \\ &= [0]^{(n+1)} \vee [\mathrm{con}(0^{(n)})] = [0]^{(n+1)}.\end{aligned}$$

We can think of the properly high_{n+1} degree as simply copying 0^* while threatening annihilation if a witness is ever found that it is high_n. Similarly, the properly low_{n+1} degree copies 0, while threatening $\skull_{0^*}$ if a witness is ever found that it is low_n. To construct an intermediate ψ, we combine both threats. Fix $[\varphi] \le [0^*]$. At stage s, assuming that we have not already performed a $\skull$ action, we check whether s is a proof of $\mathrm{tot}(\psi^{(n)}) \to \mathrm{tot}(0^{(n+1)})$ or $\mathrm{tot}(0^{(n)}) \to \mathrm{tot}(\psi^{(n)})$ for some n. If it is, we perform action $\skull_0$ (in the first case) or $\skull_{0^*}$ (in the second) on ψ. Otherwise, we define $\psi(s) = \varphi(s)$. By the same arguments as above, $[\psi] \le [0^*]$, and $[\psi]$ is not low_n or high_n for any n. $\square$

The nature of the intermediate degree we construct depends on which function φ we copy (while threatening $\skull$ actions). If we copy 0, then the constructed intermediate degree is non-escaping (since its totality is provable from the true Π^0_1-sentence saying that no $\skull$ action occurs). If we copy 0^*, then $[0^*]$ is non-escaping relative to the constructed intermediate degree (since $\mathrm{tot}(0^*)$ is provable from $\mathrm{tot}(\psi)$ plus the true Π^0_1-sentence saying that annihilation never happens). These two facts will be useful later.

Corollary 6.3. *There is an intermediate degree that is non-escaping, and there is an intermediate degree $[\psi]$ such that $[0]^*$ is non-escaping relative to $[\psi]$.* $\square$

Two sequences of degrees. In addition to the properly high_{n+1}, properly low_{n+1}, and intermediate degrees constructed in the proof of Theorem 6.2, one important sequence of degrees made an appearance: degrees of the form $[0]^* \wedge [\mathrm{con}(0^{(n)})]$. These degrees, and their complements, have some interesting properties.

Definition 6.4. Let $\pi_n = [0]^* \wedge [\mathrm{con}(0^{(n)})]$. We additionally define its complement, which we call $\rotatebox{180}{\pi}_n$. We would like to define $\rotatebox{180}{\pi}_n$ as $[0]^* \wedge [\neg\,\mathrm{con}(0^{(n)})]$. Since $\mathrm{con}(0^{(n)})$ is a Π^0_1-sentence, its negation is provably equivalent to totality of some partial computable function, which we will call $\neg\,\mathrm{con}(0^{(n)})$. This function is not total, and hence does not belong to the structure we are studying. So we actually define $\rotatebox{180}{\pi}_n = [0^* \boxtimes \neg\,\mathrm{con}(0^{(n)})]$, which is almost the same thing.

Observations.

- The sequence $\langle \pi_n \rangle$ is increasing.
- The sequence $\langle \rotatebox{180}{\pi}_n \rangle$ is decreasing.
- π_n and $\rotatebox{180}{\pi}_n$ are complements to each other below $[0]^*$, i.e., $\pi_n \vee \rotatebox{180}{\pi}_n = [0]^*$ and $\pi_n \wedge \rotatebox{180}{\pi}_n = [0]$.

Theorem 6.5. *Every π_n is low_{n+1} but not low_n, and every $\rotatebox{180}{\pi}_n$ is high_{n+1} but not high_n.*

Proof. When π_n first made an appearance, in the proof of Theorem 6.2, we proved that it was low_{n+1} and above the properly low_{n+1} degree we constructed, so π_n is properly low_{n+1}.

For $\rotatebox[origin=c]{180}{π}_n$, notice that $[\mathrm{con}(0^{(n)})] \vee \rotatebox[origin=c]{180}{π}_n \geq [0]^*$. By taking the $n+1$-st jumps of both sides and using Corollary 4.10, we see that $[\mathrm{con}(0^{(n)})] \vee \rotatebox[origin=c]{180}{π}_n^{(n+1)} \geq [0]^{(n+2)}$. Since $\rotatebox[origin=c]{180}{π}_n^{(n+1)} \geq [0]^{(n+1)} > \mathrm{con}(0^{(n)})$, we have $\rotatebox[origin=c]{180}{π}_n^{(n+1)} \geq [0]^{(n+2)}$, making it high_{n+1}.

To see that $\rotatebox[origin=c]{180}{π}_n$ is not high_n, we again refer back to Theorem 6.2. Let ψ be the properly high_{n+1} degree constructed in the proof of that theorem. We argued that $[0]^* \leq [\psi] \vee [\mathrm{con}(0^{(n)})]$, which means that the totality of ψ and $\mathrm{con}(0^{(n)})$ together imply $\mathrm{tot}(0^*)$. Equivalently, the totality of ψ implies either 0^* is total, or $T + \mathrm{tot}(0^{(n)})$ is inconsistent. So $\rotatebox[origin=c]{180}{π}_n \leq [\psi]$, which we already know is not high_n. □

Theorem 6.6. *π_1 bounds every Π^0_1-degree below $[0]^*$.*

Proof. Let P be a Π^0_1-sentence provable from $\mathrm{tot}(0^*)$. We argue within T, by contrapositive:

> All true Σ^0_1-sentences are provable (verifying the witness constitutes a proof), so if $\neg P$, then T proves $\neg P$. Furthermore, $T+\mathrm{tot}(0^*)$ proves P, so $\neg P \to \neg\,\mathrm{con}(0^*)$. Thus $T + \mathrm{con}(0^*) \vdash P$ and $T + \mathrm{tot}(0^*) \vdash P$, thus $T + \mathrm{tot}(\pi_1) \vdash P$. □

The previous result is the reason for the name π_1. However, it should be noted that π_1 is not itself Π^0_1, or even Δ^0_2.

Corollary 6.7. *There is a non-escaping degree that is not Δ^0_2.*

Proof. We claim that π_1 has the desired properties. It is below the Π^0_1-degree $[\mathrm{con}(0^*)]$, so it is non-escaping. Assume that π_1 is Δ^0_2. By Theorem 4.9, it is low, so $\pi_1^{\mathrm{con}} \leq \pi_1^* \leq [0]^*$. But π_1^{con} is Π^0_1, so $\pi_1^{\mathrm{con}} \leq \pi_1$ by Theorem 6.6, contradicting Gödel's Second Incompleteness Theorem. □

Proposition 6.8. $\pi_1^\circ \wedge [0]^* = \pi_1$.

Proof. This follows from direct calculation using the distributive property of our lattice:

$$\begin{aligned}
\pi_1^\circ \wedge [0]^* &= (\pi_1 \vee [\mathrm{con}(\pi_1)]) \wedge [0]^* \\
&= (([0]^* \wedge [\mathrm{con}(0^*)]) \vee [\mathrm{con}(\pi_1)]) \wedge [0]^* \\
&= ([0]^* \vee [\mathrm{con}(\pi_1)]) \wedge [\mathrm{con}(0^*)] \wedge [0]^* \\
&= [0]^* \wedge [\mathrm{con}(0^*)] \\
&= \pi_1.
\end{aligned}$$ □

One consequence is that, even though π_1 is low_2, a single hop is sufficient to take it outside the cone below $[0]^*$. Interestingly, $\rotatebox[origin=c]{180}{π}_1$ hops inside $[0]^*$.

Proposition 6.9. *$\rotatebox[origin=c]{180}{π}_1^{\mathrm{con}} = 0^\circ$ and so $\rotatebox[origin=c]{180}{π}_1^\circ \leq [0]^*$.*

Proof. Since $T + (A \text{ or } B)$ is consistent if and only if $T + A$ or $T + B$ is consistent, $\mathrm{con}(\rotatebox[origin=c]{180}{π}_1)$ is equivalent to $\mathrm{con}(0^*)$ or $\mathrm{Con}_T(\neg\,\mathrm{con}(0^*))$. We claim $\mathrm{Con}_T(\neg\,\mathrm{con}(0^*))$ is equivalent to $\mathrm{con}(0)$; the result follows. Obviously $\mathrm{Con}_T(\neg\,\mathrm{con}(0^*))$ implies $\mathrm{con}(0)$. For the other direction, Gödel's Second Incompleteness Theorem tells us that $\mathrm{con}(0)$ implies $\mathrm{Con}_T(\neg\,\mathrm{con}(0))$, and the latter implies $\mathrm{Con}_T(\neg\,\mathrm{con}(0^*))$. □

7. Cappability

Definition 7.1. We call $[\varphi], [\psi]$ a *minimal pair* if $[\varphi], [\psi]$ are nonzero, and $[\varphi]\wedge[\psi] = [0]$. If $[\varphi]$ forms half of a minimal pair, we say that $[\varphi]$ is *cappable.*

We showed in Theorem 5.4 that every degree is the join of a minimal pair, constructing many examples of cappable degrees. The Δ^0_2 degrees are another source of examples.

Proposition 7.2. *Every Δ^0_2-degree is cappable.*

Proof. If $[\varphi]$ is Δ^0_2, then $\neg\operatorname{tot}(\varphi)$ is provably equivalent to a Π^0_2-sentence. Therefore, there is some algorithm ψ such that $\operatorname{tot}(\psi)$ is provably equivalent to $\neg\operatorname{tot}(\varphi)$. Fix an arbitrary total algorithm θ. We have $[\varphi]\wedge[\theta\boxtimes\psi] = [0]$, and $[\varphi]\vee[\theta\boxtimes\psi] = [\varphi]\vee[\theta]$. If $[\theta] > [\varphi]$, this gives a minimal pair (and, in fact, a complement in the cone below $[\theta]$). □

The proof shows that every Δ^0_2-degree is complementable to every degree above it. This has a partial converse.

Corollary 7.3. *A degree $[\varphi]$ is Δ^0_2 if and only if it is complementable to some Π^0_1-degree $[\theta]$ above it (in other words, there is a $[\psi]$ such that $[\varphi]\vee[\psi] = [\theta]$ and $[\varphi]\wedge[\psi] = [0]$), and if and only if it is complementable to every Π^0_1-degree above it.*

Proof. Assume that $[\theta] > [\varphi]$ is Π^0_1 and $[\varphi]$ is complementable to $[\theta]$ by $[\psi]$. Then $\operatorname{tot}(\varphi)$ is equivalent to $\operatorname{tot}(\psi)\to\operatorname{tot}(\theta)$, which is Σ^0_2, so $[\varphi]$ is Δ^0_2. The other directions follow from Propositions 4.8 and 7.2. □

As a result, Δ^0_2-degrees are definable from Π^0_1-degrees.

We saw in the previous section that π_1 is not Δ^0_2, so it cannot be complementable to $\operatorname{con}(0^*)$. In particular, the degrees below $\operatorname{con}(0^*)$ do not form a boolean algebra.

Theorem 7.4. *$[0]^*$, and consequently every degree above $[0]^*$, is not cappable.*

Proof. Suppose that $[0]^*\wedge[\varphi] = [0]$. Then $0^*\boxtimes\varphi+1$ is provably total, and by padding, there are infinitely many proofs s of $\operatorname{tot}(0^*\boxtimes\varphi+1)$. For every such s, $0^*(s)$ is defined to be $(0^*\boxtimes\varphi+1)(s)$. The algorithm that 0^* follows on such input s is (first decode the proof, then) wait for a stage when either $0^*(s)$ or $\varphi(s)$ converges, and then add one to that value. It cannot be the case that $0^*(s)$ converges first, otherwise $0^*(s) = 0^*(s)+1$, so $\varphi(s)$ must converge. Therefore, we have infinitely many arguments where 0^* is copying $\varphi+1$. We can carry out the above argument in $T+\operatorname{tot}(0^*)$, which shows that $[\varphi]\le[0^*]$. It follows that $[\varphi] = [0]$. □

Corollary 7.5. *Every nonzero degree bounds a nonzero degree below $[0]^*$.* □

Theorem 7.6. *There is a $[\psi] < [0]^*$ that is not cappable.*

Proof. We define a function ψ so that $\psi(\langle p,x\rangle) = \varphi_e(x)$ if p : "$\operatorname{tot}(\psi\boxtimes\varphi_e)$" and there is no $q\le p$ with q : "$\operatorname{tot}(\psi)\to\operatorname{tot}(0^*)$", and $\psi(\langle p,x\rangle) = 0$ otherwise. In other words, if we witness a proof that $\psi\boxtimes\varphi_e$ is total, we code φ_e into a column of ψ, but if we ever witness a proof that $[0^*]\le[\psi]$, we stop any future coding of new functions (while continuing to code any functions we already started coding).

To see that $[\psi]\le[0]^*$, we argue inside $T+\operatorname{tot}(0^*)$ (which, recall, proves Π^0_2-soundness for T):

> Suppose that ψ is not total. For all e, by Π^0_2-soundness, if T proves $\operatorname{tot}(\psi \boxtimes \varphi_e)$, then $\psi \boxtimes \varphi_e$ is total, which means φ_e must be total (since we are assuming ψ is not). So for every p and x, either $\psi(\langle p, x\rangle) = 0$, or $\psi(\langle p, x\rangle) = \varphi_e(x)$ for some total function φ_e. Therefore, ψ is total.

To see that $[\psi] < [0]^*$, suppose to the contrary there is a q such that q : "$\operatorname{tot}(\psi) \to \operatorname{tot}(0^*)$". Then ψ is equivalent to a finite join of the functions φ_e that the construction began coding before finding q. The totality of each such φ_e is provable from the totality of ψ, which means that each such φ_e is provably total (because we have a proof of $\operatorname{tot}(\psi)$ or $\operatorname{tot}(\varphi_e)$). So ψ is provably total, but then it cannot prove the totality of 0^*.

By the same argument, for each e such that T proves $\operatorname{tot}(\psi \boxtimes \varphi_e)$, φ_e is always below ψ and so provably total. This shows that ψ is not cappable. □

8. Inverting jumps

In this structure, we have defined two natural jump-like operators (strictly increasing and degree invariant). We have called these the jump and the hop. In this section, we prove the existence of both jump-inverses and hop-inverses. We also prove the existence of skip-inverses. We even prove a form of pseudo-jump inversion. For the hop, the inverse has a natural self-referential definition.

Theorem 8.1. *If $[\varphi] \geq [0]^\circ$, then there is a degree $[\theta]$ such that $[\theta]^\circ = [\varphi]$.*

Proof. Suppose $[\varphi] \geq [0]^\circ$. Let $\Phi(x)$ be the number of steps required for $\varphi(x)$ to converge (where if $\varphi(x)$ diverges, then we set $\Phi(x) = +\infty$). Using the Recursion Theorem, we can define θ as follows:

$$\theta(x) = \begin{cases} 0 & (\exists p \leq \Phi(x))\ p : \text{“}\operatorname{tot}(\theta) \to 0 = 1\text{”} \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Note that T proves $\operatorname{tot}(\theta) \iff (\operatorname{tot}(\varphi)$ or "$T + \operatorname{tot}(\theta)$ is inconsistent").

By direct calculation, $[\theta]^\circ = [\theta] \vee [\operatorname{con}(\theta)] = ([\varphi] \wedge [\neg \operatorname{con}(\theta)]) \vee [\operatorname{con}(\theta)] = [\varphi] \vee [\operatorname{con}(\theta)]$. So it suffices to show that $[\varphi]$ is above $[\operatorname{con}(\theta)]$. We argue in $T + \operatorname{tot}(\varphi)$ as follows (using the fact that $T + \operatorname{tot}(\varphi) \vdash T$ is consistent):

> Suppose that $T + \operatorname{tot}(\theta)$ is inconsistent. Then there is some p : "$\operatorname{tot}(\theta) \to 0 = 1$", so there is a stage s such that for all $x < p$, $\theta(x)$ has converged by stage s, and for all $x \geq p$, $\theta(x)$ is just the straightforward 0 algorithm. So T proves $\operatorname{tot}(\theta)$, which means that T itself is inconsistent, a contradiction. □

For the skip, notice that a skip is automatically Π^0_1 and above $[con(0)]$, and so we can only possibly find skip-inverses for these degrees:

Theorem 8.2. *If $\varphi \geq_p \operatorname{con}(0)$ is Π^0_1, there is a Π^0_1-degree $[\theta]$ such that $[\operatorname{con}(\theta)] = [\varphi]$.*

Proof. Assume that T proves $\varphi \leftrightarrow (\forall n)\ P(n)$, where P has only bounded quantification. Define θ as follows. On input n, we diverge if θ has diverged on any previous input (as usual). If not, we check if there is a $p \leq n$ such that p : "$\operatorname{tot}(\theta) \to 0 = 1$". If so, we perform action $\skull_0$. (In other words, we declare that $\theta(s){\downarrow}= 0$ for all $s \geq n$.) If no such p has yet been found, we let $\theta(n){\downarrow}$ iff $P(n)$.

