

Math 748 Final Exam

Due Saturday, December 16, at 2 pm in Box 96

You must work alone on these problems. You may use your class notes, homeworks, and anything in Milne, but do not use any other references.

1. [25 points] Determine the class group (including structure as an abelian group) of $\mathbb{Q}(\sqrt{-21})$. [Possibly useful general fact: if $\mathfrak{a} = (\alpha, \beta)$ then $\mathfrak{a}^2 = (\alpha^2, \alpha\beta, \beta^2)$.]
 2. Let $K = \mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5th root of unity.
 - (a) [13 points] Let $u = -\zeta_5^2(1 + \zeta_5)$. Show that u is a unit, that u is a real number between 1 and 2, and that $K \cap \mathbb{R} = \mathbb{Q}(\sqrt{5})$. Deduce that $u = (1 + \sqrt{5})/2$.
 - (b) [12 points] Use the previous part to give an explicit description of all the units in O_K .
 3. [25 points] Find the density of the set of primes p that are both equivalent to 3 modulo 4 and have the property that 2 is a cube modulo p . Prove that your answer is correct. [Notes: You may take for granted that an irreducible cubic polynomial over \mathbb{Z} has Galois group S_3 if its discriminant is not a square, that $\text{Gal}(KL/\mathbb{Q})$ is the subgroup of $\text{Gal}(K/\mathbb{Q}) \times \text{Gal}(L/\mathbb{Q})$ consisting of (g_1, g_2) where g_1, g_2 agree on $K \cap L$ (when all extensions are Galois), and that $\sqrt{-3}$ is contained in the splitting field of $x^3 - 2$ over \mathbb{Q} .]
 4. Let α be a root of $x^3 - 7x^2 + 14x - 7$, which has discriminant 49, and let $K = \mathbb{Q}(\alpha)$ have ring of integers O_K . Note K/\mathbb{Q} is Galois since 49 is a square.
 - (a) [5 points] Show that $O_K = \mathbb{Z}[\alpha]$ and that O_K has class number 1.
 - (b) [12 points] Find the prime decomposition of $7O_K$. Use this to show that if $\beta \in K$ and $(\beta) + (7) = O_K$, then $N_{K/\mathbb{Q}}(\beta) \equiv \pm 1 \pmod{7}$. [Hint: use the fact that K/\mathbb{Q} is Galois.]
 - (c) [8 points] Deduce that if $p \not\equiv \pm 1 \pmod{7}$ then pO_K is prime. On the other hand, find the density of the set of p such that pO_K is prime and $p \equiv \pm 1 \pmod{7}$ (you don't need to prove the density is correct, only say what it is and give a brief justification).
- B1 (*Bonus Problem, 5 points, partial credit awarded. This is a lot harder than the others, so only spend time on it if you feel good about your other answers.*) Let S be the set of prime numbers for which $1/p$, when developed as a decimal fraction, has an odd period length. For example, one has $37 \in S$ because $1/37 = 0.027027027 \dots$, and $7 \notin S$ because $1/7 = 0.142857142857 \dots$. Prove that S has a density, and compute it. [Hint: one approach is to consider the cases $p \equiv 1 + 2^{n-1} \pmod{2^n}$ separately for each $n \geq 2$.]
- B2 (*Super Ultra Triple-Star Bonus Problem*¹) Solve the bonus problem using a dynamical approach. [Hint: periodicity of $0 \pmod{p}$ under iteration of $(x + 10)^2 - 10$.]

¹This doesn't mean you actually get any points for doing this, only lots of good karma that will doubtless accompany you well into the next life.