1 Twists

Let us denote complexes by vectors $y_1, y_2, \ldots$. Let $N$ = number of species.
All reaction rates $\kappa(t)$'s satisfy $\epsilon < \kappa(t) < 1/\epsilon$ for some $\epsilon > 0$. Tiers $\{T_i\}$ is always with respect to $f(x, y) = x^y$.

**Definition 1.1.** For two linkage classes $L_1$ and $L_2$ of a reaction network, $L_1$ is $\infty$-dominated by $L_2$ if there exist $y_1, y_2 \in L_1$ and $y'_1 \in L_2$ such that $y'_1 - y_1 = ay_2$ for some $a \in \mathbb{Q}_{>0}$

**Definition 1.2.** For two linkage classes $L_1$ and $L_2$ of a reaction network, $L_1$ and $L_2$ are twisted if $L_1$ is $\infty$-dominated by $L_2$ and $L_2$ is $\infty$-dominated by $L_1$.

**Definition 1.3.** Let $(S, C, R, K(t))$ be a reaction network. $x_n \in \mathbb{R}^N_{>0}$ is called a $C_2$-sequence of $(S, C, R, K(t))$ if

1. $\lim_{n \to \infty} x_{n,i} = \{\infty, 0\}$ for at least one $i = 1, 2, \ldots, N$
2. $T_1$ is a union of linkage classes.

If $x_n \in \mathbb{R}^N_{>0}$ satisfies

1. $\lim_{n \to \infty} x_{n,i} = \infty$ for at least one $i = 1, 2, \ldots, N$
2. $T_1$ is a union of linkage classes.
3. There exists at least one $T_i$ for some $i \geq 2$ that is not a union of linkage classes.

then it is called $C_2'$-sequence.

**Example 1.1.**

\[
A \Rightarrow A + B \Rightarrow C
\]
\[
2A \Rightarrow B
\]

The first linkage class $L_1$ contains $y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $y_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and the second one $L_2$ contains $y'_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $y'_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Then it holds that $y'_1 - y_1 = y_1$. Therefore $L_1$ is $\infty$-dominated by $L_2$. In addition, $y_2 - y'_2 = \frac{1}{2} y'_1$. So, $L_2$ is $\infty$-dominated by $L_1$ as well. Therefore $L_1$ and $L_2$ are twisted.

**Proposition 1.1.** Let $L_1$ and $L_2$ be different linkage classes of a weakly reversible reaction network. Let $x_n$ be a $C_2'$-sequence and $\{T_i\}$ be tiers along with $x_n$. Suppose that $L_1$ is $\infty$-dominated by $L_2$. Then,

1) If $\lim_{n \to \infty} x_{n,y} = \infty$ for some $y \in L_1$, then $L_1 \not\subseteq T_1$.

2) If $\lim_{n \to \infty} x_{n,y'} = 0$ for some $y' \in L_2$, then $L_2 \not\subseteq T_1$.

Note that $T_1$ along with $x_n$ is a union of linkage classes.
Proof. Let \( y_1, y_2 \in L_1 \) and \( y'_1 \in L_2 \) satisfy \( y'_1 - y_1 = ay_2 \) for some \( a \in \mathbb{Q}_{>0} \). Suppose \( L_1 \subset T_1 \) for some \( C2\)-sequence \( x_n \) such that \( \lim_{n \to \infty} x_n^{y_2} = \infty \) for some \( y \in L_1 \). Then \( \lim_{n \to \infty} x_n^{y_1} = \infty \) for all \( y \in L_1 \) because \( T_1 \) equals to a union of linkage classes. Then \( \lim_{n \to \infty} x_n^{y_1} = \infty \). However, \( \lim_{n \to \infty} x_n^{y'_1 - y_1} < \infty \). It is contradiction.

Now suppose \( L_2 \subset T_1 \) for some \( C2\)-sequence \( x_n \) such that \( \lim_{n \to \infty} x_n^{y'_1} = 0 \) for some \( y' \in L_2 \). Since \( T_1 \) is a union of linkage classes, \( \lim_{n \to \infty} x_n^{y'_1} = 0 \) for all \( y' \in L_2 \). We also have \( \lim_{n \to \infty} x_n^y = 0 \) for all \( y \in L_1 \).

Therefore \( C = \lim_{n \to \infty} x_n^{y'_1 - y_1} = \lim_{n \to \infty} x_n^{a_1 y_2} = 0 \) for some \( C \in (0, \infty] \). This is contradiction. \( \square \)

**Corollary 1.2.** Let \( x_n \) be a \( C2\)-sequence and \( \{T_i\}_i \) be tiers along with \( x_n \). Suppose \( L_1 \) and \( L_2 \) are twisted. Then,

1) If \( L_1 \subset T_1 \), then \( 0 < \lim_{n \to \infty} x_n^y < \infty \) for all \( y \in L_1 \)

2) If \( L_2 \subset T_1 \), then \( 0 < \lim_{n \to \infty} x_n^y < \infty \) for all \( y \in L_2 \)

**Proof.** By the symmetry of the definition of twist, we only need to prove 1).

Suppose 1) is false. Then \( L_1 \subset T_1 \) and \( \lim \inf x_n^y = 0 \) or \( \lim \sup x_n^y = \infty \) for some \( y \in L_1 \). We can choose subsequence \( x_{n_k} \) of \( x_n \) such that \( \lim k \to \infty x_n^{y_k} = \{ 1, \infty \} \). Since \( x_{n_k} \) is a sub-sequence of \( x_n \),

\[
\lim_{n \to \infty} \frac{x_n^{y_1}}{x_n^{y_2}} = \lim_{k \to \infty} \frac{x_{n_{k}}^{y_1}}{x_{n_{k}}^{y_2}} \quad \text{for all complexes} \quad y_1, y_2.
\]

This means tiers structure with \( x_{n_k} \) is same to that with \( x_n \). Therefore \( L_1 \subset T_1 \) along with \( x_{n_k} \). It is, however, contradiction by Corollary 1.2. \( \square \)

**Corollary 1.3.** Let \( \{T_i\}_{i=1}^m \) be tiers along with a \( C2\)-sequence \( x_n \). If \( L_1 \) and \( L_2 \) are twisted, either \( (L_1 \cup L_2) \cap T_1 = \emptyset \) or \( (L_1 \cup L_2) \subset T_1 \) holds.

**Proof.** There exist complexes \( y_1, y_2 \in L_1 \) and \( y'_1, y'_2 \in L_2 \) such that \( y'_1 - y_1 = ay_2 \) and \( y'_1 - y_1 = by'_2 \) for some \( a, b \in \mathbb{Q}_{>0} \).

