A fake Schottky group in \text{Mod}(S)

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Abstract. We use the classical construction of Schottky groups in hyperbolic geometry to produce non-Schottky subgroups of the mapping class group.

1. Introduction

In hyperbolic geometry, a Schottky group is a free convex cocompact Kleinian group, classically constructed as follows. Pick four pairwise disjoint closed balls \( B^-_1, B^-_2, B^+_1, B^+_2 \) in \( S^{n-1}_\infty \), the ideal boundary of hyperbolic \( n \)-space. Suppose there are isometries \( f_1 \) and \( f_2 \) so that
\[
 f_i(B^-_i) = S^{n-1}_\infty - B^+_i.
\]
Then \( \langle f_1, f_2 \rangle \) is a Schottky group isomorphic to the free group \( F_2 \) of rank two.

Now let \( S \) be a closed surface of genus \( g \geq 2 \) and let \( \text{Mod}(S) = \pi_0(\text{Homeo}^+(S)) \) be its mapping class group. By way of analogy with the theory of Kleinian groups, B. Farb and L. Mosher defined [FM] a notion of convex cocompactness for subgroups of \( \text{Mod}(S) \). In this setting, a Schottky group is a free convex cocompact subgroup of \( \text{Mod}(S) \). In [KL1, KL2], we extended Farb and Mosher’s analogy, providing several characterizations of convex cocompactness borrowed from the Kleinian setting (see also Hamenstädt [H]). The analogy is an imperfect one, see [KL3] and the references there, and we point out some new imperfections here.

Theorem 1.1. There exist pseudo-Anosov elements \( f_1 \) and \( f_2 \) in \( \text{Mod}(S) \) and pairwise disjoint closed balls \( B^-_1, B^-_2, B^+_1, B^+_2 \) in \( \text{PMCL}(S) \) for which
\[
 f_i(B^-_i) = \text{PMCL}(S) - B^+_i
\]
and yet \( \langle f_1, f_2 \rangle \cong F_2 \) is not a Schottky group.

The construction is based on work of N. Ivanov, and it is clear from his work in [I] that he was aware of this construction (see also McCarthy [Mc]). The group \( G = \langle f_1, f_2 \rangle \) contains reducible elements and so fails to be convex cocompact. It is worth noting that there are sufficiently high powers of the \( f_i \) that generate a Schottky group, as proven by Farb and Mosher [FM], see also [KL1, H].

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Part of the analogy between Kleinian groups and mapping class groups was developed by J. McCarthy and A. Papadopoulos [MP], who constructed a limit set $\Lambda_G$ and domain of discontinuity $\Delta_G \subset \mathbb{P} \mathcal{ML}(S) - \Lambda_G$ for any subgroup $G < \text{Mod}(S)$, see Section 4. Unlike in the Kleinian setting, $\Delta_G \neq \mathbb{P} \mathcal{ML}(S) - \Lambda_G$ in general. While examples illustrate the necessity of taking an open set strictly smaller than $\mathbb{P} \mathcal{ML}(S) - \Lambda_G$ as a domain of discontinuity, it is not clear that $\Delta_G$ is an optimal choice. In [KL1], we asked whether or not $\Delta_G$ is the largest open set on which $G$ acts properly discontinuously—see Question 3 there. Here, we answer this in the negative.

There is an obvious open set on which our group $G = \langle f_1, f_2 \rangle$ acts properly discontinuously and cocompactly, namely

$$\Omega = \bigcup_{g \in G} g \cdot \Theta$$

where $\Theta$ is the closure of the complement of our four balls. To see that $\Omega$ is open, note that $\Theta$ is contained in the interior $U$ of

$$f_1(\Theta) \cup f_1^{-1}(\Theta) \cup f_2(\Theta) \cup f_2^{-1}(\Theta)$$

and that

$$\Omega = \bigcup_{g \in G} g \cdot U$$

If the $f_i$ are chosen carefully, the set $\Omega$ will contain $\Delta_G$ properly, and we have the following theorem.

**Theorem 1.2.** There are irreducible subgroups $G < \text{Mod}(S)$ for which $\Delta_G$ is not the largest open set on which $G$ acts properly discontinuously.

Asymmetry of the construction provides another domain $\Omega'$ on which $G$ acts properly discontinuously, and we will also show that $G$ does not act properly discontinuously on the union $\Omega \cup \Omega'$.

Though $\Delta_G$ is not a maximal domain of discontinuity, we show in Section 5 that, for the groups in Theorem 1.2, it is nonetheless the intersection of all such maximal domains.

### 2. Surface dynamics

If $X$ is a subset of $\mathbb{P} \mathcal{ML}(S)$, we let

$$ZX = \{[\nu] \in \mathbb{P} \mathcal{ML}(S) \mid i(\nu, \mu) = 0 \text{ for some } [\mu] \in X\}$$

be the zero locus of $X$. If $X = \{[x]\}$ we sometimes write $Zx$ for $ZX$.

If $f$ is pseudo-Anosov, then it acts with north–south dynamics on $\mathbb{P} \mathcal{ML}(S)$, meaning that it has unique attracting and repelling fixed points $[\mu_f^+]$ and $[\mu_f^-]$, respectively—all other points are attracted to $[\mu_f^+]$ under iteration of $f$. In fact, for any neighborhood $U$ of $[\mu_f^+]$ and any compact set $K \subset \mathbb{P} \mathcal{ML}(S) - \{[\mu_f^-]\}$, there is a natural number $N$ so that

$$f^n(K) \subset U$$

for any $n \geq N$.

Ivanov proves that there is a similar situation for most pure reducible elements (see the Appendix of [I]). In particular, suppose $\alpha$ is a nonseparating simple closed curve in $S$ preserved by a mapping class $\phi$ that is pseudo-Anosov when restricted
to $S - \alpha$. Let $[\mu^+]$ and $[\mu^-]$ be the stable and unstable laminations for $\phi$ in $S - \alpha$ considered laminations on $S$, and note that

$$Z \mu^- = \{[s\mu + (1-s)\alpha] \in \PML(S) \mid s \in [0,1]\}.$$ 

If $K \subset \PML(S) - Z \mu^-$ is a compact set and $U$ a neighborhood of $[\mu^+]$, then there is an $N > 0$ such that for all $n \geq N$ we have

$$\phi^n(K) \subset U.$$ 

Given a mapping class $g$ of either type above, let $\lambda(g)$ denote the expansion factor of $g$, the number such that

$$g(\mu^+) = \lambda(g)\mu^+.$$ 

3. The construction

Let $\alpha$ be a nonseparating curve fixed by a mapping class $\phi$ that is pseudo-Anosov on $S - \alpha$, and let $[\mu^+]$ and $[\mu^-]$ be as in the previous section.

Let $S_\phi \subset \PML(S)$ be a bicorned $(6g - 8)$-dimensional sphere dividing $\PML(S)$ into two closed balls $A_\phi$ and $B_\phi$ containing $Z \mu^-$ and $[\mu^+]$, respectively.

According to (2.2), there is an $N > 0$ so that for all $n \geq N$ we have

$$\phi^n(B_\phi) \subset \text{int}(B_\phi).$$

So we choose an $n \geq N$, let $h = \phi^n$, $B^-_h = A_\phi$, and $B^+_h = h(B_\phi)$.

Recall H. Masur’s theorem [Ma] that the set

$$\{([\mu^+], [\mu^-]) \mid \psi \in \text{Mod}(S) \text{ pseudo-Anosov}\}$$

is dense in $\PML(S) \times \PML(S)$. So we choose a pseudo-Anosov $\psi$ whose fixed points $[\mu^+]$ and $[\mu^-]$ lie in $\PML(S) - (B^-_h \cup B^+_h)$. We let $S_\psi \subset \PML(S) - (B^-_h \cup B^+_h)$ be a bicorned $(6g - 8)$-sphere which bounds two balls: $A_\psi \subset \PML(S) - (B^-_h \cup B^+_h)$ containing $[\mu^-]$ and $B_\psi$ containing $[\mu^+]$. As $\PML(S) - (B^-_h \cup B^+_h)$ is a neighborhood of $[\mu^+]$, (2.1) provides an $M > 0$ such that for all $m \geq M$, we have $\psi^m(B_\psi) \subset \PML(S) - (B^-_h \cup B^+_h)$. Arguing as in [I], we may choose $m$ so that $\psi^m h$ is pseudo-Anosov, and we do so. We let $f = \psi^m, B^-_f = A_\psi, \text{and } B^+_f = f(B_\psi)$.

We now have elements $f, h$, and pairwise disjoint closed balls

$$B^-_h, B^+_h, B^-_f, B^+_f$$

such that

$$h(B^-_h) = \PML(S) - B^+_h \text{ and } f(B^-_f) = \PML(S) - B^+_f.$$ 

See Figure 1.

Let $G = \langle f, h \rangle$, set

$$\Theta = \PML(S) - (B^-_h \cup B^+_h \cup B^-_f \cup B^+_f),$$

and let

$$\Omega = \bigcup_{g \in G} g \cdot \Theta.$$ 

The group $G$ acts on $\Omega$ properly discontinuously and cocompactly with fundamental domain $\Theta$, and the usual ping–pong argument implies that $G \cong F_2$.

A slight modification now provides the desired example.
We let \( f_1 = fh \) and \( f_2 = f \), both pseudo-Anosov by construction. Of course, \( G = \langle f_1, f_2 \rangle \), and we need only find balls \( B_1^{\pm} \) and \( B_2^{\pm} \) with

\[
  f_i(B_i^\pm) = \mathbb{PML}(S) - B_i^\mp.
\]

Set \( B_1^- = B_h^- \) and \( B_1^+ = f(B_h^+) \). The ball \( B_2^- \) is constructed as a regular neighborhood of \( B_f^- \cup B_h^+ \cup \delta \) in \( \mathbb{PML}(S) - (B_h^- \cup B_f^+) \), where \( \delta \) is an arc in \( \Theta \) from \( B_f^- \) to \( B_h^+ \). The ball \( B_2^+ \) is defined to be \( \mathbb{PML}(S) - f(B_2^-) \). See Figure 2.

One can now check (3.1).

### 4. Proper discontinuity

Let \( G = \langle h, f \rangle \) be the group constructed in the previous section, and let \( \partial G \) be the Gromov boundary of \( G \). By the work in [MP], the limit set

\[
  \Lambda_G = \{ [\mu_g] \in \mathbb{PML}(S) | g \in G \text{ is pseudo-Anosov} \}
\]

is the unique minimal closed \( G \)-invariant subset of \( \mathbb{PML}(S) \). In [KL2] we showed that one may choose \( h \) and \( f \) as above so that \( G \) has the following property.

**Property 4.1.** There exists a continuous \( G \)-equivariant homeomorphism

\[
  \mathcal{J}: \partial G \to \Lambda_G.
\]

Moreover, for each \( x \in \partial G \) which is a fixed point of a conjugate \( g h \) of \( h \), \( \mathcal{J}(x) \) is the stable or unstable lamination of that conjugate \( g h \) (respecting the dynamics).
Otherwise \( \mathcal{I}(x) \) is a uniquely ergodic filling lamination. In particular, every element \( g \in G \) not conjugate to a power of \( h \) is pseudo-Anosov.

We henceforth assume that \( G \) satisfies Property 4.1.

The domain of discontinuity is defined to be

\[
\Delta_G = \mathcal{PML}(S) - Z\Lambda_G.
\]

This is an open set on which \( G \) acts properly discontinuously [MP], which justifies the name.

We may describe the zero locus \( Z\Lambda_G \) for \( G \) explicitly. For each conjugate \( gh \) of \( h \), we have the attracting and repelling fixed points \( x^\pm_{gh} \) in \( \partial G \). By Property 4.1, the map \( \mathcal{I} \) sends these to the stable and unstable laminations

\[
\mathcal{I}(x^\pm_{gh}) = [\mu^\pm_{gh}] = g[\mu^\pm_h].
\]

For any such point \( g[\mu^\pm_h] \in \Lambda_G \), the set \( Zg\mu^\pm_h = gZ\mu^\pm_h \) is a 1–simplex in \( Z\Lambda_G \). Since \( \mathcal{I}(x) \) is uniquely ergodic and filling for every other point \( x \in \partial G \), it follows that \( Z\Lambda_G \) is the union of \( \Lambda_G \) and all of these intervals.

The intervals \( Z\mu^-_h \) and \( Z\mu^+_h \) intersect each other at \( \alpha \), and so the union

\[
\mathcal{J}_h = Z\mu^-_h \cup Z\mu^+_h
\]
is an interval joining $\mu_h^-$ to $\mu_h^+$. All in all, we have

$$Z \Lambda_G = \Lambda_G \cup \bigcup_{g \in G} g \mathbb{J}_h$$

(4.1)

We impose one further restriction on $h$ and $f$—more precisely, on the balls $B_h^\pm$. Since the fixed points of $f$ do not meet the interval $\mathbb{J}_h$, we may replace $f$ with a power so that the balls $B_f^\pm$ are disjoint from this interval. This implies that

$$\text{int } Z \mu_h^+ = Z \mu_h^+ \cap \bigcup_{n \in \mathbb{Z}} h^n \Theta$$

and so $Z \mu_h^+$ intersects the $h^n$ translates of $\Theta$, and no other $G$–translates. As $Z \mu_h^-$ does not intersect $\Omega$, these are the only $G$–translates of $\Theta$ that $J_h$ intersects. Write $\Sigma_h^+ = \partial B_h^+$ and $\Sigma_f^+ = \partial B_f^+$.

We claim that

$$\Sigma_f^+ \cap Z \Lambda_G = \emptyset.$$  

To see this, note that if $\Sigma_f^+$ nontrivially intersected $Z \Lambda_G$, it would do so in some $g \mathbb{J}_h$, by (4.1); and then $g$ must be a power of $h$, since $\Sigma_f^+$ lies in $\Theta$. But $h \mathbb{J}_h = \mathbb{J}_h$, and so $\Sigma_f^+$ would intersect $\mathbb{J}_h$, contrary to our choice of $f$. The claim follows.

Now, Theorem 1.2 will follow from

**Theorem 4.2.** The set $\Delta_G$ is properly contained in $\Omega$. In fact,

$$\Omega = \mathbb{P}ML(S) - \left( \Lambda_G \cup \bigcup_{g \in G} g Z \mu_h^- \right).$$

First note that $\Delta_G \neq \Omega$ as $\Sigma_h^- \subset \Theta \subset \Omega$ nontrivially intersects $\mathbb{J}_h \subset Z \Lambda_G = \mathbb{P}ML(S) - \Delta_G$.

To prove the containment, we must gather some information about the complement of $\Omega$. Let $\mathfrak{X} = \mathbb{P}ML(S) - \Omega$.

**Lemma 4.3.** There is a continuous $G$–equivariant map

$$\mathfrak{X} : \mathfrak{X} \rightarrow \partial G.$$  

**Proof.** The spheres $\Sigma_h^+$ and $\Sigma_f^+$ are bicollared with collars $N(\Sigma_h^+)$ and $N(\Sigma_f^+)$. We assume as we may that

$$h(N(\Sigma_h^+)) = N(\Sigma_h^+) \text{ and } f(N(\Sigma_f^-)) = N(\Sigma_f^-)$$

and that all of the $G$–translates of these collars are pairwise disjoint.

Let $\mathcal{G}$ be the Cayley graph of $G$ and identify $\partial G = \partial \mathcal{G}$. We define a continuous $G$–equivariant map

$$\mathfrak{R}_0 : \Omega \rightarrow \mathcal{G}$$

by identifying $\mathcal{G}$ with the tree dual to the hypersurface

$$\bigcup_{g \in G} g(\Sigma_h^-) \cup \bigcup_{g \in G} g(\Sigma_f^-)$$

in $\Omega$ and projecting in the usual manner, see [Sh].

The map $\mathfrak{R}_0$ extends continuously to a $G$–equivariant map

$$\mathfrak{R} : \mathbb{P}ML(S) \rightarrow \mathcal{G} = \mathcal{G} \cup \partial G.$$
whose restriction to $\mathbf{X}$ is the map we desire. The extension is described concretely as follows.

First note that given any point $[\eta] \in \mathbf{X}$, there is a unique sequence of elements $x_1^{\epsilon_1}, x_2^{\epsilon_2}, x_3^{\epsilon_3}, \ldots$ where $x_i \in \{f, h\}$ and $\epsilon_i \in \{\pm 1\}$ with the property that $[\eta]$ is contained in the nested intersection

$$\bigcap_{i=1}^{\infty} x_1^{\epsilon_1} \cdots x_i^{\epsilon_i}(B_{x_i}^{\epsilon_i}).$$

where $B_{x_{I+1}}^{\pm 1} = B_{x_I}^{\pm 1}$ and $B_{x_{I+1}}^{\pm 1} = B_{x_I}^{\pm 1}$. Identifying $\partial G$ with the set of infinite reduced words, our map is given there by

$$K([\eta]) = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots.$$ 

To see that $K$ is continuous, let $U_g \subset G$ be the open set consisting of all infinite reduced words in $\partial G$ with prefix $g$ together with the union of the open tails of the corresponding paths in $G$. Now if $g$ ends in $x_0^{\epsilon_0}$ with $x_0 \in \{f, h\}$ and $\epsilon_0 \in \{\pm 1\}$, then

$$K^{-1}(U_g) = g x_0^{-\epsilon_0}(\text{int} B_{x_0}^{\epsilon_0})$$

which is open.

**Lemma 4.4.** $K$ is a one-sided inverse to $\mathcal{J}$. That is, $K \circ \mathcal{J} = \text{id}_{\partial G}$.

**Proof.** Since $\mathbf{X}$ is a $G$–invariant closed set, it contains $\Lambda_G$, and so $K \circ \mathcal{J}$ is well-defined. Next, suppose that $x_f^+$ is the attracting fixed point of $f$. Then $\mathcal{J}(x_f^+) = [\mu_f^+]$ is the attracting fixed point in $\mathcal{PML}(S)$ of $f$, and hence $K(\mathcal{J}(x_f^+)) = x_f^+$. The same is true for any conjugate of $f$, and hence $K \circ \mathcal{J}$ is the identity on the set of attracting fixed points of conjugates of $f$. Being $G$–invariant, this set is dense in $\partial G$, and so, by continuity, $K \circ \mathcal{J}$ is the identity on all of $\partial G$. □

Theorem 4.2 follows easily from the following lemma.

**Lemma 4.5.** For all $x \in \partial G$, we have $K^{-1}(x) \subset Z \mathcal{J}(x)$. In fact, if $x$ is the repelling fixed point $x_h^-$ of a conjugate $g h$ of $h$, then $K^{-1}(x) = g Z \mu_h^-$. Otherwise, the set $K^{-1}(x)$ is a singleton contained in $\Lambda_G$.

**Proof of Theorem 4.2 assuming Lemma 4.5.** By the first statement, $\mathbf{X} \subset Z \Lambda_G$ since

$$Z \Lambda_G = \bigcup_{x \in \partial G} Z \mathcal{J}(x).$$

So $\Omega \supset \Delta_G$ as required. Again, the containment is proper as $Z \mu_h^+$ nontrivially intersects $\Omega$.

The description of $\Omega$ follows from the second and third statements. □

We need the following general fact about sequences of laminations.

**Lemma 4.6.** Suppose $\mathcal{S} \subset \mathcal{ML}(S)$ is a compact set, $\{f_k\} \subset \text{Mod}(S)$ is an infinite sequence of distinct pseudo-Anosov mapping classes with

$$\mu_{f_k}^+ \to \mu^+$$

in $\mathcal{ML}(S)$, and that $\{\nu_k\}_{k=1}^{\infty} \subset \mathcal{S}$ and $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ are sequences with

$$t_k f_k(\nu_k) \to \eta$$
in $\mathcal{ML}(S)$.

If there is an $r > 0$ such that
$$i(\nu, \mu^\pm) > r$$
for all $\nu \in S$, then $t_k \to 0$.

**Proof.** Note that continuity of $i$ and compactness of $S$ imply that there exist $K > 0$ and $R > 1$ such that for all $k \geq K$ and all $\nu \in S$
$$\frac{1}{R} < i(\nu, \mu^\pm_{f_k}) < R.$$

By the continuity of $i$ we have
$$\lim_{k \to \infty} i(t_k f_k(\nu_k), \mu^-_{f_k}) = i(\eta, \mu^-),$$
and so, for sufficiently large $k$, we have
$$i(\eta, \mu^-) - 1 < i(t_k f_k(\nu_k), \mu^-_{f_k}) < i(\eta, \mu^-) + 1.$$ The central term of this inequality is also given by
$$i(t_k f_k(\nu_k), \mu^-_{f_k}) = i(t_k \nu_k, f_k^{-1}(\mu^-_{f_k}))$$
$$= t_k i(\nu_k, \lambda f_k^{-1}(\mu^-_{f_k}))$$
$$= t_k \lambda(\nu_k, \mu^-_{f_k}),$$
where $\lambda(f_k)$ is the expansion factor of $f_k$, and so, for all sufficiently large $k$, we have
$$\frac{i(\eta, \mu^-) - 1}{R} < t_k \lambda(f_k) < R(i(\eta, \mu^-) + 1).$$

Since the $f_k$ are all distinct, and their fixed points converge in $\mathbb{P} \mathcal{ML}(S)$, it follows that $\lambda(f_k) \to \infty$. So $t_k \to 0$ as required.

**Proof of Lemma 4.5.** First assume that $x \in \partial G$ is the fixed point of a conjugate of $h$. By the $G$-equivariance of $\mathcal{R}$, it suffices to consider the case of $h$ itself. Then, we have $x = x^+_h$ or $x = x^-_h$. In this case, the sequences of balls nesting to $\mathcal{R}^{-1}(x^+_h)$ and $\mathcal{R}^{-1}(x^-_h)$ are given by
$$\{h^k(B^+_h)\}_{k=1}^\infty \text{ and } \{h^{-k}(B^-_h)\}_{k=1}^\infty,$$
respectively.

From the discussion in Section 2, we already know that
$$\mathcal{R}^{-1}(x^+_h) = \bigcap_{k=1}^\infty h^k(B^+_h) = \{[\mu^+_h]\} \subset Z\mathfrak{I}(x^+_h)$$
and
$$\mathcal{R}^{-1}(x^-_h) = \bigcap_{k=1}^\infty h^{-k}(B^-_h) = Z\mu^-_h = Z\mathfrak{I}(x^-_h).$$

If $g \in G$ is any other element not conjugate to a power of $h$, then, by Property 4.1, $g$ is pseudo-Anosov, and the dynamical properties of pseudo-Anosov mapping classes discussed in Section 2 implies
$$\mathcal{R}^{-1}(x^\pm(g)) = \{[\mu^\pm(g)]\} = Z\mathfrak{I}(x^\pm(g)).$$

Therefore, to complete the proof of the lemma, we assume that $x \in \partial G$ is not a fixed point of any element of $G$. 

We write $x$ as an infinite reduced word

$$x = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots.$$  

Since $x$ is not the fixed point of any element of $G$, we can assume that $x_n = f$ and, say, $\epsilon_n = +1$ for infinitely many $n$ (the case that $x_n = f$ and $\epsilon_n = -1$ for infinitely many $n$ is similar). The $G$–equivariance of $R$ implies that we may also assume that $x_1 = f$ and $\epsilon_1 = 1$. Let $\{n_k\}_{k=1}^\infty$ be the increasing sequence of natural numbers for which $x_{n_k} = f$ and $\epsilon_{n_k} = +1$. Finally, set

$$f_k = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots x_{n_k}^{\epsilon_{n_k}} \in G.$$

Then, we have $R^{-1}(x)$ expressed as the nested intersection

$$R^{-1}(x) = \bigcap_{k=1}^\infty f_k(B_f^+).$$

Any point $[\eta]$ in the frontier of $R^{-1}(x)$ is a limit of a sequence in the frontiers

$$[\eta] = \lim_{k \to \infty} f_k([\nu_k])$$

where $[\nu_k] \in \text{Fr}(B_f^+) = \Sigma_f^+$. We fix any such $[\eta] \in \text{Fr}(R^{-1}(x))$ and such a sequence $\{[\nu_k]\}$.

We pass to a further subsequence so that $\mu_{f_k}^+ \to \mu^+ \in \mathcal{ML}(S)$. Since $[\mu_{f_k}^+] \in \Lambda_G$ for all $k$, we also have $[\mu^+] \in \Lambda_G$. In fact, since $f_k = x_1^{\epsilon_1} \cdots x_{n_k}^{\epsilon_{n_k}}$ is cyclically reduced, the axes for $f_k$ in $G$ all go through the origin and limit to a geodesic $\gamma \subset G$ through $1$ with positive ray ending at $x$. Therefore, $x_{f_k}^+ \to x$ as $k \to \infty$, and by continuity of $I$, it follows that

$$I(x) = [\mu^+] \in \Lambda_G.$$  

Moreover, the negative ray of $\gamma$ ends at some point $y \in \partial G$ and is described by an infinite word

$$y = y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3} \cdots$$

where $y_1^{\delta_1} \neq f$ since $x_1^{\epsilon_1} = f$ and $\gamma$ is a geodesic. Therefore, again appealing to the continuity of $I$ we see that

$$I(y) = [\mu^-] \in \Lambda_G \cap \mathbb{PML}(S) - B_f^+.$$

By similar reasoning, for any $[\mu] \in \Lambda_G \cap B_f^+$, we have

$$f_k([\mu]) \to [\mu^+] = I(x).$$

In fact, it follows from [MP, Lemma 2.7] that there is a $\mu$ (a fixed point of a pseudo-Anosov in $G$) and a sequence $s_k$ tending to zero such that

$$\lim_{k \to \infty} s_k f_k(\mu) = \mu^+ \in \mathcal{ML}(S).$$

We now let $\mathcal{G} \subset \mathcal{ML}(S)$ be the image of $\Sigma_f^+$ under some continuous section of $\mathcal{ML}(S) \to \mathbb{PML}(S)$. Since $\Sigma_f^+ \cap Z\Lambda_G = \emptyset$, there is an $r > 0$ such that

$$i(\nu, \mu^+) > r$$

for every $\nu \in \mathcal{G}$.
We take the representatives \( \nu_k \) of \([\nu_k]\) to lie in \( \mathcal{S} \). Then, according to Lemma 4.6, the sequence \( t_k \) for which
\[
\lim_{k \to \infty} t_k f_k(\nu_k) = \eta
\]
must tend to zero. So
\[
\bar{i}(\eta, \mu^+) = \lim_{k \to \infty} \bar{i}(t_k f_k(\nu_k), s_k f_k(\mu)) = \lim_{k \to \infty} t_k s_k \bar{i}(\nu_k, \mu) = 0
\]
since \( s_k \) and \( t_k \) tend to zero and \( \bar{i}(\nu_k, \mu) \) is uniformly bounded by compactness of \( \mathcal{S} \). Since \( \mu^+ \) is uniquely ergodic, we conclude that \( [\eta] = [\mu^+] = \mathcal{I}(x) \).

This means that the frontier of \( K^{-1}(x) \) is precisely \( \{ \mathcal{I}(x) \} \), and hence \( K^{-1}(x) = \mathcal{I}(x) = Z \mathcal{I}(x) \) as required.

\[ \square \]

5. Final comments

If we replace \( h \) with \( h^{-1} \) in our construction we obtain another \( G \)-invariant open set \( \Omega' \) on which \( G \) acts properly discontinuously and cocompactly. By Lemma 4.5, we have descriptions
\[
\Omega = \mathbb{PML}(S) - \left( \Lambda_G \cup \bigcup_{g \in G} gZ\mu^+_h \right)
\]
and
\[
\Omega' = \mathbb{PML}(S) - \left( \Lambda_G \cup \bigcup_{g \in G} gZ\mu^-_h \right),
\]
and it follows that
\[
\Omega \cup \Omega' = \mathbb{PML}(S) - \left( \Lambda_G \cup \bigcup_{g \in G} G \cdot \alpha \right).
\]
The group \( G \) does not act properly discontinuously on \( \Omega \cup \Omega' \), and in fact, we have the following.

**Proposition 5.1.** If \( \mathcal{U} \subset \mathbb{PML}(S) \) is any open set on which \( G \) acts properly discontinuously, then \( \mathcal{U} \subset \Omega \) or \( \mathcal{U} \subset \Omega' \).

**Proof.** Let \( \mathcal{U} \subset \mathbb{PML}(S) \) be a \( G \)-invariant open set. We will show that if \( \mathcal{U} \) is not contained in either \( \Omega \) or \( \Omega' \), then \( G \) does not act properly discontinuously on \( \mathcal{U} \).

If \( \mathcal{U} \cap \Lambda_G \neq \emptyset \), then since \( G \) acts minimally on \( \Lambda_G \) and \( \mathcal{U} \) is \( G \)-invariant, we must have \( \Lambda_G \subset \mathcal{U} \). As \( G \) clearly fails to act properly discontinuously on \( \mathcal{U} \) in this case, we assume that \( \mathcal{U} \cap \Lambda_G = \emptyset \).

So if \( \mathcal{U} \) fails to be contained in either \( \Omega \) or \( \Omega' \), there are points \([\eta^+] \in \mathcal{U} \cap Z\mu^+_h \) and \([\eta^-] \in \mathcal{U} \cap Z\mu^-_h \). Moreover, \([\eta^+] \) is in the interior of \( Z\mu^+_h \). Let \( \mathcal{Y}^\pm \) be small compact balls contained in \( \mathcal{U} \) containing \([\eta^+] \). Since \([\eta^+] \in \Omega \), we may assume that \( \mathcal{Y}^+ \subset \Omega \). Moreover, \( G \)-invariance of \( \mathcal{U} \) allows us to pick \([\eta^+] \) and \( \mathcal{Y}^+ \) to lie in \( B^-_h \).
After passing to a subsequence, we can assume that the sequence of sets \( \{h^{-k_j}(\Upsilon^+)\}_{j=1}^{\infty} \) converges in the Hausdorff topology. Moreover, we have
\[
\lim_{j \to \infty} h^{-k_j}(\Upsilon^+) \subset \bigcap_{k=1}^{\infty} h^{-k}(B^-_k) = Z_{\mu^-_h}.
\]
Note that the Hausdorff limit must be connected since \( \Upsilon^+ \) is. This limit contains \( \alpha \) as the pointwise limit of \( h^{-k}[\eta^+] \), and \( [\mu^-_h] \) as the pointwise limit of any other point of \( \Upsilon^+ \) under \( h^{-k} \). Therefore,
\[
\lim_{j \to \infty} h^{-k_j}(\Upsilon^+) = Z_{\mu^-_h}.
\]
Now, consider the compact set \( \Upsilon = \Upsilon^+ \cup \Upsilon^- \). Since \( \text{int}(\Upsilon^-) \) is a neighborhood of \( [\eta^-] \), we have
\[
h^{-k_j}(\Upsilon) \cap \Upsilon \supset h^{-k_j}(\Upsilon^+) \cap \text{int}(\Upsilon^-) \neq \emptyset
\]
for all sufficiently large \( j \). So \( G \) does not act properly discontinuously on \( U \). \( \square \)

From this we deduce that \( \Omega \) and \( \Omega' \) are the only maximal open sets on which \( G \) acts properly discontinuously. By our descriptions of \( \Omega \) and \( \Omega' \) we also have
\[
\Delta_G = \Omega \cap \Omega'.
\]
It follows that \( \Delta_G \) can be described purely in terms of the action of \( G \) on \( \mathbb{PM}(S) \), without referring to geometric structures on the surface.

Though \( \Delta_G \) may not be a maximal open set on which \( G \) acts nicely, it remains a canonically defined one, and we pose the following question.

**Question 5.2.** If \( G \) is an irreducible subgroup of \( \text{Mod}(S) \), is \( \Delta_G \) the intersection of all maximal open sets on which \( G \) acts properly discontinuously?

**References**


