Research Statement – Rafe Jones

My primary research interest is iteration of polynomials and rational functions over finite and p-adic fields. In studying these questions, I use tools from algebraic number theory, group theory, and probablility theory. In my dissertation I determine the density of the "hyperbolic Mandelbrot set" of $\overline{\mathbb{F}}_p$, that is

$$J_p = \{c \in \overline{\mathbb{F}}_p : 0 \text{ is purely periodic under iteration of } x^2 + c\}. \tag{1}$$

My main result is that

$$\lim_{k\to\infty} \frac{\# J_p \cap \mathbb{F}_{p^k}}{p^k} = 0. \tag{2}$$

This result has an application to the hyperbolic subset of the p-adic Mandelbrot set, whose complex analogue has been much studied [2, 4, 5, e.g.]. Much of the proof of (2) is an analysis of the Galois tower formed by the splitting fields of iterates of $y^2+x \in \mathbb{F}_p(x)[y]$. Similar towers have been studied recently by Morton [7], Odoni [8], and Aitken, Hajir, and Maire [1]. I introduce a stochastic process associated to any tower of Galois extensions, and show that the process associated to the tower mentioned above is a martingale. A martingale convergence theorem is then instrumental in proving (2). This method of proof appears to be highly unusual, and may well have applications to other density questions in number theory. Here I give an indication of the proof, describe the application of (2) to the p-adic Mandelbrot set, and discuss some directions for further research.

The proof of (2) begins with an examination of the inverse orbit of 0 under $f_c = x^2 + c$ for $c \in \overline{\mathbb{F}}_p(x)$. Denote the nth iterate of f_c by f_c^n. I show that 0 is periodic under f_c if and only if $f_c^n(x) = 0$ has at least one solution in $\mathbb{F}_p(c)$ for each n. I consider the numbers

$$d(n) = \lim_{k\to\infty} \frac{\#\{c \in \mathbb{F}_{p^k} : f_c^n(x) \text{ has at least one root in } \mathbb{F}_{p^k}\}}{p^k}$$

and show that $\lim\limits_{n\to\infty} d(n) = 0$ implies (2).

I next use the Tchebotarev Density Theorem for function fields to reduce the problem to Galois theory. Specifically, let $K = \mathbb{F}_p(x)$ and $f_x = y^2 + x \in K[y]$. Let K_n be the splitting field over K of f_x^n, and let G_n be the Galois group of this extension. Note that G_n acts on the roots $\mathcal{R}_n$ of f_x^n in the usual way. The Tchebotarev Density Theorem implies that

$$d(n) = \frac{\#\{g \in G_n : g \text{ fixes at least one element of } \mathcal{R}_n\}}{\# G_n}.$$

I analyze G_n through a study of $H_n = \mathrm{Gal}(K_n/K_{n-1})$, and prove two theorems about H_n. The first is that for any n, H_n contains an element with a certain "nice property." The second describes explicitly the structure of H_n when n is squarefree. I then apply these results by constructing a stochastic process associated to any tower of Galois extensions of a field. In the case of the Galois tower $K \subseteq K_1 \subseteq K_2 \subseteq \cdots$, we denote this process $(\Omega, \mathcal{F}, \mathbf{P}, X_n)$ by S, and show that

$$\mathbf{P}(X_n > 0) = d(n). \tag{3}$$

The first theorem on H_n mentioned above is used to show that S is a martingale. From the theory of martingales I obtain the convergence of the random variables X_n to a random variable X_∞. The second theorem on H_n mentioned above is then used to establish that $\mathbf{P}(X_\infty = 0) = 1$. Applying (3), this shows that $d(n) \to 0$, which proves (2).

This result has an application to the p-adic Mandelbrot Set, defined to be

$$M_p = \{c \in \mathbb{C}_p : 0 \text{ has a bounded orbit under } z^2 + c\},$$

where $\mathbb{C}_p$ is the smallest complete, algebraically closed extension of $\mathbb{Q}_p$. One can easily show that $M_p = \{|c| \leq 1\}$. However, the subset

$$A_p = \{c \in M_p : 0 \text{ is attracted to an attracting cycle of } z^2 + c\},$$

called the *hyperbolic subset* of M_p, is more interesting. The complex analogue of A_p has attracted much attention [2, 4, 5, e.g.], and is the subject of an important conjecture [6]. Let $\phi : \{|c| \leq 1\} \to \overline{\mathbb{F}}_p$ be the reduction homomorphism. One can show that A_p is the preimage under ϕ of the set J_p defined in (1). Thus (2) shows that in a certain sense the hyperbolic subset A_p of the p-adic Mandelbrot set has density 0. This is a sharp contrast from the complex case, where the hyperbolic subset is conjectured to be the interior of the Mandelbrot set [6].

This work presents several avenues for further research. One direction is to consider families of functions in $\overline{\mathbb{F}}_p(x)$ other than $x^2 + c$. For instance, the family $x^3 + c$ (assuming that $\mathbb{F}_p$ has three cube roots of unity) creates additional Galois-theoretic complexity, and it is not clear if the same argument works. Perhaps more interesting is the question of whether there exists a one-parameter family f_c of functions in $\overline{\mathbb{F}}_p(x)$ with a fixed critical point α such that

$$\{c \in \mathbb{F}_{p^k} : \alpha \text{ is purely periodic under iteration of } f_c\}$$

has positive density as $k \to \infty$.

Another direction for further research follows from considering $f_c = x^2 + c$ for a fixed c. If we take $c \in \mathbb{F}_p$, we can ask about the density as $k \to \infty$ of the set

$$\{\beta \in \mathbb{F}_{p^k} : \beta \text{ is purely periodic under } f_c\}.$$

Using the Tchebotarev Theorem for function fields, this boils down to a question about Galois groups similar to those considered in my thesis. If we take $c \in \mathbb{Z}$, we can consider the image of f_c in $\mathbb{F}_p[x]$ for various primes p. One natural question is to find the density of the set of primes p for which 0 is purely periodic under $f_c \in \mathbb{F}_p[x]$. More concretely, we can ask for the density of primes p for which 0 has at least one nth preimage in $\mathbb{F}_p$ under $f_c \in \mathbb{F}_p[x]$. A method similar to the one used in my thesis – this time using the Tchebotarev Theorem for number fields – seems promising in answering these questions.

The Galois-theoretic portion of my thesis appears to apply to fields other than $K = \mathbb{F}_p(x)$. If we put $K = \mathbb{Q}(x)$ then we can consider, for $a \in \mathbb{Z}$, specializations f_a of $f_x = y^2 + x \in \mathbb{Q}(x)[y]$. Values of a for which the Galois group of the nth iterate of f_a over $\mathbb{Q}$ is smaller than the corresponding group for f_x over $\mathbb{Q}(x)$ correspond to integral points on a certain hyperelliptic curve. It may thus be possible to use techniques from algebraic geometry to get further results. Moreover, my results imply that for most $a \in \mathbb{Z}$ a certain infinite set of elements exists in the Galois tower formed by all iterates of f_a. This builds on work of Odoni [8] and Stoll [9].

More generally, I plan to learn more about the theory of tree representations of Galois groups, which Aitken et al [1] and Boston [3] have proposed as a vehicle for studying the maximal extension of a number field unramified outside a finite set of primes. Such representations arise naturally from iterates of polynomials over number fields. I also plan to deepen my knowledge of group theory and probability theory with an eye to applying them to distribution problems in number theory and algebraic geometry. Finally, I plan to find applications of my results to the dynamics of more general p-adic rational functions.

References

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