1. Solve the following DEs:

   (a) [10 points] $\frac{dy}{dx} = y\frac{6x^2}{x^3+1}$.

$$\frac{dy}{y} = \frac{6x^2}{x^3+1}\,dx \qquad u = x^3+1, \quad du = 3x^2\,dx$$

$$\ln|y| = \int \frac{6x^2}{x^3+1}\,dx = 2\int \frac{du}{u} = 2\ln|u| + c$$

$$y = Cu^2 = C(x^3+1)^2$$

   (b) [10 points] $2x\frac{dy}{dx} = -y + 10\sqrt{x}$.

$$\frac{dy}{dx} + \frac{1}{2x}y = 5x^{-1/2} \qquad \rho(x) = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x} = x^{1/2}$$

$$\underbrace{x^{1/2}\frac{dy}{dx} + \frac{1}{2}x^{-1/2}y}_{\frac{d}{dx}(x^{1/2}y)} = 5$$

$$\Rightarrow yx^{1/2} = 5x + C$$

$$\Rightarrow \boxed{y = 5x^{1/2} + Cx^{-1/2}}$$

2. [12 points] The time rate of change of an anteater population P is proportional to $1/P^2$. There are 1 trillion anteaters initially (this is not an anteater population to be trifled with) and 4 trillion after seven months. How many trillions will there be after 11 months (you may leave your answer unsimplified)? What happens in the long run?

$$\frac{dP}{dt} = k\frac{1}{P^2}$$

$P(0) = 1$ (trillion)

$P(7) = 4$

$P(11) = ?$

$$\int P^2\,dP = \int k\,dt$$

$$\frac{P^3}{3} = kt + C$$

$$P(t) = (3kt + C)^{1/3}$$

$$1 = P(0) = (C)^{1/3} \Rightarrow C = 1$$

$$4 = P(7) = (21k + 1)^{1/3}$$

$$\Rightarrow 21k + 1 = 64 \Rightarrow k = 3$$

$$\boxed{P(t) = (9t + 1)^{1/3}} \longrightarrow \boxed{\lim_{t\to\infty} P(t) = \infty}$$

$$\cancel{P(7) = (64)^{1/3}}$$

$$P(11) = \boxed{(100)^{1/3} \text{ trillion}}$$

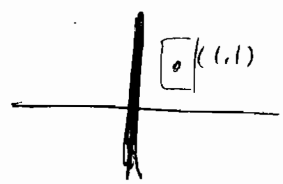
3. Consider the initial value problem

$$x\frac{dy}{dx} = 2y, \qquad y(1) = 1$$

(a) [10 points] Does there exist a unique solution in some interval containing $x = 1$?

$$\frac{dy}{dx} = \frac{2y}{x} = f(x,y) \qquad f_y(x,y) = \frac{2}{x}$$

both f and f_y are continuous everywhere except along $x=0$



$\Rightarrow$ Existence & Uniqueness Theorem applies

$\boxed{\text{YES}}$

(b) [8 points]Does there exist a unique solution on the interval $(-\infty, \infty)$, that is, defined for all x? [Hint: one such solution is

$$y(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0. \end{cases}$$

Can you find another? If so, be sure to verify that it's a solution.]

Check $y(x) = x^2$ is a solution: $y(1) = 1^2 = 1$ ✓

and $x\frac{dy}{dx} = x(2x) = 2x^2 = 2y$ ✓

$\Rightarrow$ $\boxed{\text{NO}}$

4. [12 points] For

$$\frac{dx}{dt} = x^2(9 - x^2)$$

classify all equilibrium solutions and sketch roughly a slope field and some solution curves. Then determine from your sketch the asymptotic behavior (as $t \to \infty$) for the solution with $x(0) = 2$.

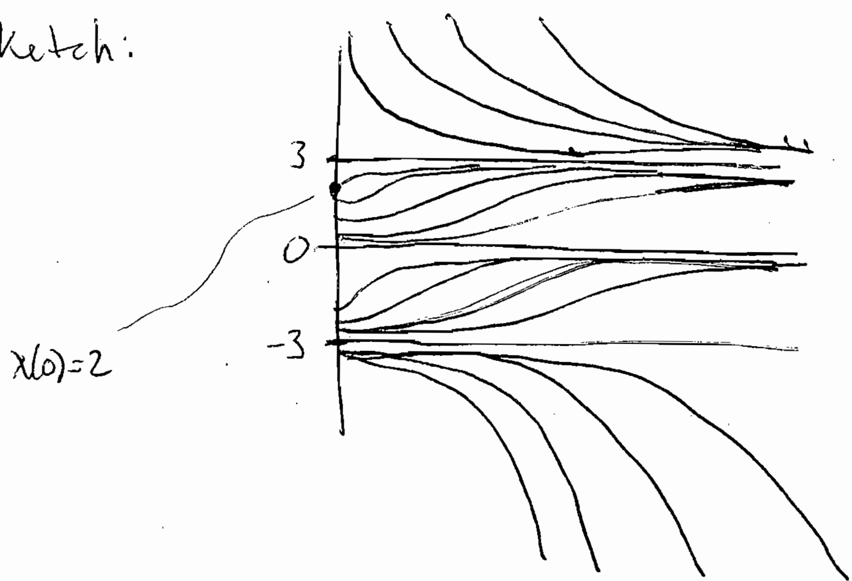
eq. solns are roots of $x^2(9-x^2)$, i.e.

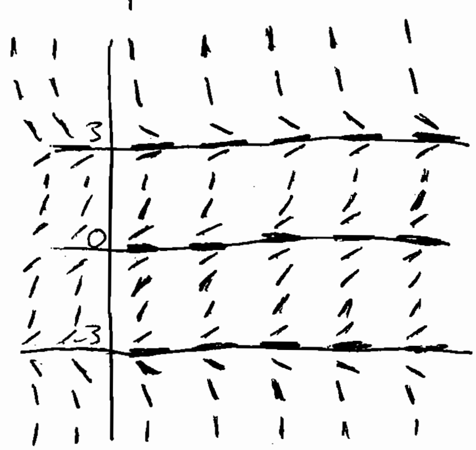
$x = 0, \quad x = \pm 3$

Phase diagram: $- \quad -3 \quad + \quad 0 \quad + \quad 3 \quad -$

slope field:

Sketch:





Soln with $x(0)=2$ approaches the eq. soln 3 as $t \to \infty$

5. (a) [10 points] Use Euler's method to approximate y(0.2), where y is the solution to the initial value problem $y' = 1 + \dfrac{3}{x+y}$, $y(0) = 1$. Use step size 0.1.

$x_0 = 0 \quad y_0 = 1$

$x_1 = 0.1 \quad y_1 = y_0 + f(x_0, y_0) \cdot 0.1 = 1 + \left(1 + \frac{3}{0+1}\right) \cdot 0.1$

$= 1 + 0.4$

$= 1.4$

$x_2 = 0.2 \quad y_2 = 1.4 + \left(1 + \frac{3}{0.1+1.4}\right) 0.1 = 1.4 + \left(1 + \frac{3}{1.5}\right) \cdot 0.1$

$= 1.4 + (3) \cdot 0.1$

$= 1.7$

(b) [3 points] Suppose now that y is the solution to the initial value problem $y' = 2$, $y(0) = 5$. Consider using Euler's method with step size 0.1 and 0.0001 to approximate $y(1)$ (but don't actually carry out these approximations). Circle the statement that's true:

i. Euler's method with step size 0.1 gives you a better approximation to $y(1)$.
ii. Euler's method with step size 0.0001 gives you a better approximation to $y(1)$.
iii. Euler's method with either step size gives you the same quality approximation to $y(1)$.

Since $y' = 2$, all solution curves are straight lines. Since Euler's method uses straight lines to approximate solns, any step size will yield the exact answer.

6. [15 points] Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, and $C = AB$.

   Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$.

   Find all solutions to the system of equations corresponding to the matrix equation $C\vec{x} = \vec{b}$.

$$C = AB = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

Augmented matrix for system $C\vec{x} = \vec{b}$ is

$$\begin{bmatrix} -1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ -2 & -1 & 0 & 4 \end{bmatrix} \xrightarrow{(-2)R_1 + R_3} \begin{bmatrix} -1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & -2 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} -1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-x_1 + x_3 = 3$

$x_2 + 2x_3 = 2$

x_3 is free var.

$x_3 = t$

$x_2 = 2 - 2t$

$-x_1 + t = 3$, so $x_1 = t - 3$

first 3 col.s

Note: Since the Red. Ech. Form of the coeff. matrix C has a row of 0s, **C is not invertible.** So trying to solve $C\vec{x} = \vec{b}$ by finding C^{-1} and multiplying through by it won't work, since there is no C^{-1}. When solving equations of the form $C\vec{x} = \vec{b}$, use the method above, which will always work and is no more effort than finding C^{-1}.

7. [10 points] Find the inverse of $A = \begin{bmatrix} 1 & 11 & 2 \\ 3 & 4 & 4 \\ 4 & 5 & 5 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 11 & 2 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 & 1 & 0 \\ 4 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 11 & 2 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - R_3]{R_2 - 3R_3} \left[\begin{array}{ccc|ccc} 0 & 10 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 4 & -3 \\ 1 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[-10R_2 + R_1]{-R_2 + R_3} \left[\begin{array}{ccc|ccc} 0 & 0 & -9 & 1 & -39 & 29 \\ 0 & 1 & 1 & 0 & 4 & -3 \\ 1 & 0 & 0 & 0 & -5 & 4 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{9}R_1]{\text{Rearrange rows}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -5 & 4 \\ 0 & 1 & 1 & 0 & 4 & -3 \\ 0 & 0 & 1 & -1/9 & 13/3 & -29/9 \end{array}\right]$$

$$\xrightarrow{-R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -5 & 4 \\ 0 & 1 & 0 & 1/9 & -1/3 & 2/9 \\ 0 & 0 & 1 & -1/9 & 13/3 & -29/9 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & -5 & 4 \\ 1/9 & -1/3 & 2/9 \\ -1/9 & 13/3 & -29/9 \end{bmatrix}$$