Note that we can computably determine if $\theta(n)\downarrow$, so $\operatorname{tot}(\theta)$ is Π^0_1. We claim that

$$\operatorname{con}(\theta) \equiv_p (\theta + \operatorname{con}(0)) \equiv_p \varphi.$$

First, arguing in $T + \operatorname{Con}(T)$:

> If action ☠$_0$ occurs, then T proves $\operatorname{tot}(\theta)$. But the action's occurrence implies that $T + \operatorname{tot}(\theta)$ is inconsistent, hence so is T. This is a contradiction, so action ☠$_0$ never occurs. Therefore, θ is total if and only φ is total.

This shows that $\theta + \operatorname{con}(0) \geq_p \varphi$. We assumed that $\varphi \geq_p \operatorname{con}(0)$, so we also have $\varphi \geq_p \theta$. Therefore, $\theta + \operatorname{con}(0) \equiv_p \varphi$.

Now argue in $T + \operatorname{Con}(T) + \operatorname{tot}(\theta)$:

> Assume $\neg \operatorname{con}(\theta)$. Then there is a least p such that p : "$\operatorname{tot}(\theta) \to 0 = 1$". The only obstacle to performing action ☠$_0$ at stage $n = p$ would be if θ diverged on an earlier input. But we know $\operatorname{tot}(\theta)$, so we must perform action ☠$_0$. We have already seen that this leads to a contradiction, hence $\operatorname{con}(\theta)$.

This shows that $\theta + \operatorname{con}(0) \geq_p \operatorname{con}(\theta)$. Since θ is Π^0_1, we know that $\operatorname{con}(\theta) \geq_p \theta$. Therefore, $\theta + \operatorname{con}(0) \equiv_p \operatorname{con}(\theta)$. Putting everything together, $\operatorname{con}(\theta) \equiv_p \varphi$. □

For the proof of jump inversion, the key idea is simply to wait, whenever we see some $[\varphi_e] \leq [\theta]$, for φ_e to give some evidence of totality, by converging on a new input. In the meantime, we ensure that the only way θ can fail to be total is if φ_e eventually provides this evidence.

Theorem 8.3. *If $[\varphi] \geq [0]^*$, then there is a degree $[\theta]$ such that $[\theta]^* = [\varphi]$.*

Proof. We will construct the jump-inverse θ. The construction of θ is divided into two types of stages, *type I stages* and *type II stages*. Stage 0 is a type I stage.

(1) If t is a type I stage, we define $\theta(t) = \varphi(t)$, and *consider* the least p not already considered. If p : "$\operatorname{tot}(\theta) \to \operatorname{tot}(\varphi_e)$", then $t + 1$ is a type (II, e) stage; otherwise, $t + 1$ is type I.
(2) If t is a type (II, e) stage, let s be the most recent type I stage. We define $\theta(t) = 0$. If $\varphi_e(s)\downarrow$ by stage t, then $t + 1$ is type I. Otherwise, $t + 1$ is type (II, e).

Recall our convention that functions converge on initial segments, so to prove totality of φ, it suffices to show that φ converges on infinitely many inputs. To show that θ^* is above φ, it suffices to show that θ^* can prove the existence of infinitely many type I stages. We argue in $T + \operatorname{tot}(\theta^*)$:

> Suppose that we enter a type (II, e) stage from a type I stage at stage $s + 1$. Then $T + \operatorname{tot}(\theta)$ proves $\operatorname{tot}(\varphi_e)$. By Π^0_2-soundness, φ_e is total, so $\varphi_e(s)$ converges, and we eventually enter another type I stage. Hence there are infinitely many type I stages.

To show that φ is above θ^*, we argue in $T + \operatorname{tot}(\varphi)$ (using $T + \operatorname{tot}(\varphi) \vdash \operatorname{tot} 0^*$):

> Suppose there is some least pair (s, e) such that s : "$\operatorname{tot}(\theta) \to \operatorname{tot}(\varphi_e)$" but $\varphi_e(s)\uparrow$. Then the construction enters a type (II, e) stage at stage $s + 1$, and then never leaves type (II, e) stages. This means θ is total, because for $x > s$, $\theta(x)$ simply outputs another 0 as we search for a stage x where $\varphi_e(s)\downarrow$, which does not exist. By simply monitoring the construction for the first $s + 1$ stages, T can prove $\varphi_e(s)\downarrow$ or $\operatorname{tot}(\theta)$.

Since T also proves $\mathrm{tot}(\theta) \to \mathrm{tot}(\varphi_e)$, we have $T \vdash \varphi_e(s)\downarrow$ (since either it converges or φ_e is total). By Π^0_2-soundness, $\varphi_e(s)$ converges, contradicting our choice of (s, e).

By induction, for all s, if s : "$\mathrm{tot}(\theta) \to \mathrm{tot}(\varphi_e)$" then $\varphi_e(s)\downarrow$. If there is one proof, there are infinitely many, so in fact if $T + \mathrm{tot}(\theta)$ proves $\mathrm{tot}(\varphi_e)$ then φ_e is total. Thus θ^* is total. □

Using the same idea to construct two functions, we can find a pair of low degrees whose join is $[0]^*$.

Theorem 8.4. *There are low degrees $[\psi_0]$ and $[\psi_1]$ such that $[\psi_0] \vee [\psi_1] = [0]^*$ and $[\psi_0] \wedge [\psi_1] = [0]$.*

Proof. We will construct ψ_0 and ψ_1. The construction is divided into three types of stages, *type I stages*, *type II stages for ψ_0*, and *type II stages for ψ_1*. Stage 0 is a type I stage.

(1) If t is a type I stage, we define $\psi_0(s) = \psi_1(s) = 0$ for all $s \le t$ where these are not already defined. We *consider* the least p not already considered. If p : "$\mathrm{tot}(\psi_i) \to \mathrm{tot}(\varphi_e)$" for some $i \le 1$, then $t + 1$ is a type (II, e) stage for ψ_i; otherwise, $t + 1$ is type I.
(2) If t is a type (II, e) stage for ψ_i, let s be the most recent type I stage. We define $\psi_i(t) = 0$, and leave $\psi_{1-i}(t)$ undefined. If $\varphi_e(s)\downarrow$ by stage t, then $t + 1$ is type I. Otherwise, $t + 1$ is type (II, e) for ψ_i.

The jumps of $[\psi_0]$ and $[\psi_1]$ are below $[0]^*$ by the same argument as in the proof of jump inversion. To see that their join is $[0]^*$, notice that both being total means that we never stay in a type II stage forever, and so every 0-provably total function is total, which implies that 0^* is total. To see that their meet is $[0]$, notice that every stage t is type I (in which case $\psi_0(t)$ and $\psi_1(t)$ are both defined) or type II for some ψ_i (in which case $\psi_i(t)$ is defined). □

Relativizing this theorem allows us to combine jump inversion with lower-cone avoidance.

Corollary 8.5. *If $[\varphi] \ge [0]^*$, and $[\theta] \not\ge [\varphi]$, then $[\varphi]$ has a jump-inverse $[\psi] \not\le [\theta]$.*

Proof. By Theorem 8.3, $[\varphi]$ has *some* jump inverse $[\gamma]$, so we can relativize Theorem 8.4 to $[\gamma]$ to get $[\psi_0], [\psi_1]$ that are low over $[\gamma]$ and join to $[\varphi]$. Hence both $[\psi_0]$ and $[\psi_1]$ are jump-inverses of $[\varphi]$, and they cannot both be below $[\theta]$. □

The following theorem gives us a form of pseudo-jump inversion.

Theorem 8.6. *Given a computable function ψ such that $[\varphi_i] \le [\varphi_{\psi(i)}] \le [\varphi_i^*]$ for every index i (in particular, if φ_i is total, then so is $\varphi_{\psi(i)}$), there is always an index e such that $[\varphi_{\psi(e)}] = [0]^*$.*

Proof. We will construct φ_e. Again, there are two types of stages. At a type I stage s, we code $\varphi_e(s) = 0^*(s)$, and then switch to a type II stage. At a type II stage x (with s as the most recent type I stage), we define $\varphi_e(x) = 0$ and check whether $\varphi_{\psi(e)}(s)$ converges at stage x. If not, the next stage is also type II. If it does converge, the next stage is type I.

For all x, $\varphi_e(x)$ copies either $0^*(x)$ or the zero function, so $[\varphi_e] \le [0]^*$. Moreover, if $\varphi_{\psi(e)}$ and φ_e are both total, then the construction is never stuck in a type II stage, so there are infinitely many type I stages, and 0^* is also total. Since $[\varphi_e] \leq [\varphi_{\psi(e)}]$, this shows that $[0^*] \leq [\varphi_{\psi(e)}]$.

It remains to show $[\varphi_{\psi(e)}] \leq [0]^*$. Fix a proof p : "$\mathrm{tot}(\varphi_e^*) \to \mathrm{tot}(\varphi_{\psi(e)})$". We argue in $T + \mathrm{tot}(0^*)$:

> The construction is never stuck in a type I stage, and so it suffices to show that every sequence of consecutive type II stages is terminated by a type I stage. Let $X_{s,k}$ be the formalized Π^0_1-sentence saying "s is a type I stage, the code of the computation process through the first s stages is k, and $\varphi_{\psi(e)}(s)$ diverges," and let $\chi_{s,k}$ be the corresponding computable function (using the correspondence between Π^0_2-sentences and functions that maps between true sentences and total functions). This sentence clearly implies that φ_e is total, because it implies that the construction of φ_e becomes stuck in type II stages after stage s, and so $\varphi_e(x) = 0$ for all $x > s$.
>
> Suppose that for some pair s, k, the sentence $X_{s,k}$ is true ($\chi_{s,k}$ is total). Then $\chi^*_{s,k}$ is also total (cf. the note following Theorem 4.9), and hence φ_e^* is total.
>
> Otherwise, $X_{s,k}$ is false for all s, k. This implies that the construction is never stuck in type II stages, and hence $\varphi_{\psi(e)}$ is total.

The above is a proof that totality of 0^* implies that either φ_e^* is total, or $\varphi_{\psi(e)}$ is total. Adjoining the proof p, we obtain a proof that $\mathrm{tot}(0^*)$ implies $\mathrm{tot}(\varphi_{\psi_e})$. □

One might imagine that for this theorem to be true we would need uniformity of proofs witnessing $\varphi_e \leq_p \varphi_{\psi(e)} \leq_p \varphi_e^*$, or else provable totality of ψ, but neither is required. On the other hand, like most of the proofs in this paper (with the notable exception of Corollary 8.5), the proof of Theorem 8.6 is uniform: e can be found uniformly from an index for ψ and an index for the enumeration of T.

We end the section with an application of Theorem 8.6 completely analogous to Jockusch and Shore's first application of pseudo-jump inversion [3].

Example. Let ψ_0 be a computable function such that $[\varphi_{\psi_0(i)}] = [\varphi_i]^\circ$. So $[\varphi_{\psi_0(i)}]$ is always low over $[\varphi_i]$. By Theorem 8.6, there is an e such that $[\varphi_{\psi_0(e)}] = [0]^*$, meaning that $[\varphi_e]$ is properly high. Using the uniformity of the proof of Theorem 8.6 with respect to the base theory, we get a computable ψ_1 such that $[\varphi_{\psi_1(i)}]$ is always properly high over $[\varphi_i]$. Applying Theorem 8.6 again, there is an e such that $[\varphi_{\psi_1(e)}] = [0]^*$. We claim that $[\varphi_e]$ is properly low$_2$. This follows from the fact that $[0]^*$ is properly high over $[\varphi_e]$, so $[0]^* < [\varphi_e]^*$ (i.e., $[\varphi_e]$ is not low) and $([0]^*)^* \geq [\varphi_e]^{**}$ (i.e., $[\varphi_e]$ is low$_2$).

Continuing in this way, we could show that the high/low hierarchy is strict below $[0]^*$, reproving all of Theorem 6.2 except the existence of intermediate degrees.

9. Jump classes and domination properties

A natural question arising in our study of the provability degrees is the relationship between escape and domination notions, on the one hand, and the jump classes on the other. In part, this is inspired by Martin's high–domination theorem from computability. In part, it is inspired by the fact that every Π^0_1-degree is $\mathbf{GL}_1$ (Theorem 4.9), which implies that non-escaping degrees must also be weak in the sense of the jump hierarchy because of the characterization of non-escaping degrees (Theorem 4.3). However, we will see that in the setting of the provability degrees, the relationship between these two notions is actually quite weak. In fact, the only interactions are the ones that follow from Π^0_1-degrees being $\mathbf{GL}_1$.

The escape and domination notions we study are "escaping", defined in Section 4, along with "dominant" and "full", defined here.

Definition 9.1. A degree is *dominant* if it contains a "dominant" function, that is, a function that dominates every provably total function. A degree $[\psi]$ is *full* if for every $\varphi \in [0]^*$, there is a $\rho \in [\psi]$ such that $\rho \sim \varphi$. (In words, full degrees contain every function that $[0]^*$ can prove to be total, but possibly computed by a different algorithm.)

We would like to apply Corollary 4.5 to conclude that a degree $[\psi]$ is full if and only if $[0]^*$ is non-escaping relative to $[\psi]$, but if we are not assuming that $[\psi] < [0]^*$, this does not follow by an immediate relativization. However, it is true.

Theorem 9.2. *The following are equivalent for a degree* $[\psi]$:

1. $[\psi]$ *is full,*
2. *There is a* Π^0_1*-degree* $[\theta]$ *such that* $[0]^* \leq [\psi] \vee [\theta]$,
3. *Every* $\varphi \in [0]^*$ *is dominated by a* $\rho \in [\psi]$, *and*
4. *There is a* $\xi \in [\psi]$ *so* $\xi \sim 0^*$.

Proof. We start by showing that (1), (2) and (3) are equivalent. Clearly, (1) implies (3). Now assume (3) and let $\rho \in [\psi]$ dominate the computing time of 0^*. Then $\mathrm{tot}(0^*)$ follows from $\mathrm{tot}(\rho)$ (which in turn, follows from $\mathrm{tot}(\psi)$) and the true Π^0_1-statement $(\forall x)[0^*(x)\downarrow$ before stage $\rho(x)]$. This proves (2). Next assume (2). Relativizing Theorem 4.3 and Corollary 4.5, we see that for every $\varphi \in [0]^*$, there is a $\rho \leq_p \psi$ such that $\rho \sim \varphi$. Take $\hat{\rho} \in [\psi]$ such that $\hat{\rho} \sim \rho$ (see Proposition 7.1 of [1]) and so $\hat{\rho} \sim \varphi$, proving (1).

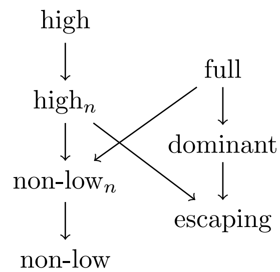
All that remains is to show that (4) is equivalent to the others. Clearly, (1) implies (4). Now assume (4). There is a primitive recursive (hence provably total) function γ such that, for every algorithm φ_e, the function $\varphi_{\gamma(e)}$ is defined by

$$\varphi_{\gamma(e)}(x) = (\mu s)(\forall y < x)\, \varphi_{e,s}(y)\downarrow .$$

Furthermore, there is another primitive recursive function τ such that if p is a proof of $\mathrm{tot}(\varphi_e)$, then $\tau(p)$ is a proof of $\mathrm{tot}(\varphi_{\gamma(e)})$, and $\tau(p) > p$.

Let Q be the sentence, "for all p, e and s, if p is a proof of $\mathrm{tot}(\varphi_e)$ and $\xi(\tau(p))$ converges to s, then $\varphi_{e,s}(p)$ converges." This sentence is clearly Π^0_1, and is true because if p is a proof of $\mathrm{tot}(\varphi_e)$, then φ_e and $\varphi_{\gamma(e)}$ are total, $\tau(p)$ is a proof of $\mathrm{tot}(\varphi_{\gamma(e)})$, and so $\xi(\tau(p)) = 0^*(\tau(p)) = \varphi_{\gamma(e)}(\tau(p)) \geq t$, where t is the number of stages for $\varphi_e(p)$ to converge. Furthermore, Q and $\mathrm{tot}(\xi)$ together imply the totality of 0^*, since 0^* is equivalent to Π^0_2-soundness of T. This implies (2). □

The following diagram summarizes the relationship between the jump classes and our escape/domination notions:



The up-down direction implications are immediate consequences of the definitions. The diagonal arrows are both consequences of Π^0_1-degrees being $\mathbf{GL}_1$.

Theorem 9.3. *Every full degree is non-low$_n$ for every n. Every high$_n$ degree is escaping.*

Proof. Let $[\psi]$ be full. By Theorem 9.2, there is a Π^0_1-degree $[\theta]$ such that $[0]^* \leq [\psi] \vee [\theta]$. Taking the n-th jump on both sides and using Corollary 4.10, we obtain $[0]^{(n+1)} \leq [\psi]^{(n)} \vee [\theta]$. If $[\psi] \in \text{low}_n$, that would imply $[0]^{(n+1)} \leq [0]^{(n)} \vee [\theta]$, and so $[0]^{(n+1)}$ would be non-escaping relative to $[0]^{(n)}$, giving us a contradiction.

The second claim is a dual version. Given a non-escaping ψ, we can find a Π^0_1-degree θ that bounds it. Then we know that $[\theta]^{(n)} \leq [0]^{(n)} \vee [\theta]$, so $[\theta]^{(n)}$ cannot be above $[0]^{(n+1)}$ for the same reason as above. □

In addition, by Corollary 6.3, we know that being full does not imply high$_n$, and being non-escaping does not imply low$_n$. It remains to show that high degrees may not be dominant, and that dominant degrees may be low. These two results are the most complicated ones in the paper. (This will then imply that the right column implications are strict, because the property of being dominant is incomparable with anything from the left column.)

Theorem 9.4. *There is a low degree that is dominant.*

Proof. We will construct an algorithm θ such that $[\theta]$ is low and dominant. The construction is a modification of the jump inversion construction used in the proof of Theorem 8.3, with top degree $[0]^*$. We modify the type II stages to additionally ensure that the jump-inverse θ being constructed is dominant, and we do so by outputting, instead of 0, the maximum of the first k provably total functions, where k gradually increases over time.

The construction of θ is divided into two types of stages, *type I stages* and *type II stages*. Stage 0 is a type I stage. Initially, the *function number* is 0.

(1) If t is a type I stage, we define $\theta(t) = 0^*(t)$, and *consider* the least p not already considered. If p : "$\text{tot}(\theta) \to \text{tot}(\varphi_e)$", then $t+1$ is a type (II, e) stage; otherwise, we increment the function number, and stage $t+1$ is type I.
(2) If t is a type (II, e) stage, let s be the most recent type I stage. We define $\theta(t)$ to be the max of the first k provably total functions, where k is the current function number. If $\varphi_e(s)\downarrow$ by stage t, then $t+1$ is type I. Otherwise, $t+1$ is type (II, e).

The argument that $[\theta]$ is low is identical to the argument that φ is above θ^* in the proof of jump inversion. The only difference is instead of copying the 0 function, we copy the max of the first k provably total functions. But since $T + \text{tot}(0^*)$ is equivalent to T together with Π^0_2-soundness of T, in the internal argument we know that these functions are actually total, so that is no obstacle.

To show that θ is dominant, consider the function $\hat{\theta}$ defined by $\hat{\theta}(x) = \theta(s)$, where s is the x^{th} type II stage in the construction of θ. The totality of θ implies that there are infinitely many type II stages in the construction of θ (since there are infinitely many proofs of totality of some function), and hence that $\hat{\theta}$ is also total. But $\hat{\theta}(x)$ is defined as the max of the first k provably total functions, where k depends on x and $k \to \infty$ as $x \to \infty$, so $\hat{\theta}$ is dominant. □

Theorem 9.5. *There is a high degree that is not dominant.*

Proof. Using the Recursion Theorem, we will simultaneously construct an algorithm θ such that $[\theta]$ is high, and a computable function γ such that if $[\varphi_e] \leq [\theta]$, then $\varphi_{\gamma(e)}$ is provably total and escapes φ_e (thus ensuring that θ is non-dominant). There are three types of stages, *type I stages*, *type II stages*, and *transition stages*, with stage 0 being a transition stage. At type I stages we ensure $[\theta]$ is non-dominant, and at type II stages we follow a jump inversion strategy to make $[\theta]$ high. Transition stages just serve to decide whether the next stage should be type I or type II.

Construction of θ:

(1) If t is a transition stage, we define $\theta(t) = 0$, and we *consider* the least proof p not already considered such that p : "$\text{tot}(\theta) \to \text{tot}(\varphi_e)$" or p : "$\text{tot}(0^*) \to \text{tot}(\varphi_e)$". In the first case, stage $t+1$ is a type (I, e) stage. In the second case, stage $t+1$ is a type (II, e) stage.
(2) If t is a type (I, e) stage, let s be the most recent transition stage. We define $\theta(t) = 0$. If $\varphi_{\gamma(e),t}(x)\downarrow > \varphi_{e,t}(x)\downarrow$ for some $x \in (s, t)$, then stage $t+1$ is a transition stage. Otherwise, stage $t+1$ is a type (I, e) stage.
(3) If t is a type (II, e) stage, let s be the most recent transition stage. We define $\theta(t) = 0^*(t)$. If $\varphi_{e,t}(s)\downarrow$, then $t+1$ is a transition stage. Otherwise, $t+1$ is type a (II, e) stage.

This construction may be carried out by a Turing machine, of course, but the "stages" of the construction will not correspond to stages of the Turing machine executing the construction. We call the stages of the Turing machine execution "beats" to distinguish them from construction "stages". This is important because T can prove that there are infinitely many beats, but the statement that there are infinitely many stages is equivalent (over T) to the totality of θ.

Construction of γ:

The function γ is defined so that, for all e, $\varphi_{\gamma(e)}$ is the function

$$\varphi_{\gamma(e)}(x) = \begin{cases} \varphi_e(x) + 1 & \text{if } \varphi_e(x) \text{ converges before there is a beat} > x \\ & \text{of the construction not in a type (I, } e\text{) stage} \\ 0 & \text{otherwise.} \end{cases}$$

Verification:

We first prove that the construction is never stuck in type I or type II, i.e., that there are infinitely many transition stages. In fact, we will carry out this proof in $T + \text{tot}(\theta^*)$, because we will need the fact later that $T + \text{tot}(\theta^*)$ proves the existence of infinitely many transition stages.

> Suppose that the construction becomes stuck in type (I, e) stages at beat s. Then afterwards $\varphi_{\gamma(e)}(s)$ cannot be defined by the second case of its definition. Therefore $\varphi_{\gamma(e)}(s)$ converges if and only if $\varphi_e(s)$ converges, and $\varphi_{\gamma(e)}(s) = \varphi_e(s) + 1$. Also, there is a proof p : "$\mathrm{tot}(\theta) \to \mathrm{tot}(\varphi_e)$". By Π^0_2-soundness of $T + \mathrm{tot}(\theta)$, the algorithm φ_e is actually total. So $\varphi_e(s)$ does converge, and $\varphi_{\gamma(e)}(s) = \varphi_e(s) + 1$. This implies that some x will eventually be found causing the construction to return to a transition stage, so it was not, in fact, stuck in type (I, e).
>
> Suppose the construction becomes stuck in type (II, e) stages at stage $s+1$. Then there is a proof p : "$\mathrm{tot}(0^*) \to \mathrm{tot}(\varphi_e)$", but $\varphi_e(s)\uparrow$. Furthermore, by monitoring the first s stages of the construction, T can prove that stage $s+1$ of the construction is type (II, e), and therefore that either $\varphi_e(s)\downarrow$ or $\mathrm{tot}(\theta) \Leftrightarrow \mathrm{tot}(0^*)$. So $T + \mathrm{tot}(\theta)$ proves $\varphi_e(s)\downarrow$ or $\mathrm{tot}(0^*)$, and must therefore also prove $\varphi_e(s)\downarrow$ or $\mathrm{tot}(\varphi_e)$ (by adjoining the proof p). Therefore, $T + \mathrm{tot}(\theta)$ proves that $\varphi_e(s)$ converges. By Π^0_2-soundness of this theory, $\varphi_e(s)$ actually converges, at which point there will be a transition stage.

The above shows that $[0]^{**} \leq [\theta]^*$, because each proof p : "$\mathrm{tot}(0^*) \to \mathrm{tot}(\varphi_e)$" puts the construction into a type (II, e) stage, so for the construction to return to a transition stage infinitely often guarantees that $\varphi_e(x)$ converges for infinitely many x, which means that φ_e is total (by our convention that functions converge on initial segments). Clearly $[\theta] \leq [0]^*$, so $[\theta]$ is high.

To show that $[\theta]$ is non-dominant, we consider some $\varphi_e \leq_p \theta$. There are infinitely many type (I, e) stages and infinitely many transition stages, so $\varphi_{\gamma(e)}$ escapes φ_e. We must now argue in T that $\varphi_{\gamma(e)}$ is total, using our fixed proof that if θ is total, so is φ_e:

> If there are infinitely many beats of the construction that are not in type (I, e) stages, then $\varphi_{\gamma(e)}$ is clearly total. Suppose instead that all beats of the construction after beat t are type (I, e). Each of these stages takes finitely many beats, because the construction only has to check for finitely many x whether $\varphi_{\gamma(e),t}(x)\downarrow > \varphi_{e,t}(x)\downarrow$. So there are infinitely many stages of the construction, hence θ is total. This implies that φ_e is total, which in turn implies that $\varphi_{\gamma(e)}$ is total. □

10. Open Questions

The Π^0_1-degrees came up naturally in our study of the provability degrees, specifically in the characterization of non-escaping. The Δ^0_2-degrees also proved useful, though their role is less clear. What else can be said about the Δ^0_2-degrees and their relationship to the Π^0_1-degrees? For example:

Question 10.1. *Is there a Δ^0_2-degree that bounds no nonzero Π^0_1-degree?*

Question 10.2. *Is every Δ^0_2 degree that is below $[0^*]$ also below π_1?*

To help motivate these questions, note that properly Δ^0_2 degrees do exist:

Proposition 10.3. *There is a Δ^0_2-degree that is not Π^0_1.*

Proof. Construct a computable function f as follows: Output 0 until we find a stage s such that s: "$\lim_{t\to\infty} f(t) = 0 \leftrightarrow P$," for some Π^0_1-statement P. Once we find such an s, and until we find that P is false (which if it is, we would eventually see because P is Π^0_1), f outputs 1. If we find that P is false, f outputs 0 forever.

It is clear that $\lim_{t\to\infty} f(t)$ exists (as it changes values at most twice) and is either 0 or 1. Consider the statement "$\lim f = 0$". It is Δ^0_2, since f is known to have a limit, but it cannot be provably equivalent to a Π^0_1 statement (if it were, it diagonalizes). □

Other natural questions concern jump inversion. We proved a number of jump inversion theorems, including the analog of the Friedberg jump inversion theorem for this structure. We ask what analog of Shoenfield jump inversion holds, and whether jump inversion can be combined with upper cone avoidance.

Question 10.4. *Is there a characterization of the degrees that are jumps of degrees below* $[0]^*$*?*

Question 10.5. *Given* $\mathbf{d} > [0]^*$ *and* $[\psi] > 0$*, is there always some* $[\varphi]$ *with* $[\varphi]^* = \mathbf{d}$ *and* $[\varphi] \not\geq [\psi]$*?*

The answer to Question 10.5 is yes in the case when $\mathbf{d} = [0]^*$, by Theorem 8.4.

Question 10.6. *Is there a characterization of the degrees that are cappable?*

Question 10.7. *Which of the following classes and operations are definable in the lattice of p-degrees?*

(1) Π^0_1,
(2) Δ^0_2,
(3) *the jump:* $\mathbf{d} \mapsto \mathbf{d}^*$,
(4) *the hop:* $\mathbf{d} \mapsto \mathbf{d}^\circ$.

Question 10.8. *Is there a high/low hierarchy for hop?*

Finally, everything we proved was independent of the theory T, only assuming that T is effectively axiomatizable, sound and extends PA^- plus Σ^0_1-induction.

Question 10.9. *To what extent, if any, does the structure of the provability degrees depend on the underlying theory* T*? Are these structures isomorphic for different theories?*

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