Math 748 Homework 11, Part 1

Due Wednesday, November 29

1. Let $L = \mathbb{Q}(\zeta_n)$, and consider the subfield $K = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

   (a) Show that K is the fixed field of complex conjugation (which is an element of $\mathrm{Gal}(L/\mathbb{Q})$. (This implies that K is the maximal real subfield of L.)

   (b) Show that $O_K = \mathbb{Z}[\zeta_n + \zeta_n^{-1}]$. [Hint: put $\omega = \zeta_n + \zeta_n^{-1}$. One method is to find an integral basis for O_L that has lots of elements fixed by complex conjugation. Be sure to prove that your new integral basis is indeed an integral basis – for this you'll want to take the determinant of the appropriate matrix.]

2. Consider the number field $K = Q(\sqrt[3]{2})$, which has discriminant $-2^2 \cdot 3^3$. Its splitting field L is a Galois extension of degree 6 over $\mathbb{Q}$, and can be constructed by adjoining to $\mathbb{Q}$ a root of the polynomial $x^6 + 108$, which has discriminant $-2^{16} \cdot 3^{21}$. For $p = 5, 7, 11, 31$, find the number of $\mathfrak{p} \subset O_K$ and $\mathfrak{P} \subset O_L$ lying over p and also the residue class degrees of each $\mathfrak{p} \subset O_K$ and $\mathfrak{P} \subset O_L$. (You'll probably want to use Magma for this, but only use it to factor polynomials over finite fields). Why can't pO_L ever be inert?

   For a bonus point, find the statistics of the factorization (including ramification degrees) of $2O_L$ and $3O_L$.