Blow up and regularity for fractal Burgers equation

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April 29, 2008

Abstract

The paper is a comprehensive study of the existence, uniqueness, blow up and regularity properties of solutions of the Burgers equation with fractional dissipation. We prove existence of the finite time blow up for the power of Laplacian $\alpha < 1/2$, and global existence as well as analyticity of solution for $\alpha \geq 1/2$. We also prove the existence of solutions with very rough initial data $u_0 \in L^p$, $1 < p < \infty$. Many of the results can be extended to a more general class of equations, including the surface quasi-geostrophic equation.

1 Introduction

The purpose of this paper is to present several results on Burgers equation with fractional dissipation

$$u_t = uu_x - (-\Delta)^\alpha u, \quad u(x,0) = u_0(x). \tag{1}$$

We will consider (1) on the circle $\mathbb{S}^1$. Equivalently, one can consider (1) on the real line with periodic initial data $u_0(x)$.

The Burgers equation with $\alpha = 0$ and $\alpha = 1$ has received an extensive amount of attention since the studies by Burgers in the 1940s (and it has been considered even earlier by Beteman [3] and Forsyth [12], pp 97–102). If $\alpha = 0$, the equation is perhaps the most basic example of a PDE evolution leading to shocks; if $\alpha = 1$, it provides an accessible model for studying the interaction between nonlinear and dissipative phenomena.

The Burgers equation can also be viewed as the simplest in the family of partial differential equations modeling the Euler and Navier-Stokes equation nonlinearity. Recently,

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there has been increased interest in models involving fractional dissipation, in particular Navier-Stokes (see [13]) and surface quasi-geostrophic equations (see e.g. [7, 4, 14, 18] for further references). Fractional dissipation also appears naturally in certain combustion models [15]. Our goal is to present here, in the most accessible framework of the Burgers equation, results and techniques that in some cases apply (with relatively straightforward adjustments) to a wider class of equations including in particular surface quasi-geostrophic. Among the results we prove are the global existence of solutions for $\alpha = 1/2$ (more generally $\alpha \geq 1/2$), space analyticity of solutions for $\alpha \geq 1/2$, the existence of solutions with very rough initial data, as well as blow up in finite time for $\alpha < 1/2$.

Let us now describe in more detail some of the results that we prove. We denote by W^s_p, $s \in \mathbb{R}$, $1 \leq p \leq \infty$ the standard Sobolev spaces. If $p = 2$ we use notation H^s, and denote by $\|\cdot\|_s$ the norm in H^s. Without loss of generality, we will consider equation (1) on the subspace of mean zero functions. This subspace is preserved by evolution and contains all non-trivial dynamics. The advantage of this subspace is that the H^s norm dominates the L^2 norm, and this simplifies the estimates. We will also always assume that the initial data (and so the solution) are real valued.

The case $\alpha > 1/2$ is subcritical, and smooth solutions exist globally. This is a fairly simple fact to prove, using the maximum principle control of the L^∞ norm for $\alpha \leq 1$, and straightforward estimates for $\alpha > 1$ (see [16, 9] for the quasi-geostrophic case and $1/2 \leq \alpha \leq 1$; the argument transfers to the dissipative Burgers equation without significant changes). One can also use the scheme of the proof suggested here for the critical case $\alpha = 1/2$. That is why we just state the result for $\alpha > 1/2$ without proof.

Theorem 1.1 *Assume that $\alpha > 1/2$, and the initial data $u_0(x)$ belongs to H^s, $s > 3/2 - 2\alpha$, $s \geq 0$. Then there exists a unique global solution of the equation (1) $u(x,t)$ that belongs to $C([0,\infty), H^s)$. Moreover, this solution is real analytic in x for $t > 0$.*

We have been unable to find the analyticity claim in the existent literature, but again the proof is parallel to that for the $\alpha = 1/2$ case, which we will do in detail. In what follows, we will consider mostly $0 < \alpha \leq 1/2$. Our first result concerns the local existence and uniqueness of classical solutions to Burgers equation with initial data in Sobolev spaces.

Let us denote $C_w([0,T], L^2)$ the class of solutions that are weakly continuous as functions with values in L^2.

Definition. We say that $u \in L^2([0,T], L^2) \cap C_w([0,T], L^2)$ and such that $du/dt \in L^1([0,T], H^{-1})$ is a weak solution of (1) for $t \in (0,T)$ if for any smooth periodic function $\varphi(x)$ we have

$$(u,\varphi)_t = -\frac{1}{2}(u^2, \varphi_x) - (u, (-\Delta)^\alpha \varphi), \quad \text{a.e. } t \in (0,T), \quad (u,\varphi)(0) = (u_0, \varphi). \qquad (2)$$

Note that then $(u,\varphi)(t)$ is absolutely continuous and

$$(u,\varphi)(t) - (u_0,\varphi) = \int\limits_0^t \Big(-\frac{1}{2}(u^2,\varphi_x) - (u,(-\Delta)^\alpha\varphi)\Big)\,ds \qquad (3)$$

for any $t \in [0,T]$.

Theorem 1.2 *Assume that $0 < \alpha \le 1/2$, and the initial data $u_0(x) \in H^s$, $s > 3/2 - 2\alpha$. Then there exists $T = T(\alpha, \|u_0\|_s) > 0$ such that there exists a weak solution of the equation (1) $u(x,t)$ satisfying $u(x,t) \in C([0,T],H^s) \cap L^2([0,T],H^{s+\alpha})$. Moreover, u can be chosen to satisfy $u(x,t) \in C^\infty$ for $0 < t < T$. If v is another weak solution of (1) with initial data u_0 such that $v \in C([0,T],L^2) \cap L^{3/2\delta}([0,T],H^\delta)$ with some $\delta \in (1/2,1]$, then v coincides with u.*

Remark 1. In particular, it follows that the solution u in Theorem 1.2 solves (1) in classical sense for every $t > 0$.
Remark 2. It is clear from Theorem 1.2 that the solution is unique in the $C([0,T],H^s)$ class.

We prove a slightly stronger version of Theorem 1.2 in Section 2.
In the case $\alpha = 1/2$, we prove a result similar to Theorem 1.1. However, the critical case is harder and requires a new nonlocal maximum principle. We handle this case in Section 3.

Theorem 1.3 *Assume that $\alpha = 1/2$, and that the initial data u_0 belongs to H^s with $s > 1/2$. Then there exists a global solution of (1) which is real analytic in x for any $t > 0$.*

The corresponding question for the quasi-geostrophic equation has been a focus of significant effort (see e.g. [6, 7, 9]) and has been recently resolved independently and by different means in [14] and [4]. The proof in [14] is similar to the argument presented here.
Next, we prove the finite time blow up for the supercritical case $\alpha < 1/2$.

Theorem 1.4 *Assume that $0 < \alpha < 1/2$. Then there exists smooth periodic initial data $u_0(x)$ such that the solution $u(x,t)$ of (1) blows up in H^s for each $s > \frac{3}{2} - 2\alpha$ in finite time.*

We will also obtain a fairly precise picture of blow up, similar to that of Burgers equation without dissipation – a shock is formed where the derivative of the solution becomes infinite. In the scenario we develop, the initial data leading to blow up is odd and needs to satisfy a certain size condition, but no other special assumptions. After this work has been completed, we became aware of the preprint [1], where a result similar to Theorem 1.4 is proved in the whole line (not periodic) setting, and for a class of initial data satisfying certain convexity assumption.

The blow up or global regularity for $\alpha < 1/2$ remains open for the surface quasi-geostrophic equation. The problem is that the conservative surface quasi-geostrophic dynamics is not well understood, in contrast to the non-viscous Burgers equation where finite time shock formation is both well-known and simple. Existence of blow up in the non-viscous quasi-geostrophic equation remains a challenging open question (see e.g. [8]). Still, some elements of the blow up construction we present here may turn out to be useful in future attempts to attack the question of blow up or regularity for the dissipative surface quasi-geostrophic equation.

In Section 5, we prove existence of solutions with rough initial data, $u_0 \in L^p$, $1 < p < \infty$, when $\alpha = 1/2$ (the case $\alpha > 1/2$ is similar). These solutions become smooth immediately for $t > 0$, however the behavior near zero may be quite singular. The uniqueness for such solutions is not known and remains an interesting open problem.

The results of Theorems 1.1, 1.2, 1.3, 1.4 remain valid for the case $s = 3/2 - 2\alpha$, with slight modifications. However, more subtle estimates are needed. This critical space case is handled in Section 6.

2 Local existence, uniqueness and regularity

Denote by P^N the orthogonal projection to the first $(2N+1)$ eigenfunctions of Laplacian, $e^{2\pi ikx}$, $k = 0, \pm 1, \dots, \pm N$. Consider Galerkin approximations $u^N(x,t)$, satisfying

$$u^N_t = P^N(u^N u^N_x) - (-\Delta)^\alpha u^N, \;\; u^N(x,0) = P^N u_0(x). \tag{4}$$

We start with deriving some a-priori bounds for the growth of Sobolev norms.

Lemma 2.1 *Assume that $s \geq 0$ and $\beta \geq 0$. Then*

$$\left| \int (u^N)^2 (-\Delta)^s u^N_x \, dx \right| \leq C \|u^N\|_q \|u^N\|^2_{s+\beta} \tag{5}$$

for any q satisfying $q > 3/2 - 2\beta$.

Proof. On the Fourier side, the integral in (5) is equal to (up to a constant factor)

$$\sum_{k+a+b=0, |k|,|a|,|b| \leq N} k|k|^{2s} \hat{u}^N(k) \hat{u}^N(a) \hat{u}^N(b) =: S.$$

Symmetrizing, we obtain

$$|S| = \frac{1}{3}\left|\sum_{k+a+b=0,|k|,|a|,|b|\le N} \left(k|k|^{2s} + a|a|^{2s} + b|b|^{2s}\right) \hat{u}^N(k)\hat{u}^N(a)\hat{u}^N(b)\right| \le \tag{6}$$
$$2\sum_{k+a+b=0,|a|\le|b|\le|k|\le N} \left| k|k|^{2s} + a|a|^{2s} + b|b|^{2s}\right| |\hat{u}^N(k)\hat{u}^N(a)\hat{u}^N(b)|.$$

Next, note that under conditions $|a| \le |b| \le |k|, \quad a+b+k=0$, we have $|a| \le |k|/2, \ |b| \ge |k|/2$ and

$$\begin{aligned}&\left|k|k|^{2s} + a|a|^{2s} + b|b|^{2s}\right| = \left|b(|b|^{2s} - |b+a|^{2s}) + a(|a|^{2s} - |k|^{2s})\right| \le \\ &C(s)|a||k|^{2s} \le C(s)|a|^{1-2\beta}|b|^{s+\beta}|k|^{s+\beta}.\end{aligned} \tag{7}$$

Thus

$$|S| \le C\sum_{k+a+b=0,|a|\le|b|\le|k|\le N} |a|^{1-2\beta}|b|^{s+\beta}|k|^{s+\beta}|\hat{u}^N(k)\hat{u}^N(a)\hat{u}^N(b)| \le$$
$$C\|u^N\|^2_{s+\beta}\sum_{|a|\le N}|a|^{1-2\beta}|\hat{u}^N(a)| \le C(\beta,q,s)\|u^N\|_q\|u^N\|^2_{s+\beta}.$$

Here the second inequality is due to Parseval and convolution estimate, and the third holds by Hölder's inequality for every $q > 3/2 - 2\beta$. ◻

Lemma 2.1 implies a differential inequality for the Sobolev norms of solutions of (4).

Lemma 2.2 *Assume that $\alpha > 0$, $q > 3/2 - 2\alpha$, and $s \ge 0$. Then*

$$\frac{d}{dt}\|u^N\|_s^2 \le C(q)\|u^N\|_q^{M(q,\alpha,s)} - \|u^N\|^2_{s+\alpha}. \tag{8}$$

If in addition $s = q$ then

$$\frac{d}{dt}\|u^N\|_s^2 \le C(\epsilon)\|u^N\|_s^{2+\frac{\alpha}{\epsilon}} - \|u^N\|^2_{s+\alpha}, \tag{9}$$

for any

$$0 < \epsilon < \min\left(\frac{2q-3+4\alpha}{4}, \alpha\right). \tag{10}$$

Proof. Multiplying both sides of (4) by $(-\Delta)^s u^N$, and applying Lemma 2.1, we obtain (here we put $\beta := \alpha - \epsilon$, with ϵ satisfying (10))

$$\frac{d}{dt}\|u^N\|_s^2 \le C(q,\epsilon,\alpha,s)\|u^N\|_q\|u^N\|^2_{s+\alpha-\epsilon} - 2\|u^N\|^2_{s+\alpha}.$$

Observe that if $q \geq s + \alpha - \epsilon$, the estimate (8) follows immediately. If $q < s + \alpha - \epsilon$, by Hölder we obtain

$$\|u^N\|_{s+\alpha-\epsilon}^2 \leq \|u^N\|_{s+\alpha}^{2(1-\delta)} \|u^N\|_q^{2\delta} \tag{11}$$

where $\delta = \dfrac{\epsilon}{s+\alpha-q}$. Applying Young's inequality we finish the proof of (8) in this case.

The proof of (9) is similar. We have

$$\frac{d}{dt}\|u^N\|_s^2 \leq C(s,\epsilon,\alpha)\|u^N\|_s\|u^N\|_{s+\alpha-\epsilon}^2 - 2\|u^N\|_{s+\alpha}^2.$$

Applying the estimate (11) with $q = s$ and $\delta = \epsilon/\alpha$ and Young's inequality we obtain

$$\frac{d}{dt}\|u^N\|_s^2 \leq C\|u^N\|_s^{1+2\epsilon/\alpha}\|u^N\|_{s+\alpha}^{2-2\epsilon/\alpha} - 2\|u^N\|_{s+\alpha}^2 \leq C\|u^N\|_s^{2+\frac{\alpha}{\epsilon}} - \|u^N\|_{s+\alpha}^2.$$

□

The following lemma is an immediate consequence of (9) and local existence of the solution to the differential equation $y' = Cy^{1+\alpha/2\epsilon}$, $y(0) = y_0$.

Lemma 2.3 *Assume $s > 3/2 - 2\alpha$, $\alpha > 0$ and $u_0 \in H^s$. Then there exists time $T = T(s,\alpha,\|u_0\|_s)$ such that for every N we have the bound (uniform in N)*

$$\|u^N\|_s(t) \leq C(s,\alpha,\|u_0\|_s), \quad 0 \leq t \leq T. \tag{12}$$

Proof. From (9), we get that $z(t) \equiv \|u^N\|_s^2$ satisfies the differential inequality $z' \leq Cz^{1+\alpha/2\epsilon}$. This implies the bound (12) for time T which depends only on coefficients in the differential inequality and initial data. □

Now, we obtain some uniform bounds for higher order H^s norms of the Galerkin approximations.

Theorem 2.4 *Assume $s > 3/2 - 2\alpha$, $s \geq 0$, $\alpha > 0$ and $u_0 \in H^s$. Then there exists time $T = T(s,\alpha,\|u_0\|_s)$ such that for every N we have the bounds (uniform in N)*

$$t^{n/2}\|u^N\|_{s+n\alpha} \leq C(n,s,\alpha,\|u_0\|_s), \quad 0 < t \leq T, \tag{13}$$

for any $n \geq 0$. Here time T is the same as in Lemma 2.3.

Proof. We are going to first verify (13) by induction for positive integer n. For $n = 0$, the statement follows from Lemma 2.3. Inductively, assume that $\|u^N\|_{s+n\alpha}^2(t) \leq Ct^{-n}$ for $0 \leq t \leq T$. Fix any $t \in (0,T]$, and consider the interval $I = (t/2,t)$. By (8) with s replaced by $s + n\alpha$ and q by s, we have for every $n \geq 0$

$$\frac{d}{dt}\|u^N\|_{s+n\alpha}^2 \leq C\|u^N\|_s^M - \|u^N\|_{s+(n+1)\alpha}^2. \tag{14}$$

Due to Lemma 2.3 and our induction assumption,

$$\int_{t/2}^{t} \|u^N\|_{s+(n+1)\alpha}^2\,ds \le Ct + C\|u^N(t/2)\|_{s+n\alpha}^2 \le Ct^{-n}.$$

Thus we can find $\tau \in I$ such that

$$\|u^N(\tau)\|_{s+(n+1)\alpha}^2 \le C|I|^{-1}t^{-n} \le Ct^{-n-1}.$$

Moreover, from (14) with n changed to $n+1$ we find that

$$\|u^N(t)\|_{s+(n+1)\alpha}^2 \le \|u^N(\tau)\|_{s+(n+1)\alpha}^2 + Ct \le Ct^{-n-1},$$

concluding the proof for integer n. Non-integer n can be covered by interpolation:

$$\|u^N\|_{s+r\alpha} \le \|u^N\|_s^{1-\frac{r}{n}}\|u^N\|_{s+n\alpha}^{\frac{r}{n}}, \quad 0 < r \le n.$$

□

Now we are ready to prove existence and regularity of a weak solution of Burgers equation (1).

Theorem 2.5 *Assume $s > 3/2 - 2\alpha$, $s \ge 0$, $\alpha > 0$, and $u_0 \in H^s$. Then there exists $T(s, \alpha, \|u_0\|_s) > 0$ and a solution $u(x,t)$ of (1) such that*

$$u \in L^2([0,T], H^{s+\alpha}) \cap C([0,T], H^s); \tag{15}$$

$$t^{n/2}u \in C((0,T], H^{s+n\alpha}) \cap L^\infty([0,T], H^{s+n\alpha}) \tag{16}$$

for every $n > 0$.

Corollary 2.6 *If $\alpha > 0$ and $u_0 \in H^s$ with $s > 3/2 - 2\alpha$, $s \ge 0$, then there exists a local solution $u(x,t)$ which is C^∞ for any $0 < t \le T$.*

Proof. The proof of Theorem 2.5 is standard. It follows from (4) and (13) that for every small $\epsilon > 0$ and every $r > 0$ we have uniform in N and $t \in [\epsilon, T]$ bounds

$$\|u_t^N\|_r \le C(r, \epsilon). \tag{17}$$

By (16) and (17) and the well known compactness criteria (see e.g. [5], Chapter 8), we can find a subsequence u^{N_j} converging in $C([\epsilon, T], H^r)$ to some function u. Since ϵ and r are arbitrary one can apply the standard subsequence of subsequence procedure to find a subsequence (still denoted by u^{N_j}) which converges to u in $C((0,T], H^r)$, for any $r > 0$. The limiting function u must satisfy the estimates (13) and it is straightforward to check that it solves the Burgers equation on $(0,T]$. Thus, it remains to show that u can be made to converge to u_0 strongly in H^s as $t \to 0$.

We start by showing that u converges to u_0 as $t \to 0$ weakly in H^s. Let $\varphi(x)$ be arbitrary C^∞ function. Consider

$$g^N(t,\varphi) \equiv (u^N,\varphi) = \int u^N(x,t)\varphi(x)\,dx.$$

Clearly, $g^N(\cdot,\varphi) \in C([0,\tau])$, where $\tau \equiv T/2$. Also, taking inner product of (4) with φ we can show that for any $\delta > 0$,

$$\int_0^\tau |g_t^N|^{1+\delta}\,dt \le C\left(\int_0^\tau \|u^N\|_{L^2}^{2+2\delta}\|\varphi\|_{W^1_\infty}^{1+\delta}\,dt + \int_0^\tau \|u^N\|_{L^2}^{1+\delta}\|\varphi\|_{2\alpha}^{1+\delta}\,dt\right). \tag{18}$$

Due to (9), the definition of τ, and the condition $s \ge 0$, we have that $\|u^N\|_{L^2} \le C$ on $[0,\tau]$, and thus $\|g_t^N(\cdot,\varphi)\|_{L^{1+\delta}} \le C(\varphi)$. Therefore the sequence $g^N(t,\varphi)$ is compact in $C([0,\tau])$, and we can pick a subsequence $g^{N_j}(t,\varphi)$ converging uniformly to a function $g(t,\varphi) \in C([0,\tau])$. Clearly, by choosing an appropriate subsequence we can assume $g(t,\varphi) = (u,\varphi)$ for $t \in (0,\tau]$. Next, we can choose a subsequence $\{N_j\}$ such that $g^{N_j}(t,\varphi)$ has a limit for any smooth function φ from a countable dense set in H^{-s}. Given that we have uniform control over $\|u^{N_j}\|_s$ on $[0,\tau]$, it follows that $g^{N_j}(t,\varphi)$ converges uniformly on $[0,\tau]$ for every $\varphi \in H^{-s}$. Now for any $t > 0$,

$$|(u-u_0,\varphi)| \le |(u-u^{N_j},\varphi)| + |(u^{N_j}-u_0^{N_j},\varphi)| + |(u_0^{N_j}-u_0,\varphi)|. \tag{19}$$

The first and the third terms in RHS of (19) can be made small uniformly in $(0,\tau]$ by choosing sufficiently large N_j. The second term tends to zero as $t \to 0$ for any fixed N_j. Thus $u(\cdot,t)$ converges to $u_0(\cdot)$ as $t \to 0$ weakly in H^s. Consequently,

$$\|u_0(\cdot)\|_s \le \liminf_{t\to 0} \|u(\cdot,t)\|_s. \tag{20}$$

Furthermore, it follows from (9) that for every N the function $\|u^N\|_s^2(t)$ is always below the graph of the solution of the equation

$$y_t = Cy^{1+\frac{\alpha}{2\epsilon}}, \quad y(0) = \|u_0\|_s^2.$$

By construction of the solution u, the same is true for $\|u\|_s^2(t)$. Thus, $\|u_0\|_s \ge \limsup_{t\to 0} \|u\|_s(t)$. From this and (20), we obtain that $\|u_0\|_s = \lim_{t\to 0} \|u\|_s(t)$. This equality combined with weak convergence finishes the proof. □

We next turn to the uniqueness. First of all, we obtain some identities which hold for every weak solution of the Burgers equation (1). Let u be a solution of the Burgers equation in a sense of (3), (2). Then for any function f of the form

$$f(x,t) := \sum_{k=1}^{K} \varphi_k(x)\psi_k(t),$$

where $\varphi_k \in C^\infty(\mathbb{T})$ and $\psi_k \in C_0^\infty([0,T])$, we have (see (2))

$$\sum_{k=1}^{K}(u,\varphi_k)_t\psi_k = -\frac{1}{2}(u^2, f_x) - (u, (-\Delta)^\alpha f), \quad \text{a.e. } t \in (0,T).$$

Integrating and using integration by parts on the left hand side we obtain

$$-\int_0^T (u, f_t)\, dt = -\frac{1}{2}\int_0^T (u^2, f_x)\, dt - \int_0^T (u, (-\Delta)^\alpha f)\, dt. \tag{21}$$

Applying closure arguments to (21) and using inclusion $u \in L^1([0,T], L^2)$ we derive the following statement.

Lemma 2.7 *Let u be a weak solution of the Burgers equation* (1) *in the sense of* (3), (2)*. Then for every function $f \in C_0^\infty([0,T], C^\infty(\mathbb{T}))$ we have*

$$-\int_0^T (u, f_t)\, dt = -\frac{1}{2}\int_0^T (u^2, f_x)\, dt - \int_0^T (u, (-\Delta)^\alpha f)\, dt, \quad f \in C_0^\infty([0,T], C^\infty(\mathbb{T})). \tag{22}$$

Now, we are ready to prove

Theorem 2.8 *Assume $v(x,t)$ is a weak solution of* (1) *for $0 < \alpha \le 1/2$, and initial data $u_0 \in H^s$, $s > 3/2 - 2\alpha$. If*

$$v(x,t) \in C([0,T], L^2) \cap L^{3/2\delta}([0,T], H^\delta), \quad 1 \ge \delta > 1/2, \tag{23}$$

then $v(x,t)$ coincides with the solution $u(x,t)$ described in Theorem 2.5.

Remark. Theorems 2.8 and 2.5 imply Theorem 1.2.

Proof. We will need an auxiliary estimate for $\|v^2\|_{1-\delta}$. Recall that by integral characterization of Sobolev spaces,

$$\|v^2\|_{1-\delta} \le C\left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|v(x)^2 - v(y)^2|}{|x-y|^{1-\delta}}\, dxdy + \|v^2\|_{L^2}\right) \le C\|v\|_{L^\infty}\|v\|_{1-\delta}.$$

Recall that for $1/2 < \delta \le 1$, we have $\|v\|_{L^\infty} \le C\|v\|^{1-1/2\delta}\|v\|_\delta^{1/2\delta}$. Applying this inequality the L^∞ norm and Hölder inequality to $H^{1-\delta}$ norm above we obtain

$$\|v^2\|_{1-\delta} \le C'\|v\|^{2-k}\|v\|_\delta^k, \quad k := \frac{3}{2\delta} - 1. \tag{24}$$

Now, let us obtain a bound for $\frac{dv}{dt}$. Since v is a weak solution of the equation (1), due to Lemma 2.7 for every function $f(x,t) \in C_0^\infty([0,T], C^\infty(\mathbb{T}))$ we have

$$\left|\int_0^T (v, f_t)\, dt\right| \le \frac{1}{2}\int_0^T |(D^{1-\delta}(v^2), D^\delta f)|\, dt + \int_0^T |(D^{2\alpha-\delta} v, D^\delta f)|\, dt \le \left(\int_0^T \|f\|_\delta^\gamma dt\right)^{1/\gamma} \left(\frac{1}{2}\left(\int_0^T \|v^2\|_{1-\delta}^{\gamma'} dt\right)^{1/\gamma'} + \left(\int_0^T \|v\|_{1-\delta}^{\gamma'} dt\right)^{1/\gamma'}\right). \tag{25}$$

Here $D := (-\Delta)^{1/2}$, $\gamma := \frac{3}{2\delta}$ and $\gamma^{-1} + (\gamma')^{-1} = 1$. It follows from (23), (24) and equality $\gamma' k = \gamma$ that the integral $\int_0^T \|v^2\|_{1-\delta}^{\gamma'} dt$ is convergent. The estimate for $\int_0^T \|v\|_{1-\delta}^{\gamma'} dt$ is similar and even simpler. Thus, it follows from (25) that $\frac{dv}{dt}$ belongs to $L^{\gamma'}([0,T], H^{-\delta})$. Certainly, the same (and even more) is true for $\frac{du}{dt}$. Thus, $((u-v)_t, (u-v)) \in L^1([0,T])$. Moreover, it follows from the definition of a weak solution, our assumptions and estimates for v_t and v^2 that for a.e. $t \in [0,T]$

$$u_t - v_t = \frac{1}{2}(u^2)_x - \frac{1}{2}(v^2)_x - (-\Delta)^\alpha (u-v), \quad \text{a.e. } t \in [0,T],$$

where the equality is understood in $H^{-\delta}$ sense. Thus,

$$2((u-v)_t, (u-v)) = ((u^2)_x - (v^2)_x, u-v) - 2\|u-v\|_\alpha^2, \quad \text{a.e. } t \in [0,T]. \tag{26}$$

For every fixed $t \in [0,T]$ where (26) holds we approximate v in H^δ by smooth functions v_n. Direct calculations using integration by parts give

$$((u^2)_x - (v_n^2)_x, u - v_n) = 2(u_x, (u-v_n)^2) + 2((u-v_n)_x, v_n(u-v_n)) =$$
$$2(u_x, (u-v_n)^2) - 2((u-v_n)_x, (u-v_n)^2) + 2((u-v_n)_x, u(u-v_n)) = (u_x, (u-v_n)^2).$$

Due to (23) and (24) we may apply closure arguments in H^δ to obtain

$$((u^2)_x - (v^2)_x, u-v) = (u_x, (u-v)^2).$$

Substituting into (26) we get

$$2((u-v)_t, (u-v)) = (u_x, (u-v)^2) - 2\|u-v\|_\alpha^2, \quad \text{a.e. } t \in [0,T]. \tag{27}$$

Note that $2((u-v)_t, (u-v)) = \partial_t \|u-v\|^2$. Indeed, denote $u - v =: g$. Recall that $g_t \in L^{\gamma'}([0,T], H^{-\delta})$ and $g \in C([0,T], L^2) \cap L^\gamma([0,T], H^\delta)$. Approximate g in $W_{\gamma'}^1([0,T], H^{-\delta}) \cap C([0,T], L^2) \cap L^\gamma([0,T], H^\delta)$ by smooth functions g_n. Then $2((g_n)_t, g_n) = \partial_t \|g_n\|^2$ and for every $t \in (0,T]$ we have

$$\int_0^t 2((g_n)_t, g_n)\, dt = \|g_n\|^2(t) - \|g_n\|^2(0).$$

Now, we can take the limit to obtain

$$\int_0^t 2(g_t, g)\,dt = \|g\|^2(t) - \|g\|^2(0).$$

This proves the desired identity.

Finally,

$$|(u_x, (u-v)^2)| = \left|\int u_x(u-v)^2\,dx\right| \le \|u-v\|^2_{L^{2p}}\|u\|_{W^1_{p'}},$$

where $p = \frac{1}{1-\alpha}$ and p' is the Hölder conjugate exponent to p. The exponent p was chosen so that, by Gagliardo-Nirenberg inequality,

$$\|u-v\|^2_{L^{2p}} \le C\|u-v\|\|u-v\|_\alpha.$$

Also by Sobolev inequality

$$\|u\|_{W^1_{p'}} \le C\|u\|_r,$$

where $r = 3/2 - \alpha$. Thus

$$\left|\int u_x(u-v)^2\,dx\right| \le C\|u-v\|\|u-v\|_\alpha\|u\|_r. \tag{28}$$

Then from (27), (28) we find

$$\|u-v\|^2(t) - \|u-v\|^2(0) = 2\int_0^t ((u-v)_s, (u-v))\,ds \le C\int_0^t \|u-v\|^2\|u\|_r^2\,ds, \text{ for every} t \in [0,T]. \tag{29}$$

Now we are in position to apply Gronwall inequality. Notice that $\int_0^T \|u\|^2_{s+\alpha}\,dt$ is controlled by $\|u\|^2_{C([0,T],H^s)}$ because of (9). But if $s > 3/2 - 2\alpha$, then $\int_0^T \|u\|_r^2\,dt$ is also under control. This proves the Theorem. □

3 Global existence and analyticity for the critical case $\alpha = 1/2$

As we have already mentioned in the introduction, if $\alpha > 1/2$, then the local smooth solution can be extended globally. In Section 4, we show that a blow up can happen in finite time if $\alpha < 1/2$. Thus the only remaining case to consider is the critical one, $\alpha = 1/2$. In this section, we will prove

Theorem 3.1 *Assume* $\alpha = 1/2$, *and* $u_0 \in H^s$, $s > 1/2$. *Then there exists a global solution* $u(x,t)$ *of* (1) *which belongs to* $C([0,\infty), H^s) \cap C((0,\infty), C^\infty)$. *If* v *is another weak solution of* (1) *with initial data* u_0 *such that* $v \in C([0,T], L^2) \cap L^{3/2\delta}([0,T], H^\delta)$ *with some* $\delta \in (1/2, 1]$, *then* v *coincides with* u *on* $[0,T]$.

We also state the following result separately to break the otherwise unwieldy proof:

Theorem 3.2 *The solution of Theorem 3.1 is real analytic for every* $t > 0$.

Together, Theorems 3.1 and 3.2 imply Theorem 1.3. We first discuss the global existence. Much of the discussion follows [14]; we reproduce the argument here for the sake of completeness. Recall that a modulus of continuity is an arbitrary increasing continuous concave function $\omega : [0,+\infty) \to [0,+\infty)$ such that $\omega(0) = 0$. Also, we say that a function $f : \mathbb{R} \to \mathbb{R}$ has modulus of continuity ω if $|f(x) - f(y)| \leq \omega(|x-y|)$ for all $x, y \in \mathbb{R}$.

The term uu_x in the dissipative Burgers equation tends to make the modulus of continuity of u worse while the dissipation term $(-\Delta)^{1/2}u$ tends to make it better. Our aim is to construct some special moduli of continuity for which the dissipation term always prevails and such that every periodic C^∞-function u_0 has one of these special moduli of continuity.

Our moduli of continuity will be derived from one single function $\omega(\xi)$ by scaling: $\omega_B(\xi) = \omega(B\xi)$. Note that the critical ($\alpha = \frac{1}{2}$) equation has a simple scaling invariance: if $u(x,t)$ is a solution, then so is $u(Bx, Bt)$. This means that if we prove that the modulus of continuity ω is preserved by the evolution, then the whole family $\omega_B(\xi) = \omega(B\xi)$ of moduli of continuity will also be preserved (provided that we look at the initial data of all periods). Also observe that if ω is unbounded, then every C^∞ periodic function has modulus of continuity ω_B if $B > 0$ is sufficiently large.

We will eventually have an explicit expression for ω. For now, we show how preservation of ω is used to control the solution.

Lemma 3.3 *Assume that* $\omega(\xi)$ *satisfies*

$$\omega'(0) < \infty, \;\; \omega''(0) = -\infty. \tag{30}$$

Then if a smooth function f *has modulus of continuity* ω, *it must satisfy* $\|f'\|_{L^\infty} < \omega'(0)$.

Proof. Indeed, take a point $x \in \mathbb{R}$ at which $\max |f'|$ is attained and consider the point $y = x + \xi$. Then we must have $f(y) - f(x) \leq \omega(\xi)$ for all $\xi \geq 0$. But the left hand side is at least $|f'(x)|\xi - C\xi^2$ where $C = \frac{1}{2}\|f''\|_{L^\infty}$ while the right hand side can be represented as $\omega'(0)\xi - \rho(\xi)\xi^2$ with $\rho(\xi) \to +\infty$ as $\xi \to 0+$. Thus $|f'(x)| \leq \omega'(0) - (\rho(\xi) - C)\xi$ for all $\xi > 0$ and it remains to choose some $\xi > 0$ satisfying $\rho(\xi) > C$. $\square$

Given this observation, Theorem 3.1 will be proved as follows. By Theorem 2.5, if $u_0 \in H^s$, $s > 1/2$, then the solution immediately becomes C^∞ and stays smooth at least till time T. Hence it will preserve one of our moduli of continuity, ω_B. This will imply that $\|u_x\|_{L^\infty}$ remains bounded by $\omega_B'(0)$. Starting at time T we construct again Galerkin approximations which define our solution for one more step in time. And we continue this process inductively. Since our solution remains smooth and its $W^{1,\infty}$ norm satisfies uniform bound, it implies uniform boundedness of, for instance, H^1 norm for which our local existence result is valid. Therefore our time step can be chosen to be fixed. Thus we obtain smooth global solution of the equation (1).

We next proceed with the construction of ω and the proof of preservation. Let us outline the only scenario how the modulus of continuity satisfying (30) can be lost.

Lemma 3.4 *Assume that a smooth solution $u(x,t)$ has modulus of continuity ω at some time t_0. The only way this modulus of continuity may be violated is if there exists $t_1 \geq t_0$ and y, z, $y \neq z$, such that $u(y,t_1) - u(z,t_1) = \omega(|y-z|)$, while for all $t < t_1$, the solution has modulus of continuity ω.*

Proof. Assume that $u(x,t)$ loses modulus of continuity ω. Define

$$\tau = \sup\{t : \forall x, y, \quad |u(x,t) - u(y,t)| \leq \omega(|x-y|).\}$$

Then u remains smooth up to τ, and, by local existence and regularity theorem, for a short time beyond τ. Suppose that $|u(x,\tau) - u(y,\tau)| < \omega(|x-y|)$ for all $x \neq y$. We claim that in this case u has modulus of continuity ω for all $t > \tau$ sufficiently close to τ. Indeed, by Lemma 3.3 at the moment τ we have $\|u'\|_{L^\infty} < \omega'(0)$. By continuity of derivatives and compactness in space variable, this also holds for $t > \tau$ close to τ, which immediately takes care of the inequality $|u(x,t) - u(y,t)| < \omega(|x-y|)$ for small $|x-y|$. Observe that we only need to consider x, y within a fixed bounded domain since u is periodic and ω increasing. