DEFINING TOTALITY IN THE ENUMERATION DEGREES

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Abstract. We show that if A and B form a nontrivial $\mathcal{K}$-pair, then there is a semi-computable set C such that $A \leq_e C$ and $B \leq_e \overline{C}$. As a consequence, we obtain a definition of the total enumeration degrees: a nonzero enumeration degree is total if and only if it is the join of a nontrivial maximal $\mathcal{K}$-pair. This answers a longstanding question of Hartley Rogers, Jr. We also obtain a definition of the "c.e. in" relation for total degrees in the enumeration degrees.

Two of the most fundamental concepts in computability theory are effective computation and effective enumeration, i.e., computation and enumeration of sets of natural numbers as achieved by computer programs. Each notion induces a natural measure of the complexity of sets of natural numbers. *Turing reducibility* was defined by Turing in the late 1930's to capture the relative complexity of computing sets of natural numbers: $A \leq_T B$ means that a set A is *computable* from another set B. We use the word "reducibility" because the problem of computing A is being reduced to the problem of computing B. In the same way, *Enumeration reducibility* was defined by Friedberg and Rogers in the late 1950's to capture the relative complexity of enumerating sets: $A \leq_e B$ means that there is a way to effectively produce an enumeration of a set A from any enumeration of a set B.

Turing reducibility has been studied extensively, but enumeration reducibility also arises naturally in various contexts. Enumeration reducibility restricted to partial functions is equivalent to Kleene's [12] notion of relative effective computability of partial functions. Scott [20] used enumeration reducibility to give a countable graph model of λ-calculus. C. F. Miller (unpublished manuscript) and M. Ziegler [28] applied enumeration reducibility in group theory, to state and prove an extension of Higman's embedding theorem for finitely generated groups. J. Miller [15] used enumeration reducibility in his work on computable analysis; he answered a question of Pour El and Lempp by showing that Turing reducibility is not sufficient to measure the complexity of continuous functions on $\mathbb{R}$ (but enumeration reducibility is). Similarly, Richter [16] constructed a wide variety of countable structures for which enumeration reducibility, but not Turing reducibility, is sufficient to measure the complexity of their isomorphism type. There are several other occasions in

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computable model theory when enumeration reducibility was used to explain a certain phenomenon. An example is Soskov's [24] simplification of Coles, Downey and Slaman's [4] result characterizing the jump complexity of the isomorphism types of torsion free abelian groups of rank 1. A more recent example is Soskov's [25] negative solution to the α-jump inversion problem for spectra of structures at limit computable ordinals α.

Both Turing and enumeration reducibility are pre-partial orders on the powerset of ω, hence they induce partial orders on the equivalence classes, i.e., the sets that are equally hard to compute, or equally hard to enumerate, respectively. These equivalence classes (and the induced partial orders) are called the *Turing degrees* and the *enumeration degrees*. We would like to understand these structures and the relationship between them from a logical perspective.

One can view the enumeration degrees as an extension of the Turing degrees. In other words, there is a natural embedding of the Turing degrees into the enumeration degrees; simply map the Turing degree of A to the enumeration degree of $A \oplus \overline{A}$. In the 1960's, Rogers [17, 18] asked if the members of this embedded copy of the Turing degrees—the *total* degrees—are in some way distinguishable within the enumeration degrees. Specifically, he asked whether the total degrees are definable in the enumeration degrees by a first order formula in the language of partial orders. We not only give an affirmative answer to this question, but we actually provide a natural, and relatively succinct, definition.

There have been a few partial solutions to Rogers' question. Kalimullin [11] gave a definition of the enumeration jump operator. Combining this with Friedberg jump-inversion [6] gives us a definition of the total degrees above $\mathbf{0}_e'$: they are the range of the jump operator. Ganchev and Soskova [9] defined the total degrees in the local structure (i.e., the degrees below $\mathbf{0}_e'$). Putting these together, the total degrees comparable to $\mathbf{0}_e'$ are definable. Finally, Soskova [27] gave a definition of the total degrees using a parameter. In other words, she proved that by naming a certain (possibly undefinable) enumeration degree, we can write down a property true of exactly the total degrees.

Soskova's result relies on a method that was introduced for the Turing degrees by Slaman and Woodin [22]. She used facts about the automorphism group of the enumeration degrees to translate a definition in second order arithmetic of a class of enumeration degrees to a definition (with a parameter) in the language of partial orders. The resulting definition of totality is complicated and does not obviously isolate a natural structural property that distinguishes the total degrees from the other enumeration degrees. By contrast, Kalimullin's definition of the enumeration jump is relatively natural. For his definition, Kalimullin introduced $\mathcal{K}$-pairs, an elegant notion that we discuss in detail in the next section. For now, it is sufficient to say that $\mathcal{K}$-pairs have a simple definition in the enumeration degrees. Ganchev and Soskova also used $\mathcal{K}$-pairs in their definition of totality in the local structure. In particular, they proved that the nonzero total degrees below $\mathbf{0}_e'$ are exactly the joins of nontrivial maximal $\mathcal{K}$-pairs.

The key difficulty in Ganchev and Soskova's proof is showing that the joins of nontrivial maximal $\mathcal{K}$-pairs below $\mathbf{0}_e'$ are total. This is the result that we generalize in Section 2 to obtain a global definition of totality. We prove that if A and B are a nontrivial $\mathcal{K}$-pair, then there is a semi-computable set C such that $A \leq_e C$ and $B \leq_e \overline{C}$. From this it follows that joins of nontrivial maximal $\mathcal{K}$-pairs are total, and hence that a nonzero enumeration degree is total if and only if it is the join of a nontrivial maximal $\mathcal{K}$-pair.

In Section 3, we give some applications and leave some open questions. Using previous results of Ganchev and Soskova [9] and of Cai and Shore [3], we show that the relation *computably enumerable in* between Turing degrees is definable in the enumeration degrees, which by a result of Cai [1] (extending work of Cai and Shore [2]), also gives a definition of the *array nonrecursive* Turing degrees in the enumeration degrees. Note that it is not known whether "c.e. in" or array nonrecursive are definable in the Turing degrees. We also observe that a nontrivial automorphism of the enumeration degrees would induce a nontrivial automorphism of the Turing degrees (the existence of which is a persistent open question). Slaman and Woodin [22] proved that every automorphism of the Turing degrees fixes the cone above $\mathbf{0}_T''$. Pulling that back to the enumeration degrees, we see that every automorphism of the enumeration degrees is the identity on the cone above $\mathbf{0}_e''$, improving a result of Ganchev and Soskov [26]. Finally, we give a criterion for an automorphism of the Turing degrees to be extendible to an automorphism of the enumeration degrees.

1. Preliminaries

Enumeration reducibility was introduced in 1959 by Friedberg and Rogers [7, p. 124] and Rogers (unpublished mimeographed notes, forming the basis for [18]). Intuitively, a set of natural numbers A is enumeration reducible to a set of natural numbers B if every enumeration of B can be effectively transformed into an enumeration of A. Formally:

Definition 1.1. A set A is *enumeration reducible* ($\leq_e$) to a set B if there is a c.e. set Φ such that

$$A = \Phi(B) = \{n \colon (\exists u)\ [(n, u) \in \Phi \text{ and } D_u \subseteq B]\},$$

where D_u denotes uth finite set under the standard coding of finite sets. We will refer to the c.e. set Φ as an *enumeration operator*.

In contrast to a Turing reduction, which generates a complete description of the membership relation of a set from a complete description of the membership relation of another set, an enumeration reduction generates positive information from positive information. An intermediate relation is "*c.e. in*", which generates positive information about a set from a complete description of another set. The following easy proposition gives the precise relationship between these notions.

Proposition 1.2. *Let* $A, B \subseteq \omega$.

(1) A *is c.e. in* $B \iff A \leq_e B \oplus \overline{B}$.
(2) $A \leq_T B \iff A \oplus \overline{A}$ *is c.e. in* $B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

Enumeration reducibility is a reflexive and transitive relation on sets of natural numbers and so, just like Turing reducibility, it induces a degree structure. A set A is *enumeration equivalent* ($\equiv_e$) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation $\equiv_e$ is the enumeration degree of A, which we write as $\mathbf{d}_e(A)$. The structure of the enumeration degrees $(\mathcal{D}_e, \leq)$ is the class of all enumeration degrees with the relation $\leq$ defined by $\mathbf{d}_e(A) \leq \mathbf{d}_e(B)$ if and only if $A \leq_e B$. Its least element is $\mathbf{0}_e = \mathbf{d}_e(\emptyset)$, the set of all c.e. sets. The least upper bound of $\mathbf{d}_e(A)$ and $\mathbf{d}_e(B)$ is $\mathbf{d}_e(A) \vee \mathbf{d}_e(B) = \mathbf{d}_e(A \oplus B)$.

Cooper [5] defined a jump operation on the enumeration degrees.

Definition 1.3. The *enumeration jump* of a set A is denoted by $J_e(A)$ and is defined as $K_A \oplus \overline{K_A}$, where $K_A = \{(e, x)\colon\ e \in \omega \text{ and } x \in \Phi_e(A)\}$. This induces a jump operation on the enumeration degrees: the enumeration jump of $\mathbf{d}_e(A)$ is $\mathbf{d}_e(A)' = \mathbf{d}_e(J_e(A))$.

Thus the structure of the enumeration degrees is an upper semilattice with least element and jump operation, just like the Turing degrees. The relationship between Turing and enumeration reducibility, described in Proposition 1.2, gives rise to the following notion.

Definition 1.4. A set A is *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set. The collection of all total degrees is denoted by $\mathcal{TOT}$.

The structure $\mathcal{TOT}$ is an isomorphic copy of the Turing degrees. The map ι, defined by

$$\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A}).$$

is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$ with range $\mathcal{TOT}$. It preserves the order, the least upper bound, and the least element. McEvoy [14] showed that it also preserves the jump operation.

A natural question, first posed by Rogers [17, 18], is whether $\mathcal{TOT}$ is first order definable in $\mathcal{D}_e$. We give a positive answer to this longstanding question. Our methods are based on a long line of research, starting with an article by Jockusch [10] that introduced the notion of a semi-computable set.

Definition 1.5 (Jockusch [10])**.** We say that $A \subseteq \omega$ is *semi-computable* if there is a total computable selector function $s_A\colon \omega \times \omega \to \omega$ such that for any $x, y \in \omega$, we have $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$, then $s_A(x, y) \in A$.

Arslanov, Cooper and Kalimullin [13] investigated the properties of semi-computable sets in the context of enumeration reducibility. They showed that for every semi-computable set A such that A and $\overline{A}$ are not c.e., the enumeration degrees $\mathbf{d}_e(A)$ and $\mathbf{d}_e(\overline{A})$ form a minimal pair in a very strong sense, namely:

$$(\forall \mathbf{x} \in \mathcal{D}_e)\ (\mathbf{d}_e(A) \vee \mathbf{x}) \wedge (\mathbf{d}_e(\overline{A}) \vee \mathbf{x}) = \mathbf{x}.$$

Kalimullin [11] characterized all enumeration degrees with this strong minimal pair property.

Definition 1.6 (Kalimullin [11])**.** A pair of sets $A, B \subseteq \omega$ are a *Kalimullin pair* (*$\mathcal{K}$-pair*) if and only if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Kalimullin originally called such pairs of sets *e-ideals*. The reason for this can be seen from the following property of $\mathcal{K}$-pairs:

Proposition 1.7 (Kalimullin [11])**.** *Let $A \subseteq \omega$. The set of all B such that $\{A, B\}$ is a $\mathcal{K}$-pair is closed downwards under enumeration reducibility as well as closed under join.*

Thus the relation "$\{A, B\}$ is a $\mathcal{K}$-pair" is enumeration degree invariant and induces a relation on enumeration degrees: if $\{A, B\}$ is a $\mathcal{K}$-pair of sets then $\{\mathbf{d}_e(A), \mathbf{d}_e(B)\}$ is a *$\mathcal{K}$-pair* of enumeration degrees.

A c.e. set A forms a $\mathcal{K}$-pair with every set of natural numbers. We call $\mathcal{K}$-pairs with a c.e. member *trivial.* A more interesting example comes from a semi-computable set. Indeed, if A is semi-computable, then $\{A, \overline{A}\}$ form a $\mathcal{K}$-pair, witnessed by the c.e. set $\{(x,y)\colon s_A(x,y)=x\}$, where s_A is the selector function for A. We shall call $\mathcal{K}$-pairs of this form *semi-computable* $\mathcal{K}$-pairs.

Nontrivial $\mathcal{K}$-pairs possess nontrivial properties.

Proposition 1.8 (Kalimullin [11])**.** *If $A, B \subseteq \omega$ are a nontrivial $\mathcal{K}$-pair witnessed by W, then*

$$A = \{n\colon (\exists k \notin B)\ (n,k) \in W\},\ \textit{and}$$
$$B = \{k\colon (\exists n \notin A)\ (n,k) \in W\}.$$

Similarly,

$$\overline{A} = \{n\colon (\exists k \in B)\ (n,k) \in \overline{W}\},\ \textit{and}$$
$$\overline{B} = \{k\colon (\exists n \in A)\ (n,k) \in \overline{W}\}.$$

The enumeration degrees of a $\mathcal{K}$-pair satisfy the same minimal pair property as the enumeration degrees of a semi-computable set and its complement. In fact, this gives a characterization of $\mathcal{K}$-pairs.

Theorem 1.9 (Kalimullin [11])**.** *Two sets $A, B \subseteq \omega$ are a $\mathcal{K}$-pair if and only if:*

$$(\forall \mathbf{x} \in \mathcal{D}_e)\ (\mathbf{d}_e(A) \vee \mathbf{x}) \wedge (\mathbf{d}_e(B) \vee \mathbf{x}) = \mathbf{x}.$$

Kalimullin used the definability of $\mathcal{K}$-pairs to show that the enumeration jump is first order definable in $\mathcal{D}_e$. Ganchev and Soskova investigated properties of $\mathcal{K}$-pairs bounded by $\mathbf{0}_e'$, the first jump of the least enumeration degree. The structure of all enumeration degrees bounded by $\mathbf{0}_e'$, denoted by $\mathcal{D}_e(\leq \mathbf{0}')$, is also known as the *local structure* of the enumeration degrees. In [8], Ganchev and Soskova showed that the $\mathcal{K}$-pairs in $\mathcal{D}_e(\leq \mathbf{0}')$ are also first order definable. Then in [9], they introduced the notion of a maximal $\mathcal{K}$-pair.

2. Maximal $\mathcal{K}$-pairs

Definition 2.1 (Ganchev, Soskova [9])**.** A $\mathcal{K}$-pair of enumeration degrees $\{\mathbf{a}, \mathbf{b}\}$ is a *maximal* $\mathcal{K}$-pair if for every $\mathcal{K}$-pair of enumeration degrees $\{\mathbf{c}, \mathbf{d}\}$ such that $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$, we have $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

Every nontrivial semi-computable $\mathcal{K}$-pair is maximal. In fact, every nontrivial $\mathcal{K}$-pair of the form $\{A, \overline{A}\}$ is maximal. To see this, consider a $\mathcal{K}$-pair $\{A, \overline{A}\}$ and let $\{C, D\}$ be a $\mathcal{K}$-pair such that $A \leq_e C$ and $\overline{A} \leq_e D$. By Proposition 1.7, $\mathcal{K}$-partners are downwards closed under enumeration reducibility, hence A forms a $\mathcal{K}$-pair with D. By Proposition 1.8, $D \leq_e \overline{A}$. Similarly, $\overline{A}$ forms a $\mathcal{K}$-pair with C and hence $C \leq_e \overline{\overline{A}} = A$.

Jockusch [10] showed that every nonzero Turing degree contains a semi-computable set that is neither c.e. nor co-c.e. Translated into enumeration degree terms, this means that every nonzero total enumeration degree contains the set $A \oplus \overline{A}$ for some semi-computable set A such that neither A nor $\overline{A}$ is c.e. But $\{A, \overline{A}\}$ is a maximal $\mathcal{K}$-pair, so we can restate Jockusch's result as follows:

Theorem 2.2 (Jockusch [10])**.** *Every nonzero total enumeration degree is the least upper bound of a maximal $\mathcal{K}$-pair.*

Ganchev and Soskova [9] show that every nontrivial $\mathcal{K}$-pair in the local structure can be extended to a semi-computable $\mathcal{K}$-pair. Thus every maximal $\mathcal{K}$-pair in the local structure is represented by a semi-computable $\mathcal{K}$-pair. Consequently, the nontrivial total degrees in the local structure are exactly the least upper bounds of maximal $\mathcal{K}$-pairs, hence first order definable in $\mathcal{D}_e(\leq \mathbf{0}')$. Our main result extends this to arbitrary nontrivial $\mathcal{K}$-pairs in $\mathcal{D}_e$.

Theorem 2.3. *If $A, B \subseteq \omega$ form a nontrivial $\mathcal{K}$-pair, then there is a semi-computable set C such that $A \leq_e C$ and $B \leq_e \overline{C}$.*

This theorem immediately allows us to prove our main result:

Theorem 2.4. *The set of total enumeration degrees is first order definable in the structure of the enumeration degrees: a degree is total if and only if it is $\mathbf{0}_e$ or the join of a maximal $\mathcal{K}$-pair.*

Proof. By Theorem 2.2, every nonzero total enumeration degree is the least upper bound of a maximal $\mathcal{K}$-pair. For the other direction, Theorem 2.3 implies that every maximal $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is represented by a semi-computable $\mathcal{K}$-pair $\{C, \overline{C}\}$. This means that $\mathbf{a} \vee \mathbf{b}$ contains the set $C \oplus \overline{C}$, hence is total. $\square$

Proof of Theorem 2.3. Assume that $A, B \subseteq \omega$ form a nontrivial $\mathcal{K}$-pair as witnessed by the c.e. set W. We dynamically label elements of $\mathbb{Q}$ with natural numbers for the sake of both A and B, though it is important to note that the labeling will only depend on W. Rationals may not have more than one label. If $q \in \mathbb{Q}$ is labeled with k for the sake of B, we write the label as $\overline{k}$. We do not decorate labels for A. At any stage of the construction, let the *current* n-labeled rational $q(n)$ be the leftmost labeled with n. (There will only be finitely many labeled rationals at any stage, so if some rational is labeled with n, then $q(n)$ is well-defined.) Similarly, let the *current* $\overline{k}$-labeled rational $q(\overline{k})$ be the rightmost labeled $\overline{k}$. Two labeled rationals are said to be currently *adjacent* if no rational between them is labeled.

It might be easier to think of the following construction dynamically: at any stage we have an order on (some of) the elements of two copies of ω, the set natural numbers. For notational convenience we label the numbers from one of the copies by the overhead bar. The numbers without the bar (e.g., n) start from the right and can only move left; the numbers with the bar (e.g., $\overline{k}$) start from the left and can only move right; of course, both obey certain rules that we will specify. The $\mathbb{Q}$-labeling in the above paragraph is an alternative description of such a dynamic process: $q(n)$ is the current position of n and $q(\overline{k})$ of $\overline{k}$.

Call $C \subseteq \mathbb{Q}$ a *cut* if it is downwards closed with respect to the standard ordering of the rationals. Every cut is semi-computable; the selector function simply picks the lesser of its two arguments. Let

$$A_C = \{n \colon (\exists q \in C)\ q \text{ is } n\text{-labeled}\}, \text{ and}$$
$$B_C = \{k \colon (\exists q \in \overline{C})\ q \text{ is } \overline{k}\text{-labeled}\}.$$

Note that $A_C \leq_e C$ and $B_C \leq_e \overline{C}$. The goal is to define the labeling so that there must be a cut C for which $A = A_C$ and $B = B_C$. To this end, if $(n, k) \notin W$, then we will *never* allow an n-labeled rational to be left of any $\overline{k}$-labeled rational. This ensures that for any cut C, if $n \in A_C$, then every $\overline{k}$-labeled rational is in C, so $k \notin B_C$. Hence, A_C and B_C are consistent with the fact that $(n, k) \notin W$. If $(n, k) \in W$, then ideally we would like to have an n-labeled rational left of some

$\overline{k}$-labeled rational. If we could do this, then for every cut C we would either have $n \in A_C$ or $k \in B_C$, ensuring consistency with the fact that $(n, k) \in W$.

As it turns out, we cannot necessarily put an n-labeled rational left of some $\overline{k}$-labeled rational. For example, consider the following "checkerboard scenario":

W	n	m
k	$\checkmark$	
j		$\checkmark$

If (m, j) enters W first and there are no other obstacles, we would have

$$q(\overline{k}) < q(m) < q(\overline{j}) < q(n).$$

Now when (n, k) enters W, we cannot ensure that $q(n) < q(\overline{k})$ because $q(n)$ cannot move past $q(\overline{j})$ and $q(\overline{k})$ cannot move past $q(m)$. This has two consequences:

- Not every cut C will result in sets A_C, B_C that are a $\mathcal{K}$-pair witnessed by W. In particular, it is possible that a cut C contains every $\overline{k}$-labeled rational but no n-labeled rational (for example, any cut that is between $q(m)$ and $q(\overline{j})$), so $n \notin A_C$ and $k \notin B_C$. This is not consistent with $(n, k) \in W$.
- Every cut C that contains an n-labeled rational contains every $\overline{k}$-labeled rational. Therefore, it is not possible that both $n \in A_C$ and $k \in B_C$.

The first observation is not problematic; we can simply avoid inconsistent cuts. However, the second observation requires careful attention. If $(n, k) \in W$ but we are not able to put an n-labeled rational left of some $\overline{k}$-labeled rational, we must prove that if A and B are a $\mathcal{K}$-pair witnessed by W, then either $n \notin A$ or $k \notin B$. (The consistency of A and B with W means that either $n \in A$ or $k \in B$, so in fact we need $n \in A$ iff $k \notin B$.) This is our key difficulty, which we address in full generality in Lemma 2.3.2.

In preparation, let us consider the simple checkerboard scenario in more detail. We have stages $s < t$ such that the following two tables describe all memberships from the set $\{n, m\} \times \{k, j\}$ in W_s and W_t:

W_s	n	m
k		
j		$\checkmark$

$$q(\overline{k}) < q(m) < q(\overline{j}) < q(n)$$

W_t	n	m
k	$\checkmark$	
j		$\checkmark$

$$q(\overline{k}) < q(m) < q(\overline{j}) < q(n).$$

First assume that W_t correctly describes membership in W of the four pairs of natural numbers under consideration, i.e., that $(n, j) \notin W$ and $(m, k) \notin W$. We reason as follows: if $n \in A$, then $j \notin B$ because $(n, j) \notin W$. If $j \notin B$, then $m \in A$ because $(m, j) \in W$. If $m \in A$, then $k \notin B$ because $(m, k) \notin W$. And finally, if $k \notin B$, then $n \in A$ because $(n, k) \in W$. Putting it all together, we have proved what was required: $n \in A$ if and only if $k \notin B$.

On the other hand, assume that $n \in A$ and $k \in B$. As we have shown, W_t does not correctly describe membership in W of the four pairs under consideration. So suppose, for example, that (m, k) enters W at stage $r > t$. Then the obstacle for $q(\overline{k})$ is removed at stage r and we can correctly order the labels once again.

W_r	n	m
k	✓	✓
j		✓

$$q(m) < q(\bar{j}) < q(n) < q(\bar{k}).$$

These examples suggest that a naive dynamic strategy might work. In other words, what if we move A-labels as far to the left as possible and B-labels as far to the right as possible, always respecting the rule: "if $(n,k) \notin W$ then all B-labels for k are to the left of all A-labels for n"? This strategy can fail, as the following "infinite checkerboard scenario" shows.

Suppose that we replace m in the previous scenario by a sequence of distinct natural numbers $\{m_i\}_{i<\omega}$. Suppose also that we have an increasing list of stages

$$s_0 < t < s_1 < r_0 < s_2 < r_1 < s_3 < r_2 < \cdots$$

such that the pair (n,k) enters W at stage t, each pair of the form (m_i,j) enters W at stage s_i, and each pair of the form (m_i,k) enters W at stage r_i. Consider the how the current labels move if we use the naive strategy:

W_{s_0}	n	m_0
k		
j		✓

$$q(\bar{k}) < q(m_0) < q(\bar{j}) < q(n)$$

W_t	n	m_0
k	✓	
j		✓

$$q(\bar{k}) < q(m_0) < q(\bar{j}) < q(n)$$

W_{s_1}	n	m_0	m_1
k	✓		
j		✓	✓

$$q(\bar{k}) < q(m_1) < q(m_0) < q(\bar{j}) < q(n)$$

W_{r_0}	n	m_0	m_1
k	✓	✓	
j		✓	✓

$$q(m_0) < q(\bar{k}) < q(m_1) < q(\bar{j}) < q(n)$$

Note that $q(m_0)$ is temporarily an obstacle to moving $q(\bar{k})$ to the right and that, by the time it ceases to be an obstacle, it is replaced by $q(m_1)$ as a new temporary obstacle. The order on the stages ensures that the pair (n,k) is part of infinitely many impermanent checkerboard scenarios, each new one appearing before the previous one is resolved. It is consistent with our description of W that both $n \in A$ and $k \in B$. (In this case, $j \notin B$ and each $m_i \in A$.) However, the order of the enumeration of W prevents us from moving $q(n)$ to the left of $q(\bar{k})$, so there would be no cut C for which $A_C = A$ and $B_C = B$.

To resolve this conflict, we introduce priority between the pairs of natural numbers. Fix a priority ordering of pairs of natural numbers of order type ω. A pair that has been assigned higher priority can restrict an interval, so that pairs of lower priority may not label any rational in that interval. In the scenario described above, the pair (n,k) will restrict the interval $[q(\bar{k}), q(n)]$ at stage t. As there are only finitely many pairs (m_i,j) of higher priority than (n,k), there will be a stage after which (n,k) will not be involved in new checkerboard scenarios, all of the old checkerboard scenarios will be resolved, and we will be able to correctly order the current labels for n and k.

With this modification in mind, we are ready to describe the construction. It is in stages. At stage s, we consider the first s pairs (n,k) in the priority ordering and let them act in order of priority from highest to lowest.

Strategy for (n,k) at stage s. An (n,k) strategy first ensures that there are rationals labeled n and $\overline{k}$. If no rational is labeled n, choose a q to the right of all labeled rationals and label it n. If no rational is labeled $\overline{k}$, choose a q to the left of all labeled rationals and label it $\overline{k}$.

If $(n,k) \notin W_s$, there is nothing else to do. If $(n,k) \in W_s$, then we want $q(n)$ to be left of $q(\overline{k})$. As we have discussed, this may not be possible while respecting W_s. If $(n,j) \notin W_s$, then $q(\overline{j})$ is an obstacle; we may not label any rational to the left of $q(\overline{j})$ with n. Let $q(\overline{j})$ be the rightmost such obstacle for n. Similarly, let $q(m)$ be the leftmost such obstacle for $\overline{k}$. In particular, $(m,k) \notin W_s$. Now, move $q(n)$ as far left as possible. In other words,

- if there are no obstacles, make it the leftmost label;
- if possible, put it adjacent to $q(\overline{j})$;
- if it is inside an interval containing $q(\overline{j})$ that is protected by higher priority strategies, leave it in place;
- otherwise, put it adjacent to the right endpoint of a the maximal interval containing $q(\overline{j})$ that is protected by higher priority strategies.

(If $q(n)$ is already—qualitatively speaking—as far left as it can be, it should not be moved.) Similarly, move $q(\overline{k})$ as far right as possible.

If $q(n) < q(\overline{k})$, then we are done and this strategy will never act again. On the other hand, if $q(n) > q(\overline{k})$, then mark the interval $[q(\overline{k}), q(n)]$ as a "dead zone" and restrict lower priority strategies from labeling any rational in this interval. This dead zone is not necessarily permanent: At later stages, this strategy will again analyze the obstacles to moving $q(n)$ left of $q(\overline{k})$, possibly succeeding in this goal or at least shrinking the dead zone. Before completing the actions for this strategy, we want to clear current labels out of $[q(\overline{k}), q(n)]$ as much as possible. Take the union of all dead zones currently declared by this and higher priority strategies. Let $[x,y]$ be the maximal interval in this union that extends $[q(\overline{k}), q(n)]$. If $q(a) \in [q(\overline{k}), q(n)]$ and it is consistent to move it left of x, then do so. Similarly, move each $q(\overline{b}) \in [q(\overline{k}), q(n)]$ left of y, if possible.

Verification. We need to show that for any nontrivial $\mathcal{K}$-pair A, B witnessed by W, there is a cut C such that $A = A_C$ and $B = B_C$. First, we show that every strategy settles down, which is straightforward by a priority argument. After that, we prove that permanent dead zones are well-behaved. Finally, we construct C and prove that it works.

Lemma 2.3.1. *Every strategy eventually stops acting.*

Proof. Consider the (n,k) strategy. If $(n,k) \notin W$, then the strategy never acts. So assume that $(n,k) \in W$. If there is ever a stage such that $q(n) < q(\overline{k})$, then the (n,k) strategy will never act again, so also assume that this is not the case. By induction, there is a stage s such that all higher priority strategies have stopped acting. In addition, let s be large enough such that $(n,k) \in W_s$.

At every stage $t \geq s$, the (n,k) strategy declares $[q(\overline{k}), q(n)]$ to be a dead zone. At stage t, take the union of all dead zones declared by this and higher priority strategies and let $[x_t, y_t]$ be the maximal interval in this union that extends $[q(\overline{k}), q(n)]$. Let $M_t = \{m \colon q(m) \in [x_t, y_t]\}$ and $J_t = \{j \colon q(\overline{j}) \in [x_t, y_t]\}$. Note that no lower priority strategy can label any rational in $[x_t, y_t]$ at stage t. Also, no higher priority strategy acts after stage s. Therefore, $M_{t+1} \subseteq M_t$ and $J_{t+1} \subseteq J_t$ for all $t \geq s$.

Moreover, if $m \in M_{t+1} \smallsetminus \{n\}$, then $q(m)$ cannot have been moved between stage t and $t+1$. The same is true for J_{t+1}. So every time the (n, k) strategy acts at a stage $t+1 > s$, it either moves $q(n)$ or $q(\overline{k})$ past a label that had previously been an obstacle, or it moves a current label out of $[x_{t+1}, y_{t+1}]$. In both cases, this corresponds to an element of $M_t \smallsetminus M_{t+1}$ or $J_t \smallsetminus J_{t+1}$. Of course, M_s and J_s are finite, so the (n, k) strategy can only act finitely often. □

Call a dead zone $[q(\overline{k}), q(n)]$ *permanent* if the (n, k) strategy and all higher priority strategies have stopped acting. Note that a current label in a permanent dead zone will remain there forever; if it could be moved out of the dead zone, then the (n, k) strategy would do so, but it has stopped acting.

We say that two current labels are *connected* if they are contained in a connected union of permanent dead zones. We have observed that if a cut $C \subseteq \mathbb{Q}$ splits a dead zone, then A_C and B_C do *not* form a $\mathcal{K}$-pair witnessed by W. Therefore, we are restricted to cuts C with the property that for every dead zone D either $D \subseteq C$ or $D \subseteq \overline{C}$. In particular if $q(a)$ and $q(\overline{b})$ are connected, then either $q(a)$ and $q(\overline{b})$ are in C and so $a \in A_C$ and $b \notin B_C$, or $q(a)$ and $q(\overline{b})$ are in $\overline{C}$ and so $a \notin A_C$ and $b \in B_C$. The following lemma, which is the key component in our verification, shows that this restriction is compatible with all possible $\mathcal{K}$-pairs that are witnessed by W.

Lemma 2.3.2. *Assume that A and B are a $\mathcal{K}$-pair witnessed by W.*

- *If $q(a)$ and $q(\overline{b})$ are connected, then $a \in A \iff b \notin B$.*
- *If $q(a)$ and $q(c)$ are connected, then $a \in A \iff c \in A$.*
- *If $q(\overline{b})$ and $q(\overline{d})$ are connected, then $b \notin B \iff d \notin B$.*

Proof. It is enough to prove the lemma under the assumption that the current labels are contained in the same permanent dead zone. This is because intersecting dead zones share a current label, so the conclusion of the lemma follows, by transitivity, for current labels contained in a connected union of permanent dead zones.

We use induction on the priority of the permanent dead zone. Suppose that a permanent dead zone $[q(\overline{k}), q(n)]$ is declared by the strategy for (n, k) and let $q(\overline{j})$ and $q(m)$ be the corresponding permanent obstacles. In particular, we know that $(n, k) \in W$ and $(n, j), (m, k) \notin W$. We make the inductive assumption that the lemma holds for current labels connected by higher priority dead zones.

Case 1: $q(\overline{j}) < q(m)$. In this case, $q(\overline{j})$ and $q(m)$ must be connected by higher priority dead zones, otherwise the (n, k) strategy would have moved both $q(\overline{k})$ and $q(n)$ between $q(\overline{j})$ and $q(m)$ and made $q(n) < q(\overline{k})$. So we may assume, by induction, that $m \in A \iff j \notin B$. So $n \in A \implies j \notin B \implies m \in A \implies k \notin B \implies n \in A$, using the fact that $(n, j), (m, k) \notin W$ and $(n, k) \in W$. So $n \in A \iff j \notin B \iff m \in A \iff k \notin B$ and Lemma 2.3.2 holds for $q(n), q(m), q(\overline{k})$, and $q(\overline{j})$.

Now consider any other $q(a) \in [q(\overline{k}), q(n)]$. We claim that it is connected to $q(\overline{j})$ by higher priority dead zones. If not, then it would either be greater than $q(\overline{j})$ and there would be nothing to prevent us from putting $q(n)$ to its left, or it would be less that $q(m)$. In this case either $(a, k) \notin W$, so $q(a)$ would actually have been the leftmost obstacle to $q(\overline{k})$, or $(a, k) \in W$ and there would be nothing to prevent us from putting $q(\overline{k})$ to its right. So $q(a)$ must be connected to $q(\overline{j})$ by higher priority dead zones, hence by induction $a \in A \iff j \notin B$. In the same way, for any other $q(\overline{b}) \in [q(\overline{k}), q(n)]$, we have $b \notin B \iff m \in A$.

Case 2: $q(m) < q(\overline{j})$. This implies that $(m, j) \in W$, so we are in the checkerboard scenario discussed above. Because $(n, k), (m, j) \in W$ and $(n, j), (m, k) \notin W$, we have $n \in A \implies j \notin B \implies m \in A \implies k \notin B \implies n \in A$. Therefore, either $m, n \in A$ and $k, j \notin B$, or $m, n \notin A$ and $k, j \in B$, so Lemma 2.3.2 holds for $q(n)$, $q(m)$, $q(\overline{k})$, and $q(\overline{j})$.

Now consider any other $q(a) \in [q(\overline{k}), q(n)]$. If it is connected to any of $q(n)$, $q(m)$, $q(\overline{k})$, or $q(\overline{j})$ by higher priority dead zones, then we can apply the inductive hypothesis, so assume not. Note that $q(a)$ cannot be greater than $q(\overline{j})$, because there would be nothing to prevent us from putting $q(n)$ to its left. So we must have $q(a) < q(\overline{j})$, and hence $(a, j) \in W$. Because we never clear $q(a)$ out of $[q(\overline{k}), q(n)]$, there must a $q(\overline{b})$ preventing us from moving $q(a)$, i.e., $(a, b) \notin W$. There are four cases to consider:

- If $b = k$, then $a \in A \implies k \notin B$.
- If $q(\overline{b})$ is connected to $q(\overline{k})$ by higher priority dead zones, then $a \in A \implies b \notin B \implies k \notin B$.

Note that if neither of these cases hold, then we must have $q(\overline{b}) > q(\overline{k})$, or $q(\overline{b})$ would not be an obstacle to moving $q(a)$ out of $[q(\overline{k}), q(n)]$.

- If $q(\overline{b})$ is connected to $q(m)$ by higher priority dead zones, then $a \in A \implies b \notin B \implies m \in A \implies k \notin B$.
- If none of the other cases hold, then we must have $q(\overline{b}) > q(m)$, otherwise there would be nothing to prevent us from putting $q(\overline{k})$ to the right of $q(\overline{b})$. Therefore, $(m, b) \in W$. So $a \in A \implies b \notin B \implies m \in A \implies k \notin B$.

In each case we have shown that $a \in A \implies k \notin B$. But $(a, j) \in W$, so $a \notin A \implies j \in B \implies k \in B$. Therefore, $a \in A \iff k \notin B$. In the same way, for any $q(\overline{b}) \in [q(\overline{k}), q(n)]$, we can show $b \notin B \iff n \in A$. □

Defining the cut. Let A and B be a nontrivial $\mathcal{K}$-pair witnessed by W. We are ready to define the cut $C \subseteq \mathbb{Q}$ such that $A = A_C$ and $B = B_C$. Note that if such a cut exists, then C and B form a nontrivial $\mathcal{K}$-pair, hence $C \leq_e \overline{B}$. This observation points us in the right direction; we define C using the rationals that are labeled $\overline{k}$ for $k \notin B$. Let C be the downward closure of such rationals, taking permanent dead zones into account:

$$C = \{q \in \mathbb{Q} \colon (\exists k \notin B)(\exists r \in \mathbb{Q})\ [r \text{ is } \overline{k}\text{-labeled and } r \geq q, \text{ or} \\ q \text{ is in a permanent dead zone with } q(\overline{k})]\}.$$

In the same way, let

$$D = \{q \in \mathbb{Q} \colon (\exists n \notin A)(\exists r \in \mathbb{Q})\ [r \text{ is } n\text{-labeled and } r \leq q, \text{ or} \\ q \text{ is in a permanent dead zone with } q(n)]\}.$$

The set C is a cut. To see this we note that if q is in the same dead zone as $q(\overline{k})$ and $r \leq q$, then either $r \leq q(\overline{k})$ or else $r \in [q(\overline{k}), q]$ and hence is also in the same dead zone as $q(\overline{k})$. Similarly, D is an upwards closed set with respect to the ordering of the rationals. It turns out that $D = \overline{C}$ (see Observation 2.5). We only need to prove that they are disjoint.

Lemma 2.3.3. $C \cap D = \emptyset$.

Proof. Assume, for a contradiction, that $q \in C \cap D$. Because $q \in C$, there is a $k \notin B$ such that either q is in a permanent dead zone with $q(\overline{k})$, or there is some $\overline{k}$-labeled rational $r(\overline{k}) \geq q$. Similarly, $q \in D$ implies that there is an $n \notin A$ such that either q is in a permanent dead zone with $q(n)$, or there is some n-labeled rational $r(n) \leq q$. Note that $(n, k) \notin W$ (as $n \notin A$ and $k \notin B$), so by construction, every $\overline{k}$-label must be to the left of every n-label. There are four cases:

Case 1: $r(n) \leq q \leq r(\overline{k})$. This is impossible, as noted above.

Case 2: q is in a permanent dead zone with $q(\overline{k})$ and $r(n) \leq q$. We must have $q(\overline{k}) \leq q(n) \leq r(n) \leq q$, so $q(n)$ is stuck in the same permanent dead zone that contains $q(\overline{k})$ and q. This contradicts Lemma 2.3.2.

Case 3: $r(\overline{k}) \geq q$ and q is in a permanent dead zone with $q(n)$. This case is symmetric to Case 2 and also not possible.

Case 4: q is in permanent dead zones with both $q(n)$ and $q(\overline{k})$. This implies that $q(n)$ and $q(\overline{k})$ are connected, which again contradicts Lemma 2.3.2. □

Lemma 2.3.4. *Assume that $C \subseteq E$ and $D \subseteq \overline{E}$. Then $A = A_E$ and $B = B_E$.*

Proof. We will show that

$$A = A_E = \{n \colon (\exists q \in E)\ q \text{ is } n\text{-labeled}\}.$$

First, consider $n \notin A$. Every n-labeled rational is in $D \subseteq \overline{E}$, hence $n \notin A_E$.

Now assume that $n \in A$. Because A and B are a nontrivial $\mathcal{K}$-pair, by Proposition 1.8, there is a $k \notin B$ such that $(n, k) \in W$. By Lemma 2.3.1, there is a stage s such that the (n, k) strategy and all higher priority strategies have stopped acting. Either $q(n) < q(\overline{k})$ at stage s, or $[q(\overline{k}), q(n)]$ is a permanent dead zone. In each case we have $q(n)[s] \in C \subseteq E$, so $n \in A_E$.

This proves that $A = A_E$. The proof that $B = B_E$ is symmetric. □

By Lemma 2.3.3, $E = C$ satisfies the hypothesis of Lemma 2.3.4, so $A = A_C \leq_e C$ and $B = B_C \leq_e \overline{C}$. This completes the proof of Theorem 2.3. □

Observation 2.5. Although it was not needed for the proof of Theorem 2.3, we can show that $D = \overline{C}$. We know from Lemma 2.3.3 that $C \cap D = \emptyset$. To see that $C \cup D = \mathbb{Q}$, assume that there is a rational $q_0 \notin C \cup D$. Let $E = \{q \colon q < q_0\}$. Then $C \subseteq E$ and $D \subseteq \overline{E}$, so by Lemma 2.3.4, $A = A_E \leq_e E$. But this implies that A is c.e., contradicting the assumption that A and B are a nontrivial $\mathcal{K}$-pair. □

3. Applications and open questions

3.1. **Defining "c.e. in".** The relation "c.e. in" is traditionally considered as a relation between sets of natural numbers. A set A is c.e. in X if A is the domain of a function computable in X. Clearly, if A is c.e. in X, then A is c.e. in any set Y that is Turing equivalent to X. In this sense we can view "c.e. in" as a relation between a set of natural numbers and a Turing degree. A set A is *c.e. in* a Turing degree $\mathbf{d}_T(X)$ if and only if A is c.e. in X.

We can further extend the relation "c.e. in" to a binary relation on Turing degrees: for $\mathbf{a}, \mathbf{x} \in \mathcal{D}_T$, we say that $\mathbf{a}$ is *c.e. in* $\mathbf{x}$ if and only if there is a set of natural numbers $A \in \mathbf{a}$ such that A is c.e. in $\mathbf{b}$. In [9], this relation is characterized as follows:

Proposition 3.1 (Ganchev and Soskova[9]). *Let* $\mathbf{a}$ *and* $\mathbf{x}$ *be Turing degrees such that* $\mathbf{a}$ *is not c.e. Then* $\mathbf{a}$ *is c.e. in* $\mathbf{x}$ *if and only if there is a nontrivial* $\mathcal{K}$*-pair* $\{C, \overline{C}\}$ *such that* $\mathbf{d}_e(C) \leq_e \iota(\mathbf{x})$ *and* $\iota(\mathbf{a}) = \mathbf{d}_e(C) \vee \mathbf{d}_e(\overline{C})$.

From the proof of Theorem 2.3, it follows that the nontrivial $\mathcal{K}$-pairs of the form $\{\mathbf{d}_e(C), \mathbf{d}_e(\overline{C})\}$ are exactly the maximal $\mathcal{K}$-pairs, and hence first order definable in $\mathcal{D}_e$. We can use the obtained characterization to show the following extension of Friedberg's Jump Inversion Theorem [6].

Proposition 3.2. *For every nonzero Turing degree* $\mathbf{a}$ *there exists a Turing degree* $\mathbf{x}$ *such that* $\mathbf{x}' = \mathbf{a} \vee \mathbf{0}_T'$ *and* $\mathbf{a}$ *is c.e. in* $\mathbf{x}$.

Proof. If $\mathbf{a}$ is c.e., then the statement follows from Friedberg's Jump Inversion Theorem. Suppose $\mathbf{a}$ is not c.e. and consider its image $\iota(\mathbf{a})$ in the enumeration degrees. Let A be a semi-computable set such that $\iota(\mathbf{a}) = \mathbf{d}_e(A) \vee \mathbf{d}_e(\overline{A})$. Consider the jump of A, $J_e(A) = K_A \oplus \overline{K_A}$. As $K_A \equiv_e A$, we have that K_A forms a $\mathcal{K}$-pair with $\overline{A}$. Let W be a c.e. set that witnesses this. By Proposition 1.8, we have that $\overline{K_A} \leq_e \overline{A} \oplus \overline{W}$, hence $J_e(A) = K_A \oplus \overline{K_A} \leq_e A \oplus \overline{A} \oplus \overline{W}$. From this, we conclude that $\mathbf{d}_e(A)' = \iota(\mathbf{a}) \vee \mathbf{0}_e'$. Finally we apply Soskov's Jump Inversion Theorem [23] to conclude that there is a total enumeration degree $\iota(\mathbf{x}) \geq \mathbf{d}_e(A)$ with the same jump as $\mathbf{d}_e(A)$. It follows that $\mathbf{x}' = \mathbf{a} \vee \mathbf{0}_T'$ and $\mathbf{a}$ is c.e. in $\mathbf{x}$. $\square$

If $\mathbf{a}$ is not a c.e. degree, we have an order theoretic way of expressing the relation "$\mathbf{a}$ is c.e. in $\mathbf{x}$" in terms of the images of $\mathbf{a}$ and $\mathbf{x}$ in the enumeration degrees: $\mathbf{a}$ is c.e. in $\mathbf{x}$ if and only if $\iota(\mathbf{a})$ is the least upper bound of a maximal $\mathcal{K}$-pair, one of whose elements is bounded by $\iota(\mathbf{x})$. Note that this characterization fails if $\mathbf{a}$ is c.e. For every total enumeration degree $\iota(\mathbf{a})$ there is a total nonzero enumeration degree $\iota(\mathbf{x})$ such that $\{\iota(\mathbf{a}), \iota(\mathbf{x})\}$ form a minimal pair. Thus if $\mathbf{a}$ is c.e., then $\mathbf{a}$ is c.e. in $\mathbf{x}$, but $\iota(\mathbf{a})$ is not the least upper bound of a maximal $\mathcal{K}$-pair, one of whose elements is bounded by $\iota(\mathbf{x})$.

Nevertheless, the image of the c.e. degrees in the enumeration degrees is also first order definable. To see this, we use a special case of a result of Cai and Shore [3, Lemma 3.3].

Lemma 3.3 (Cai and Shore [3]). *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be Turing degrees. If* $\mathbf{b}$ *is 2-generic in* $\mathbf{a}$ *and* $\mathbf{a}$ *is not c.e., then* $\mathbf{a} \vee \mathbf{b}$ *is not c.e. in* $\mathbf{b}$.

Theorem 3.4. *The set* $\mathcal{CE} = \{\iota(\mathbf{a})\colon \mathbf{a} \text{ is a c.e. Turing degree}\}$ *is first order definable in* $\mathcal{D}_e$.

Proof. If $\mathbf{a}$ is c.e., then for every Turing degree $\mathbf{b}$ we have that $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$. In particular this is true for every Turing degree $\mathbf{b} \not\leq \mathbf{0}'$. If $\mathbf{b} \not\leq \mathbf{0}'$ then $\mathbf{a} \vee \mathbf{b}$ is not c.e. and we have a way to express that $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$ by a formula in $\mathcal{D}_e$.

In the other hand, if $\mathbf{a}$ is not c.e., then we apply Lemma 3.3—noting that 2-generic degrees are not below $\mathbf{0}'$—to see that there is a Turing degree $\mathbf{b} \not\leq \mathbf{0}'$, such that $\mathbf{a} \vee \mathbf{b}$ is not c.e. in $\mathbf{b}$.

Therefore, $\mathbf{x} \in \mathcal{CE}$ if and only if $\mathbf{x}$ is total and for every total $\mathbf{y} \not\leq \mathbf{0}_e'$ we have that $\mathbf{x} \vee \mathbf{y}$ is the least upper bound of a maximal $\mathcal{K}$-pair one of whose elements is bounded by $\mathbf{y}$. $\square$

Corollary 3.5. *The image of the relation "c.e. in" between Turing degrees is first order definable in* $\mathcal{D}_e$.

3.2. **Automorphisms.** Selman [21] showed that enumeration reducibility can be characterized in terms of the Turing degrees and the relation "c.e. in".

Theorem 3.6 (Selman [21])**.** *If $A, B \subseteq \omega$, then A is enumeration reducible to B if and only if $\{\mathbf{x} \in \mathcal{D}_T : A$ is c.e. in $\mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{D}_T : B$ is c.e. in $\mathbf{x}\}$.*

Translating Selman's Theorem to the language of enumeration reducibility (using part (1) of Proposition 1.2), we see that every enumeration degree $\mathbf{a}$ is characterized by the set of total degrees above it. A stronger version of this relationship was given by Rozinas [19], who showed that every enumeration degree is the greatest lower bound of two total enumeration degrees. In particular, the set of all total enumeration degrees is an automorphism base for the enumeration degrees.

Now consider an automorphism $\pi_e : \mathcal{D}_e \to \mathcal{D}_e$ of the enumeration degrees. We now know that the total enumeration degrees are a definable substructure of the enumeration degrees, thus π_e induces an automorphism of the Turing degrees, $\pi_T : \mathcal{D}_T \to \mathcal{D}_T$. In particular, $\pi_T(\mathbf{x}) = \iota^{-1}(\pi_e(\iota(\mathbf{x})))$, where $\iota : \mathcal{D}_T \to \mathcal{D}_e$ is the standard embedding of the Turing degrees into the enumeration degrees.

Definition 3.7. Call an automorphism of the Turing degrees π *enumeration induced* if there exists an automorphism of the enumeration degrees π_e such that for every Turing degree $\mathbf{x}$, $\pi(\mathbf{x}) = \iota^{-1}(\pi_e(\iota(\mathbf{x})))$.

As the set of total enumeration degrees forms an automorphism base for the enumeration degrees, every nontrivial automorphism of the enumeration degrees induces a nontrivial automorphism of the Turing degrees. A structure is called *rigid* if it has no nontrivial automorphisms. The existence of a nontrivial automorphism of the Turing degrees has been a central open question in degree theory. Slaman and Wooden [22] formulated the *Bi-interpretability Conjecture*, stating that the structure of the Turing degrees is bi-interpretable with Second Order Arithmetic, and proved that it is equivalent to the rigidity of the structure of the Turing degrees. Similarly, Soskova [27] showed that the rigidity of the structure of the enumeration degrees is equivalent to its own Bi-interpretability Conjecture. Now we see that the two conjectures are connected.

Corollary 3.8. *If the structure of the Turing degrees is rigid, then so is the structure of the enumeration degrees.*

Furthermore, we obtain more information about the behavior of an automorphism of the enumeration degrees. Ganchev and Soskov [26] showed that every automorphism of the enumeration degrees is the identity on the cone above $\mathbf{0}_e^{(4)}$. The definability of the automorphism base of the total enumeration degrees combined with the result of Slaman and Woodin [22] that every automorphism of the Turing degrees fixes the cone above $\mathbf{0}_T''$ allows us to improve the result of Ganchev and Soskov.

Corollary 3.9. *Every automorphism of the enumeration degrees is the identity on the cone above $\mathbf{0}_e''$.*

Proof. Let π_e be any automorphism of the enumeration degrees and let π_T be the induced automorphism of the Turing degrees. Fix any enumeration degree $\mathbf{x}$ above $\mathbf{0}_e''$. Let $\mathbf{a}$ and $\mathbf{b}$ be total enumeration degrees such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{x}$. As $\mathbf{a}$ is a total enumeration degree, we can represent $\pi_e(\mathbf{a})$ as $\iota(\pi_T(\iota^{-1}(\mathbf{a})))$. However, as $\mathbf{a} \geq \mathbf{x} \geq \mathbf{0}_e''$ it follows that $\iota^{-1}(\mathbf{a}) \geq \mathbf{0}_T''$ and hence by the result of Slaman and

Woodin that $\pi_T(\iota^{-1}(\mathbf{a})) = \iota^{-1}(\mathbf{a})$. From this, we conclude $\pi_e(\mathbf{a}) = \mathbf{a}$ and similarly that $\pi_e(\mathbf{b}) = \mathbf{b}$. Finally, $\pi_e(\mathbf{x}) = \pi_e(\mathbf{a} \wedge \mathbf{b}) = \pi_e(\mathbf{a}) \wedge \pi_e(\mathbf{b}) = \mathbf{a} \wedge \mathbf{b} = \mathbf{x}$. □

It is natural to ask if every automorphism of the Turing degrees can be extended to an automorphism of the enumeration degrees. A positive answer to this question would have important consequences. The automorphism groups of the Turing degrees and the enumeration degrees would be isomorphic. Furthermore, the relation "c.e. in" between Turing degrees would be invariant under automorphisms of the Turing degrees. Slaman and Woodin [22] showed that this is equivalent to the first order definability of this relation in the Turing degrees. Therefore, it makes sense to investigate the precise relationship between automorphisms of the enumeration degrees and automorphisms of the Turing degrees.

Fix an automorphism π_e of the enumeration degrees and let π_T be the induced automorphism of the Turing degrees. The automorphism π_e also induces a function $P\colon 2^\omega \to 2^\omega$ such that if A is a set of natural numbers and $\mathbf{x}$ is a Turing degree, then A is c.e. in $\mathbf{x}$ if and only if $P(A)$ is c.e. in $\pi_T(\mathbf{x})$. To see this, consider a presentation P of the automorphism π_e, i.e., a function from 2^ω to 2^ω such that $\pi_e(\mathbf{d}_e(A)) = \mathbf{d}_e(P(A))$ for every set A. (Soskova [27] showed that there is even an arithmetically definable such presentation.) Now let $\mathbf{x}$ be a Turing degree and $X \in \mathbf{x}$ a set. We know that A is c.e. in $\mathbf{x}$ if and only if $A \leq_e X \oplus \overline{X}$. As P preserves $\leq_e$, we have that $A \leq_e X \oplus \overline{X}$ if and only if $P(A) \leq_e P(X \oplus \overline{X})$. We know that $P(X \oplus \overline{X})$ has total enumeration degree, so there is a $Y \subseteq \omega$ such that $P(X \oplus \overline{X}) \equiv_e Y \oplus \overline{Y}$. Note that $\pi_T(\mathbf{x}) = \deg_T(Y)$, so $P(A) \leq_e Y \oplus \overline{Y}$ if and only if $P(A)$ is c.e. in $\pi_T(\mathbf{x})$.

Definition 3.10. Call an automorphism π of the Turing degrees *enumeration extendible* if there exists a function $P\colon 2^\omega \to 2^\omega$ such that for every set of natural numbers A and every Turing degree $\mathbf{x}$, we have that A is c.e. in $\mathbf{x}$ if and only if $P(A)$ is c.e. in $\pi(\mathbf{x})$.

From our discussion above we see that every enumeration induced automorphism of the Turing degrees is enumeration extendible. The next proposition shows that the converse is almost true.

Proposition 3.11. *Let $\pi\colon \mathcal{D}_T \to \mathcal{D}_T$ be an automorphism of the Turing degrees. If π and π^{-1} are enumeration extendible, then they are enumeration induced.*

Proof. Let $P, Q\colon 2^\omega \to 2^\omega$ be functions such that for every set of natural numbers A and Turing degree $\mathbf{x}$, A is c.e. in $\mathbf{x}$ if and only if $P(A)$ is c.e. in $\pi(\mathbf{x})$ if and only if $Q(A)$ is c.e. in $\pi^{-1}(\mathbf{x})$.

We show first that the function P preserves enumeration reducibility. Let A and B be sets of natural numbers such that $A \leq_e B$. Fix a Turing degree $\mathbf{x}$ such that $P(B)$ is c.e. in $\mathbf{x}$. It follows that B is c.e. in $\pi^{-1}(\mathbf{x})$. By Selman's theorem, A is c.e. in $\pi^{-1}(\mathbf{x})$, and hence $P(A)$ is c.e. in $\pi(\pi^{-1}(\mathbf{x})) = \mathbf{x}$. Thus we have that $\{\mathbf{x} \in \mathcal{D}_T\colon P(B) \text{ is c.e. in } \mathbf{x}\} \subseteq \{\mathbf{x} \in \mathcal{D}_T\colon P(A) \text{ is c.e. in } \mathbf{x}\}$, so by Selman's Theorem $P(A) \leq_e P(B)$. Similarly, Q preserves enumeration reducibility. Let $\pi_e(\mathbf{d}_e(A)) = \mathbf{d}_e(P(A))$. It follows that π_e is a well-defined, order preserving function on the enumeration degrees.

Next we show that for every set A, $A \equiv_e Q(P(A))$. Indeed, A is c.e. in $\mathbf{x}$ if and only if $P(A)$ is c.e. in $\pi(\mathbf{x})$ if and only if $Q(P(A))$ is c.e. in $\pi^{-1}(\pi(\mathbf{x})) = \mathbf{x}$. Similarly, for every set B, $B \equiv_e P(Q(B))$. From here it follows that π_e is onto and hence an automorphism of the enumeration degrees. Furthermore, $\pi_e^{-1}(\mathbf{d}_e(B)) = \mathbf{d}_e(Q(B))$.

Finally, fix a Turing degree $\mathbf{a}$ and $A \in \mathbf{a}$. We know that for every Turing degree $\mathbf{x}$, if $A \oplus \overline{A}$ is c.e. in $\mathbf{x}$ then $\mathbf{a} \leq \mathbf{x}$. From here it follows that $\pi(\mathbf{a})$ is the least Turing degree such that $P(A \oplus \overline{A})$ is c.e. in it. By Selman's theorem, it follows that $P(A \oplus \overline{A}) \in \iota(\pi(\mathbf{a}))$, hence $\pi(\mathbf{a}) = \iota^{-1}(\pi_e(\iota(\mathbf{a})))$. □

This leaves us with a new open question.

Question 3.12. Is every automorphism of the Turing degrees enumeration extendible?

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