Case 1) \( L_1 \cap T_1 \neq \emptyset \)

This assumption implies \( 0 < \lim_{n \to \infty} x_n^y < \infty \) for all \( y \in L_1 \) by Corollary 1.2. If \( L_2 \cap T_1 = \emptyset \), then

\[
0 = \lim_{n \to \infty} x_n^{y'_1 - y_1} = \lim_{n \to \infty} x_n^{a_1 y_2} > 0
\]

This is a contradiction. Therefore \( L_2 \cap T_1 = \emptyset \). This implies \( (L_1 \cup L_2) \subset T_1 \)

Case 2) \( L_1 \cap T_1 = \emptyset \)

If \( L_2 \cap T_1 \neq \emptyset \), then \( 0 < \lim_{n \to \infty} x_n^{y'_1} < \infty \) for all \( y' \in L_2 \) by Corollary 1.2. Then

\[
0 = \lim_{n \to \infty} x_n^{y'_1 - y'_1} = \lim_{n \to \infty} x_n^{a_1 y'_2} > 0
\]

This is a contradiction. Therefore \( L_2 \cap T_1 = \emptyset \). This implies \( (L_1 \cup L_2) \cap T_1 = \emptyset \).

\( \square \)

**Remark 1.1.** Corollary 1.2 and Corollary 1.3 also hold with \( C2'\)-sequence.
Lemma 1.4. Let \((S, C, R, K(t))\) be reaction network and \(D\) be a non-empty positive stoichiometric compatibility class. Let \(x_n\) be a \(C^2\)-sequence with tiers \(\{T_i\}_i\). Then, if every tier \(T_i\) is a union of linkage classes for any \(C^2\)-sequence \(x_n \in D\) of \((S, C, R, K(t))\), the reaction network has only bounded trajectories.

Proof. We show \(C1\) condition with \(D\) of Lemma 3.10 in Anderson(2011) [1] must hold. This implies every trajectory is bounded by same proof of Theorem 3.12 in [1]. Suppose \(C1\) does not hold. Then \(C2\) holds by Lemma 3.10 in [1], so there exists a \(C^2\)-sequence \(x_n \in D\) of \((S, C, R, K(t))\). By theorem 3.9 in [1], there is a conservation relation \(\omega \in \mathbb{R}^N\) with respect to \((U, V, \{T_i\})\) where \(U = \{ i \colon \lim_{n \to \infty} x_{n,i} = 0 \}\) and \(V = \{ i \colon \lim_{n \to \infty} x_{n,i} = \infty \}\). Note that, if \(V\) is non-empty, \(\omega \cdot w_n = \infty\). Furthermore, if \(V\) is empty, then \(U\) is non-empty. This implies \(\lim_{n \to \infty} \omega \cdot x_n = 0\). Since every tier is equal to union of linkage classes, \(\omega\) is perpendicular to all reaction vectors. Therefore \(\omega \cdot x_n\) is constant, so \(V\) must be empty. Then, \(\omega \cdot x_n = 0\) for all \(n\). But, by the definition of \(\omega\) and \(x_n \in \mathbb{R}^N\), \(\omega \cdot x_n > 0\). Contradiction. Therefore, \(C1\) must holds. This means all trajectories are bounded. \(\square\)

Proposition 1.5. Let \((S, C, R, K(t))\) be a two linkage classes, weakly reversible, reaction network. If two linkage classes \(L_1\) and \(L_2\) are twisted, then every trajectory of this reaction network is bounded.

Proof. Let \(D\) be a positive stoichiometric compatibility class and \(x_n\) be a \(C^2\)-sequence of \((S, C, R, K(t))\) with tiers \(\{T_i\}_i\). We show \((L_1 \cup L_2) = T_1\). By Corollary 1.3, either \(L_1 \cup L_2 = T_1\) or \(L_1 \cup L_2 \not\subseteq T_1\) holds. If \(L_1 \cup L_2 \not\subseteq T_1\), then \(T_1 = \emptyset\). That is contradiction to the definition of tiers. Therefore \(L_1 \cup L_2 = T_1\) for any \(C^2\)-sequence. The conclusion follows by Lemma 1.4. \(\square\)

Furthermore, we can prove persistence of twisted two linkage weakly reversible reaction network.

Theorem 1.6. Let \((S, C, R, K(t))\) be a weakly reversible reaction network that has only two linkage classes. If the two linkage classes \(L_1\) and \(L_2\) are twisted, then the reaction network is persistent.

Proof. Let \(D\) be a positive stoichiometric compatibility class and \(x_0 \in D\). We prove the trajectory \(x(t) = \phi(t,x_0)\) does not approach the boundary \(\partial \mathbb{R}^N_{>0}\). More precisely,

\[
\liminf_{n \to \infty} \phi_i(t,x_0) > 0 \quad \text{for all} \quad i \in \{1, \ldots, N\}
\]

To show that, we prove the \(C1\) condition of Lemma 4.7 in [3] holds for this reaction network with \(\phi(t,x_0)\). This \(C1\) is different from \(C1\) of Lemma 3.10 in [1] that is used in Lemma 1.4. If this \(C1\) condition is true, \(\omega\)-limit of trajectory is a single point by Lemma 4.9 in [3]. This implies a persistence of this network by Theorem 4.10 in [3]. To show \(C1\) condition, let’s assume \(C2\) of Lemma 4.7 in [3] holds. Then there exists a sequence \(x_n = \phi(t_n,x_0) \in \mathbb{R}^N_{>0}\) for some \(t_i\’s\) of which \(T_j\) is a union of linkage classes. Furthermore, \(\lim_{n \to \infty} x_{n,i} = 0\) for at least one \(i = 1, \ldots, N\). Therefore this sequence is \(C^2\)-sequence. As the proof of Proposition 1.5, \(L_1 \cup L_2 = T_1\) by Corollary 1.3. Then there exists conservation relation \(\omega\) with respect to the triple \((U, V, \{T_i\})\) by Theorem 3.9 in [1]. Where \(U = \{ i \colon \lim_{n \to \infty} x_{n,i} = 0 \}\) and \(V = \{ i \colon \lim_{n \to \infty} x_{n,i} = \infty \}\).

Note that since every trajectory is bounded, \(V = \emptyset\). Therefore, \(w_i > 0\) for all \(i \in U\) and \(w_i = 0\) otherwise. By the definition of conservation relation, \(\omega \cdot x_n = C\) for some constant \(C\). Since \(\lim_{n \to \infty} \omega \cdot x_n = 0\), \(C\) must be 0. However, \(\omega \cdot x_n > 0\) because of \(x_n \in \mathbb{R}^N_{>0}\). This is contradiction. Therefore \(C2\) cannot hold and \(C1\) must hold. \(\square\)

Definition 1.4. For two linkage classes \(L_1\) and \(L_2\) of a reaction network, \(L_1\) and \(L_2\) are completely twisted if there exist complexes \(y_1, y_2 \in L_1\) and \(y_1', y_2' \in L_2\) such that \(y_1 = ac_i\), \(y_1' = ce_i\) for some strictly positive integers \(a > c\) and \(y_2 = be_j\), \(y_2' = de_j\) for some strictly positive integers \(b < d\). Where \(i \neq j\), \(i, j \in \{1, 2, \ldots, N\}\) and \(e_i\) is a a vector in \(\mathbb{R}^N\) whose \(i\)-th coordinate is one and the others are zero.
Remark 1.2. Note that if $L_1$ and $L_2$ are completely twisted, then $L_1$ is $\infty$-dominated by $L_2$ because $y_2 - y_2' = -(d - b)e_2 = -\frac{d - b}{a}y_2$. In same way, $L_2$ is $\infty$-dominated by $L_1$. Therefore $L_1$ and $L_2$ are twisted.

Now, let $(S, C, R, K(t))$ be a weakly reversible reaction binary network which has 3 species and completely twisted linkage classes. We show this reaction network only has bounded trajectories. In order to show that following three lemmas are needed.

Lemma 1.7. Let $(S, C, R, K(t))$ be a 3 species binary, weakly reversible reaction network with more that or equal to three linkage classes. Assume that there exists at least one $C2'$-sequence of this reaction network. If there are two linkage classes $L_1, L_2$ that are completely twisted, then for any $C2'$-sequence $x_n$ with tiers $\{T_i\}$, there exists a linkage classes $L_3$ such that $L_3 = T_1$.

Proof. In 3 species binary reaction network, completely twisted two linkages classes only come with one of following cases
1) $\{A, 2B\} \in L_1 \text{ and } \{2A, B\} \in L_2$
2) $\{A, 2C\} \in L_1 \text{ and } \{2C, A\} \in L_2$
3) $\{C, 2B\} \in L_1 \text{ and } \{2B, C\} \in L_2$

With out loss of generality, we assume there are two linkage classes $L_1, L_2$ such that $\{A, 2B\} \in L_1$ and $\{2A, B\} \in L_2$. Let $D$ be a stoichiometric compatibility class.

Assume, there exists a $C2'$-sequence $x_n'$ with tiers $\{T_i\}$ such that $L_1 \cup L_2 \subseteq T_1$. Then $a_n, b_n \sim 1$ by the Corollary 1.2. Then $c_n \to \infty$ because of the first property of $C2'$-sequence. If at least one of $C, 2C, A + C$ and $B + C$ is involved in this reaction network, then $(L_1 \cup L_2) \not\subseteq T_1$ because $c_n \to \infty$. This means that the species $C$ is not involved in this reaction network. This is contradiction to $c_n \to \infty$.

That is, for any $C2'$-sequence $x_n$ with tiers $\{T_i\}$, $(L_1 \cup L_2) \not\subseteq T_1$. Then $(L_1 \cup L_2) \cap T_1 = \emptyset$.

i) If $A + B \in T_1$, then $a_n b_n \gg b_n^2, a_n^2$ because $2A, 2B \not\in T_1$. Therefore $a_n \gg b_n$ and $b_n \gg a_n$. Contradiction.

ii) If $2C$ and $B + C \in T_1$, then $c_n \sim b_n, b_n c_n \gg b_n^2$. Therefore, $c_n \sim b_n$ and $c_n \gg b_n$. Contradiction.

iii) If $2C$ and $A + C \in T_1$, then $c_n \sim a_n, a_n c_n \gg a_n$. Therefore $c_n \sim a_n$ and $c_n \gg a_n$. Contradiction.

iv) If $2C$ and $C \in T_1$, then $c_n \sim c_n$. Therefore $0 < \lim_{n \to \infty} c_n < \infty$. If $a_n \to \infty$ or $b_n \to \infty$ as $n \to \infty$, $c_n \to \infty$ as $n \to \infty$ because $A, B \in (L_1 \cup L_2)$ and $(L_1 \cup L_2) \cap T_1 = \emptyset$. Therefore $\lim_{n \to \infty} a_n, b_n < \infty$.

This contradicts to the first property of $C2'$-sequence.

v) If $\emptyset$ and $2C \in T_1, c_n \sim 1$. Therefore either $a_n \to \infty$ or $b_n \to \infty$. This implies $2A \in T_1$ or $2B \in T_1$.

Thus $2C \not\in T_1$. Contradiction.

vi) The case of $\emptyset$ and $C \in T_1$ is impossible by exactly same reasoning of v).

From i) to vi), we know that $2C \not\in T_1$ and $A + B \not\in T_1$. So the only possible case is that complexes of $T_1$ only come from $\{C, A + C, B + C, \emptyset\}$ except the case of $\emptyset$ and $C \in T_1$.

Now let’s check which linkage classes always equals to $T_1$.

i') Suppose there is only one linkage class $L_3$ that contains some of 3 complexes among $\{C, A + C, B + C, \emptyset\}$ except the case of $L_3$ containing $C$ and $\emptyset$. Since $T_1$ is a union of linkage classes and by i) ~ vi), $T_1 = L_3$. 

4
ii') Suppose there is a linkage class $L_3$ whose complexes are $C, A + C$. If $\lim_{n \to \infty} c_n < \infty$, then $\lim_{n \to \infty} a_n < \infty$. Then, since $B \not\in T_1$, $\lim_{n \to \infty} b_n < \infty$. This is contradiction to the definition of $C2'$-sequence. Therefore $\lim_{n \to \infty} c_n = \infty$. If there is another linkage class $L_4$ such that $L_4 \subseteq T_1$, then $L_4$ must contain either $\emptyset$ or $B + C$. However this is impossible by iv) and the fact that $T_1$ is a union of linkage classes. Thus $L_3 = T_1$. If there is a linkage class $L_3$ whose complexes are $C, B + C$, then $L_3 = T_1$ by same reasoning.

iii') Suppose there exists a linkage class $L_3$ whose complexes are $A + C, \emptyset$ and there is no linkage class whose complexes are $C, B + C$. Then the only possible linkage case is $L_3 = T_1$. For the case of $\emptyset$ and $B + C$, $L_3 = T_1$ by same reasoning.

iv') Suppose there exists a linkage class $L_4$ which contains $C, \emptyset$. If there is a linkage class $L_3$ whose complexes are $A + C, B + C$, then $L_3 = T_1$ by iv).

v') Suppose there exist a linkage class $L_4$ which contains $C, \emptyset$. If there is no linkage class whose complexes are $A + C, B + C$, then no $C2'$-sequence exists by i) $\sim$ vi). Therefore, this case is ruled out by our hypothesis.

vi') In the rest cases, there is no linkage classes whose complexes are only some of $\{C, A + C, B + C, \emptyset\}$. Then there is no $C2'$-sequence by i) $\sim$ vi) and the fact that $T_1$ is a union of linkage classes.

Since we assume that there exists at least one $C2'$-sequence for given system, the only possible cases are the case of i'), ii'), iii') and iv'). We can check that the deficiency of $L_3 = T_1$ for these four cases is always zero.

\[\square\]

**Definition 1.5.** Let $y, y'$ be complexes. We call $y$ is larger than $y'$ along a sequence $x_n \in \mathbb{R}_{\geq 0}^N$ if only if $\lim_{n \to \infty} x_n^y > \lim_{n \to \infty} x_n^{y'}$. In similar way, we define "smaller than" and "equal to".

**Lemma 1.8.** Let $(S, C, R, K(t))$ be a 3 species binary, weakly reversible reaction network with more that or equal to three linkage classes. Assume that there exists at least one $C2'$-sequence of this reaction network. If there are two linkage classes $L_1, L_2$ that are completely twisted, then for any $C2'$-sequence $x_n$ with tiers $\{T_i\}_i$, $T_2$ is not a union of linkage classes.

**Proof.** We also assume $\{A, 2B\} \in L_1$ and $\{B, 2A\} \in L_2$. By the Lemma 1.7, for a $C2'$-sequence $x_n$ with tiers $\{T_i\}_i$, there is linkage class $L_3$ which is $T_1$. As we showed in the proof of Lemma 1.7, complexes of $T_1 = L_3$ is a proper subset of $\{C, A + C, B + C, \emptyset\}$ because $\{C, \emptyset\} \not\subseteq T_1$. Let $x_n = (a_n, b_n, c_n)^T$. If $c_n \not\to \infty$, $a_n \to \infty$ or $b_n \to \infty$. Without loss of generality, we assume $a_n \to \infty$. Then, the complex $2A$ is largest complex along $x_n$. Therefore, $2A \in T_1$. This is contradiction. Therefore, $c_n \to \infty$.

i) Suppose $T_1 = \{A + C, B + C, \emptyset\}$. Since $A$ and $B$ are not in $T_1$ and $\emptyset \in T_1$, $a_n, b_n \not\to \infty$. Therefore $c_n \to \infty$. Moreover $\lim_{n \to \infty} \frac{a_n c_n}{b_n c_n} = c$ for some constant $c$. That is, $a_n \sim b_n$. In addition, since $A + C, \emptyset \in T_1$, $\lim_{n \to \infty} a_n c_n = c$. That implies $a_n, b_n \to 0$. Therefore, except complexes in $T_1$, $A$ and $B$ are largest complexes along this $C2'$-sequence $x_n$. So $T_2 = \{A, B\}$. Since $2B \in L_2$, $T_2$ is not a union of linkage classes.

ii) Suppose $T_1 = \{A + C, B + C\}$ and $a_n \to \infty$. First of all, $\lim_{n \to \infty} \frac{a_n c_n}{b_n c_n} = c$ for some constant. Therefore $a_n \sim b_n$ so that $b_n \to \infty$. If $2B \not\in T_2$, then it is only possible that either $C, 2C = T_2$ because $A + B, A, B$ and $2A$ are smaller than or equal to $2B$ along with $x_n$. But, since they cannot be in same tier, it is contradiction. Therefore $2B$ must be in $T_2$ so that $2B \in T_2$ and $A \not\in T_2$ because $a_n \sim b_n$. Therefore $T_2$ is not a union of linkage classes.
iii) Suppose \( T_1 = \{ A + C, B + C \} \) and \( a_n \not\to \infty \). As we show in ii), \( b_n \sim a_n \) so that \( b_n \not\to \infty \). Then, \( C \) and \( 2C \) cannot be involved in this reaction network because they are larger than or equal to \( A + C \) along with \( x_n \). If \( a_n, b_n \sim 1 \), then union of all linkage classes except \( L_3 \) is \( T_2 \) because nowhere \( C \) and \( 2C \) exist in this reaction network. If \( a_n, b_n \sim 0 \) for large \( n \), then \( \{ \emptyset \} = T_2 \) or \( \{ A.B \} = T_2 \). For both cases, \( T_2 \) is not a union of linkage classes.

iv) Suppose \( T_1 = \{ A + C, C \} \). Since \( c_n \to \infty \), \( 2C \) cannot exist in this reaction network. Since

\[
\lim_{n \to \infty} \frac{a_n c_n}{c_n} = c
\]

for some constant \( c \), \( a_n \to c \). If \( b_n c_n \ll a_n \) for large \( n \), then \( b_n \to 0 \) so that \( A \) is larger than \( 2B \). That is, \( A \in T_2 \) and \( T_2 \) is not a union of linkage classes. Now, suppose \( b_n c_n \gg a_n \).

\[
\lim_{n \to \infty} b_n c_n = \infty
\]

and \( B + C \) is involved in this reaction network. Then \( B + C \in T_2 \). The only possible complex which can be in \( T_2 \) is \( A + B \). But if \( A + B \in T_2 \), then \( b_n \to \infty \). So \( B + C \) is larger than \( A + C \) along this sequence \( x_n \). It is contradiction to \( T_1 = \{ A + C, C \} \). Therefore when \( b_n c_n \gg a_n \) and the complex \( B + C \) cannot be involved in this reaction network, \( \{ B + C \} = T_2 \). Now, suppose \( b_n c_n \gg a_n \) and \( B + C \) is not involved in this reaction network. In this case, if \( b_n \to \infty \), then \( 2B \in T_2 \) and \( A \notin T_2 \). If \( 0 < \lim_{n \to \infty} b_n < \infty \), union of all linkage classes except \( T_1 \) is \( T_2 \) because both \( b_n \sim a_n \) and there are no \( B + C \) and \( 2C \). This is contradiction to the fact that \( x_n \) is a \( C2' \)-sequence. If \( \lim_{n \to \infty} b_n = 0 \), \( A \notin T_2 \) and \( 2B \notin T_2 \).

Lemma 1.9. If reaction network \( (S, C, R, K) \) is deficiency zero with complexed balanced equilibria, then

\[
\sum_{y \to y'} \kappa_{y \to y'} x^y (y' - y) \cdot (\ln x - \ln \bar{x}) \leq 0
\]

for all \( x \in \mathbb{R}^N_\geq 0 \). Where \( \bar{x} \) is a complex balanced equilibrium.

Proof. This conclusion follows from Proposition 5.3 in Feinberg’s Lecture notes 5 [2].

Lemma 1.10. Let \( (S, C, R, K) \) be a weakly reversible reaction network and \( D \) be a positive stoichiometric compatibility class. Let \( x(t) = \phi(x_0, t) \in D \) be a trajectory of this system with \( \phi(x_0, 0) = x_0 \in D \) and \( \bar{x} \in \mathbb{R}^N_\geq 0 \). Then, at least one of following is true.

\( C1' \) There exists an \( T > 0 \), such that for all \( t > T \), we have

\[
\sum_{y \to y'} \kappa_{y \to y'} x(t)^y (y' - y) \cdot (\ln x - \ln \bar{x}) < 0
\]

\( C2' \) There exists a sequence of times \( t_n \) such that \( \lim_{n \to \infty} \phi_i(x_0, t_n) = \infty \) for at least one \( i \) and

(i) \( C \) is partitioned along \( x_n = \phi(x_0, t_n) \) with tiers \( \{ T_i \} \) and constant \( c \), and

(ii) \( T_1 \) consists of a union of linkage classes, and

\[
\sum_{y \to y'} \kappa_{y \to y'} x_n^y (y' - y) \cdot (\ln x_n - \ln \bar{x}) \geq 0
\]

Proof. The proof is exactly same to the proof of Lemma 4.7 in [3]
Lemma 1.11. Let \((S, C, R, K(t))\) be a weakly reversible reaction network and \(D\) be a positive stoichiometric compatibility class. For a trajectory \(x(t) = \phi(x_0, t) \in D\) with \(\phi(x_0, 0) = x_0 \in D\), if \(C1'\) in Lemma 1.10 holds for some \(\bar{x} \in \mathbb{R}^N_{\geq 0}\), then the trajectory is bounded.

Proof. Let define \(V_{\bar{x}} : \mathbb{R}^N \rightarrow R_{\geq 0}\) such that

\[
V_{\bar{x}}(x) = \sum_{i=1}^{N} \left[ x_i(\ln x_i - \ln \bar{x}_i + 1) + \bar{x}_i \right]
\]

Since \(C1'\) holds, there exists \(T > 0\) such that \(\frac{\partial}{\partial t} V_{\bar{x}}(x(t)) < 0\) for \(t > T\). Therefore \(\lim_{t \rightarrow \infty} V(x(t)) \neq \infty\). Since \(V(x) \rightarrow \infty\) as \(x \rightarrow \infty\), \(x(t)\) must be bounded.

Lemma 1.12. Let \((S, C, R, K(t))\) be a weakly reversible reaction network and \(D\) be a positive stoichiometric compatibility class. Assume this reaction network satisfies either of followings

1) For any \(C2'\)-sequence \(x_n\) with tiers \(\{T_i\}\), there exists \(L_3\) such that \(L_1 \cup L_2 \cup L_3 = T_1\) and no any other linkage class exists.

2) For any \(C2'\)-sequence \(x_n\) with tiers \(\{T_i\}\), there exists a linkage classes \(L_3\) such that \(L_1 = T_1\), then all trajectories are bounded.

Proof. If 1) holds, then boundedness of trajectories follows by Lemma 1.4 Now, assume 2) holds. Let \(x(t) = \phi(x_0, t)\) be a trajectory of this reaction network. We show \(C1'\) in Lemma 1.10 holds with \(\bar{x}\) that is complexed balanced equilibria of \(L_3\).

Suppose \(C1'\) does not hold. Then \(C2'\) holds, so there exists sequences \(t_n > 0\) such that, for \(x_n = \phi(x_0, t_n) \in D\) \(\lim_{n \rightarrow \infty} x_{n,i} = \infty\) for at least one \(i\) and

\[
\sum_{y \rightarrow y'} \kappa_{y \rightarrow y'} x_{n}^y (y' - y) \cdot (\ln x_n - \ln \bar{x}) \geq 0
\]

for all \(n\). Let \(\{T_i\}\) be tiers along with \(\{x_n\}\). By \(C2'\), \(T_1\) is a union of linkage classes. If all tiers are union of linkage classes, then every trajectory is bounded by Lemma 1.4 and Remark 1.2. We also assume that there exists at least one tier \(T_i\) for some \(i \geq 2\) that is not union of linkage classes. Thus \(x_n\) is a \(C2'\)-sequence satisfying (1).

We decompose summation (1) into three parts

\[
\sum_{y \rightarrow y'} \kappa_{y \rightarrow y'} x_{n}^y (y' - y) \cdot (\ln x_n - \ln \bar{x}) = \sum_{i} P \left[ \sum_{y \rightarrow y'} \kappa_{y \rightarrow y'} (t_n) x_{n}^y (y' - y) \cdot (\ln x_n - \ln \bar{x}) \right] \quad (2)
\]

\[
+ \sum_{m=1}^{P-1} \sum_{i} \kappa_{y \rightarrow y'} (t_n) x_{n}^y (y' - y) \cdot (\ln x_n - \ln \bar{x}) \quad (3)
\]

\[
+ \sum_{m=1}^{i-1} \sum_{i} \kappa_{y \rightarrow y'} (t_n) x_{n}^y (y' - y) \cdot (\ln x_n - \ln \bar{x}) \quad (4)
\]

The first summation (2) equals to

\[
\sum_{y \rightarrow y'} \kappa_{y \rightarrow y'} x_{n}^y \left( \ln \frac{y'}{x_n} - \ln \frac{\bar{y}'}{\bar{x}} \right) + \sum_{i=2}^{P} \sum_{y \rightarrow y'} \kappa_{y \rightarrow y'} x_{n}^y \left( \ln \frac{y'}{x_n} - \ln \frac{\bar{y}'}{\bar{x}} \right) \quad (5)
\]
Since \( x_n \) is a \( C2 \)-sequence, \( L_3 = T_1 \). Thus the first term is non-positive by Lemma 1.9. We bring the second term to next stage.

As we assume, there is at least one tier \( T_i \) for some \( i \geq 2 \) which is not union of linkage classes. This fact and weak reversibility of reaction network imply that (3) and (4) are not trivial summations. By similar statement of Lemma 4.7 in Anderson(2011) [3], we can show that summation of (3), (4) and second term of (5) is strictly negative for large \( n \). That is,

\[
\sum_{i=2}^{P} \left[ \sum_{j \to i} \kappa_{j \to i} \cdot x_n \ln \frac{x_n^y}{x_n^y} \right] + \sum_{i=1}^{P} \left[ \sum_{j \to i} \kappa_{j \to i} \cdot x_n \ln \frac{x_n^y}{x_n^y} \right] + \sum_{i=1}^{N} \kappa_{i} \cdot x_n \ln \frac{x_n^y}{x_n^y} < 0
\]

for large \( n \). Therefore,

\[
\sum_{i=1}^{N} \kappa_{i} \cdot x_n \ln \frac{x_n^y}{x_n^y} < 0
\]

It contradicts to

\[
\sum_{i=1}^{N} \kappa_{i} \cdot x_n \ln \frac{x_n^y}{x_n^y} \geq 0
\]

for all \( n \). Therefore \( C1' \) must hold. By Lemma 1.11, \( \phi(x_0, t) \) is bounded.

**Proposition 1.13.** Let \((S, C, R, K)\) be a 3 species binary, weakly reversible reaction network. If there are linkage classes \( L_1 \) and \( L_2 \) that are completely twisted, then all trajectories are bounded.

**Proof.** First of all, if there are only two linkage classes, then the conclusion follows by Proposition 1.5 and Remark 1.2. Let’s assume there are more than two linkage classes. Suppose there exists at least one \( C2' \)-sequence. Then, by Lemma 1.7 and Lemma 1.12, all trajectories are bounded. Suppose, then, there exists no \( C2' \)-sequence for this reaction network. Then, for any trajectory, \( C1' \) condition with \( \bar{x} = 0 \in \mathbb{R}^N \) must hold because \( C2' \)-sequence does not exists so \( C2' \) cannot hold. So all trajectories are bounded by Lemma 1.11.

\[
\square
\]

### 2 Persistence of completely twisted network

In this section, we define \( V_{\bar{x}} : \mathbb{R}^N_{>0} \rightarrow \mathbb{R} \) such that \( V_{\bar{x}}(x) = \sum_{i=1}^{N} x_i (\ln x_i - \ln \bar{x}_i - 1) - \bar{x}_i \) for some \( \bar{x} \in \mathbb{R}^N_{>0} \).

**Lemma 2.1.** Let \((S, C, R, K)\) be a reaction network with \( N \) species. Let \( x(t) \) be a trajectory of this network and \( \bar{x} = (1, 1, \cdots, 1) \). If there exists a positive number \( \epsilon \) such that \( \frac{\partial}{\partial t} V_{\bar{x}}(x(t)) < 0 \) for all \( t \) at which \( x(t) \in B_{r}(0) \), then the set of omega points \( \omega(x(t)) \) does not include the origin.

**Proof.** Without loss of generality, we assume \( \epsilon < \min \{x(0)_i, 1|1 = 1, 2, \cdots, N\} \). Suppose there exists a sequence of time \( t_n \rightarrow \infty \) such that \( x(t_n) \rightarrow 0 \). Since \( x(\ln x - 1) \rightarrow 1 \) as \( x \rightarrow 0 \), we can fine \( \delta > 0 \) such that

\[
\delta(\ln \delta - 1) + 1 > 1 + \frac{\epsilon}{N} (\ln \epsilon^2 - 1) \quad (6)
\]
Choose \( x(t_M) \) such that \( 0 < x(t_M)_i < \delta \) for \( i = 1, 2, 3, \cdot \cdot \cdot , N \). Then, by the continuity of trajectory, there should exist \( t_0 \) such that \( x(t_0) \in \partial B_\varepsilon(0) \) and \( x(t) \in B_\varepsilon(0) \) for \( t \in [t_0, t_M] \). Since \( x(t_0) \in \partial B_\varepsilon(0) \), for at least one \( k \in \{1, 2, \cdot \cdot \cdot , N\} \), \( x(t_0)_k > \frac{\varepsilon}{2} \). Now let’s consider \( V_\varepsilon(x) \) with \( \bar{x} = (1, 1, 1, \cdot \cdot \cdot , 1) \). Then,

\[
V_\varepsilon(x(t_0)) = \sum_{i=1}^{N} x(t_0)_i (\ln x(t_0)_i - 1) + 1 < \frac{\varepsilon}{2} (\ln \frac{\varepsilon}{2} - 1) + 1 + (N - 1) \tag{7}
\]

because \( x(\ln x - 1) + 1 \) is decreasing on \([0, 1]\) thus \( b(\ln b - 1) + 1 < x(\ln x - 1) + 1 < 1 \) for \( b < x < 1 \). That is, the inequality comes from \( x(t_0)_k (\ln x(t_0)_k - 1) + 1 < \frac{\varepsilon}{2} (\ln \frac{\varepsilon}{2} - 1) + 1 \) and \( x(t_0)_i (\ln x(t_0)_i - 1) + 1 < 1 \) for \( i \neq k \). Moreover, since \( \frac{\partial}{\partial t} V_\varepsilon(x(t)) < 0 \) for \( t \in [t_0, t_M] \),

\[
V_\varepsilon(x(t_M)) \leq V_\varepsilon(x(t_0)) \tag{8}
\]

But, since \( x(t_M)_i < \delta \) for all \( i \),

\[
V_\varepsilon(x(t_M)) > N[1 + \frac{1}{N} \left( \frac{\varepsilon}{2} (\ln \frac{\varepsilon}{2} - 1) \right)] = N + \frac{\varepsilon}{2} (\ln \frac{\varepsilon}{2} - 1)
\]

by (6) and the fact that \( x(\ln x - 1) + 1 \) is decreasing on \([0, 1]\). So this contradicts to (7) and (8).

\[\square\]

**Lemma 2.2.** Let \((S, C, K, R)\) be a reaction network with \( N \) species. Let \( x(t) \) be a bounded trajectory of this network. Let \( \sup_i |x(t)| = L \). For any \( \bar{x} \in \mathbb{R}_{>0}^N \) and \( \gamma > 0 \), if there exists a positive number \( \varepsilon_{\bar{x}, \gamma} \) such that \( \frac{\partial}{\partial t} V_\varepsilon(x(t)) < 0 \) for all \( t \) at which \( x(t) \in [\gamma, L] \times [0, \varepsilon_{\bar{x}, \gamma}]^{N-1} \), then \((a, 0, \cdot \cdot \cdot , 0) \notin \omega(x(t)) \) for any \( a > 0 \).

**Proof.** Suppose \( A = (a, 0, \cdot \cdot \cdot , 0) \in \omega(x(t)) \) for some \( a > 0 \). (Since \( |x(t)| < L \) for all \( t, a < L \). Then there exists a sequence \( t_n \not\to \infty \) such that \( x(t_n) \to A \). Let \( \bar{y} = (a/2, 1, 1, \cdot \cdot \cdot , 1) \) and \( \bar{z} = (1, 1, 1, \cdot \cdot \cdot , 1) \). Let \( \varepsilon = \min(a/2, \varepsilon_{\bar{y}, a/2}, \varepsilon_{\bar{z}, a/2}, 1, x(t_0)_{i} i = 1, 2, \cdot \cdot \cdot , N) \). Let \( f_q : \mathbb{R} \to \mathbb{R} \) such that \( f_q(x) = x(\ln x - L) \). Note that

\[
f_q(x) \text{ is decreasing on } [0, q] \text{ and increasing on } [q, \infty) \tag{9}
\]

Then we have followings by (9)

\[
f_1(\frac{\varepsilon}{2}) = \frac{\varepsilon}{2} (\ln \frac{\varepsilon}{2} - 1) < 0
\]

\[
f_{a/2}(a - \frac{\varepsilon}{2}) - f_{a/2}(a) < 0
\]

\[
f_L(a + \frac{\varepsilon}{2}) - f_L(a) < 0
\]
As the proof of Lemma 2.1, we take $0 < \delta$

$$f_1(\delta) > \frac{1}{2(N-1)}f_1\left(\frac{\varepsilon}{2}\right)$$  \hspace{1cm} (10)

$$f_1(\delta) > \frac{1}{2(N-1)}(f_{a/2}(a_{-\frac{\varepsilon}{2}}) - f_{a/2}(a))$$  \hspace{1cm} (11)

$$f_1(\delta) > \frac{1}{2(N-1)}(f_L(a + \frac{\varepsilon}{2}) - f_L(a))$$  \hspace{1cm} (12)

$$f_{a/2}(a - \delta) > f_{a/2}(a) + \frac{1}{2}f_1\left(\frac{\varepsilon}{2}\right)$$  \hspace{1cm} (13)

$$f_{a/2}(a - \delta) > f_{a/2}(a) + \frac{1}{2}(f_{a/2}(a_{-\frac{\varepsilon}{2}}) - f_{a/2}(a))$$  \hspace{1cm} (14)

$$f_L(a + \delta) > f_L(a + \delta) + \frac{1}{2}(\ln\left(\frac{\varepsilon}{2}\right) - 1)$$ \hspace{1cm} (15)

$$f_L(a + \delta) > f_L(a + \delta) + \frac{1}{2}(f_L(a + \frac{\varepsilon}{2}) - f_L(a))$$  \hspace{1cm} (16)

hold. We can find such $\delta$ satisfying (10)–(16) above due to

$$f_1(x) \not\nearrow 0 \text{ as } x \searrow 0$$

$$f_{a/2}(x) \not\nearrow f_{a/2}(a) \text{ as } x \nearrow a$$

$$f_L(x) \not\nearrow f_L(a) \text{ as } x \searrow a$$

Now, we choose $x(t_M)$ such that $(t_M) \in B_\delta(A)$. Then by continuity of $x(t)$, there exists $t_0 > 0$ such that

1) $x(t_0) \in \partial B_r(0)$ and
2) $x(t) \in B_r(A)$ for $t \in [t_0, t_M]

Condition 1) implies that there exists $k \in \{2, 3, \cdots, N\}$ such that $x(t_0)_k > \frac{x}{2}$, $a - x(t_0)_1 > \frac{x}{2}$ or $x(t_0)_1 - a > \frac{x}{2}$

Now, we take care of two cases: $a/2 < x(t_0)_1 \leq a$ and $a < x(t_0)_1 \leq L$. (Since $\varepsilon < a/2$, we don’t need to care the case of $x(t_0)_1 \leq a/2$.

Case 1, we assume $a/2 < x(t_0)_1 \leq a$.

Remark followings

1. $x(t_0)_1 \leq a$ implies $f_{a/2}(x(t_0)_1) \leq f_{a/2}(a)$
2. If $a - x(t_0)_1 > \frac{x}{2}$, $f_{a/2}(x(t_0)_1) < f_{a/2}(a - \frac{x}{2})$ because $a/2 < x < a - \frac{x}{2}$
3. If $x(t_0)_k > \frac{x}{2}$ for some $k \in \{2, 3, \cdots, N\}$, then $f_1(x(t_0)_k) < f_1\left(\frac{x}{2}\right)$
4. Since $x(t_0)_i < 1$ for $i \in \{2, 3, \cdots, N\}$, $f_1(x(t_0)_i) < 0$
5. Since $a - \delta < x(t_M)_1$, $f_{a/2}(x(t_M)_1) > f_{a/2}(a - \delta)$
6. Since $x(t_M)_i < \delta$ for $i = 2, 3, \cdots, N$, $f_1(x(t_M)_i) > f_1(\delta)$

These facts above also came from (9). Now we suppose $k \neq 1$. Then,

$$V(\delta)(x(t_0)) = f_{a/2}(x(t_0)_1) + a/2 + f_1(x(t_0)_k) + 1 + \sum_{i \neq 1, k} (f_1(x(t_0)_i) + 1) < f_{a/2}(a) + a/2 + f_1(\frac{x}{2}) + N - 1$$  \hspace{1cm} (17)
by 1, 3 and 4
Moreover, since \( \frac{\partial}{\partial t} V_{\hat{g}}(x(t)) < 0 \) for \( t \in [t_0, t_M] \).

\[ V_{\hat{g}}(x(t_M)) \leq V_{\hat{g}}(x(t_0)) \tag{18} \]

However,

\[
V_{\hat{g}}(x(t_M)) = f_{a/2}(x(t_M)_1) + a/2 + \sum_{i \geq 2}^N (f_1(x(t_M)_i) + 1)
\]

\[
> f_{a/2}(a - \delta) + a/2 + (N - 1)(f_1(\delta) + 1)
\]

\[
> f_{a/2}(a) + a/2 + \frac{1}{2} f_1(\frac{\epsilon}{2}) + (N - 1)(\frac{1}{2(N - 1)} f_1(\frac{\epsilon}{2}) + 1)
\]

\[
= f_{a/2}(a) + a/2 + f_1(\frac{\epsilon}{2}) + N - 1 = V_{\hat{g}}(x(t_0))
\]

by 5, 6, (10) and (13). This is contradiction to (18).

Now we suppose there is no such \( k \in \{2, 3, \cdots, N\} \). Then \( a/2 < x(t_0)_1 < a - \frac{\epsilon}{2} \). Therefore

\[
V_{\hat{g}}(x(t_0)) = f_{a/2}(x(t_0)_1) + a/2 + \sum_{i \geq 2}^N (f_1(x(t_0)_i) + 1) \leq f_{a/2}(a - \frac{\epsilon}{2}) + a/2 + N - 1
\]

by 2.

However,

\[
V_{\hat{g}}(x(t_M)) = f_{a/2}(x(t_M)_1) + a/2 + \sum_{i \geq 2}^N (f_1(x(t_M)_i) + 1)
\]

\[
> f_{a/2}(a - \delta) + a/2 + (N - 1)(f_1(\delta) + 1)
\]

\[
> f_{a/2}(a) + a/2 + \frac{1}{2} (f_{a/2}(a - \frac{\epsilon}{2}) - f_{a/2}(a)) + (N - 1)(\frac{1}{2(N - 1)} (f_{a/2}(a - \frac{\epsilon}{2}) - f_{a/2}(a) + 1)
\]

\[
= f_{a/2}(a - \frac{\epsilon}{2}) + a/2 + N - 1 \geq V_{\hat{g}}(x(t_0))
\]

by 5, 6, (11) and (14). This is also contradiction to (18).

Case 2: we assume \( a < x(t_0)_1 < L \). We also remark followings

7. \( a < x(t_0)_1 < L \) implies \( f_L(x(t_0)_1) \leq f_L(a) \)

8. If \( x(t_0)_1 - a > \frac{\epsilon}{2} \), \( f_L(x(t_0)_1) < f_L(a + \frac{\epsilon}{2}) \)

9. If \( x(t_0)_k > \frac{\epsilon}{2} \) for some \( k \in \{2, 3, \cdots, N\} \), then \( f_1(x(t_0)_k) < f_1(\frac{\epsilon}{2}) \)

10. Since \( x(t_0)_i < 1 \) for \( i = \{2, 3, \cdots, N\}, f_1(x(t_0)_i) < 0 \)

11. Since \( a + \delta > x(t_M)_1, f_L(x(t_M)_1) > f_L(a + \delta) \)

12. Since \( x(t_M)_i < \delta \) for \( i = 2, 3, \cdots, N, f_1(x(t_M)_i) > f_1(\delta) \)

These facts above also came from (9). Now we suppose \( k \neq 1 \). Then,

\[
V_{\hat{g}}(x(t_0)) = f_{L}(x(t_0)_1) + L + f_1(x(t_0)_k) + 1 + \sum_{i \neq 1, k} (f_1(x(t_0)_i) + 1) < f_{L}(a) + L + f_1(\frac{\epsilon}{2}) + 1 + N - 2 \tag{19}
\]
by 7,9 and 10
Moreover, since \( \frac{\partial}{\partial t} V_{\bar{y}}(x(t)) < 0 \) for \( t \in [t_0, t_M] \).

\[
V_{\bar{y}}(x(t_M)) \leq V_{\bar{y}}(x(t_0)) \tag{20}
\]

However,

\[
V_{\bar{y}}(x(t_M)) = f_L(x(t_M)) + L + \sum_{i \geq 2}^{N} (f_1(x(t_M)) + 1)
> f_L(a + \delta) + L + (N - 1)(f_1(\delta) + 1)
> f_L(a) + L + \frac{1}{2} f_1(\epsilon) + (N - 1) \left( \frac{1}{2(N - 1)} f_1(\epsilon) + 1 \right)
= f_L(a) + L + f_1(\epsilon) + N - 1 = V_{\bar{y}}(x(t_0))
\]

by (10) and (15). This is contradiction to (20).
Now we suppose there is no such \( k \in \{2, 3, \cdots, N\} \). Then \( a + \frac{\epsilon}{2} < x(t_0) < L \). Therefore

\[
V_{\bar{y}}(x(t_0)) = f_L(x(t_0)) + L + \sum_{i \geq 2}^{N} (f_1(x(t_0)) + 1) \leq f_L(a + \frac{\epsilon}{2}) + L + N - 1
\]

by 8. However,

\[
V_{\bar{y}}(x(t_M)) = f_L(x(t_M)) + L + \sum_{i \geq 2}^{N} (f_1(x(t_M)) + 1)
> f_L(a + \delta) + L + (N - 1)(f_1(\delta) + 1)
> f_{a/2}(a) + L + \frac{1}{2} (f_L(a + \frac{\epsilon}{2}) - f_L(a)) + (N - 1) \left( \frac{1}{2(N - 1)} (f_L(a - \frac{\epsilon}{2}) - f_L(a)) + 1 \right)
= f_L(a + \frac{\epsilon}{2}) + L + N - 1 \geq V_{\bar{y}}(x(t_0))
\]

by 11,12,(12) and (16). This is also contradiction to (20).

\[\square\]

**Theorem 2.3.** Let \((S,C,K,R)\) be a binary reaction network with 3 species. Suppose that there are completely twisted linkage classes \( L_1 \) and \( L_2 \). Then, this networks is persistence.

**Proof.** Let \( x(t) \) be a trajectory of this network. By the result of [5], we only suppose \( \lim \inf x(t)_i = 0 \) for \( i = 1, 2, 3 \) (If at least one concentration of species remains bounded, we take care reduced network by regarding bounded species as reaction rates. Then we have 2 or single species reduced network system. The paper proved every 2 species weakly reversible reaction network is persistence. And it is easy to check single that species binary network is persistence) Without loss of generality, we also assume \( L_1 \) contains \( A, 2B \) and \( L_2 \) contains \( B, 2A \). We show the existence of \( \epsilon \) in Lemmas 2.1 and 2.2. Suppose there does not exist \( \epsilon > 0 \) such that \( \frac{\partial}{\partial t} V_{(1,1,1)}(x(t)) < 0 \) for all \( t \) at which \( x(t) \in B_{\epsilon}(0) \). Then there exists a sequence \( x_n = (a_n, b_n, c_n) \) such that

1. \( \sum_{y \to y'} k_{y \to y'} x_n^y(y' - y) \ln x_n \geq 0 \)
2. $x_n \to (0,0,0)$ as $n \to \infty$

Let $\{T_i\}$ be tiers along with this sequence. If $T_1$ is not a union of linkage classes, then 1) cannot be true by the same argument in the proof of Lemma 4.7 in [3]. That is, $x_n$ is $C^2$-sequence. Since $a_n \to 0$, $L_1 \cup L_2 \not\subset T_1$ by the Corollary 1.2. $T_1$ cannot contain any complex of $\{A+B, A+C, B+C\}$ because they are smaller than $A$ or $B$ along with $x_n$ which are not contained $T_1$. $T_1$ also cannot contain $\emptyset$ because no complex has same order with $\emptyset$ with respect to $x_n$. (That is, $\lim_{n \to \infty} 1/x_n^i = \infty$ for all complexes but $x$ except $\emptyset$). So the only possibility is $T_1$ contains $C, 2C$ only. But it is also impossible because $\lim n \to \infty c_n e^{x_n^2} = \infty$.

That is, contradiction to the definition of $T_i$. Therefore there exists an $\epsilon$ which we look for. Therefore $(0,0,0) \not\in \omega(x(t))$ by the Lemma 2.1.

Now, let $\bar{x} \in \mathbb{R}_{>0}$ and $\gamma > 0$. We suppose there does not exist a positive number $\epsilon_{\bar{x}, \gamma}$ such that $\frac{\partial}{\partial t} V_{\bar{x}}(x(t)) < 0$ for all $t$ at which $x(t) \in [\gamma, L] \times [0, \epsilon_{\bar{x}, \gamma}] N^{-1}$. Then there exists a sequence $x_n = (a_n, b_n, c_n)$ such that

1. $\sum_{y \to y'} k_{y \to y'} x_n^y (y' - y) (\ln x_n - \ln \bar{x}) \geq 0$.
2. $b_n \to 0$ and $c_n \to 0$ as $n \to \infty$.
3. $\lim \inf a_n > \gamma > 0$

Let $\{T_i\}$ be tiers along with $x_n$. By the same reasoning above, $T_1$ must be union of linkage classes. Similarly, $b_n \to 0$ implies $L_1 \cup L_2 \not\subset T_1$ by the Corollary 1.2. But $A$ is largest complex along with $x_n$ because of condition 3). So $T_1$ must contain $A \in L_1$. This is contradiction. Therefore we showed the existence of $\epsilon_{\bar{x}, \gamma}$. Therefore $(a, 0, 0) \not\in \omega(x(t))$ for any $a > 0$. We can show $(0, b, 0) \not\in \omega(x(t))$ by exactly same argument because of symmetry of $A$ and $B$.

Finally, the Corollary 3.3 in [4] implies if $(c, d, 0) \in \omega(x(t))$ for some $c, d, > 0$, then $\omega(x(t))$ contains $(a, 0, 0)$ or $(0, b, 0)$ for some $a, b > 0$. We already showed it is impossible. Therefore all the statement above make contradictions to $\lim \inf x(t)_i = 0$ for $i = 1, 2, 3$ especially the case of $i = 3$.

References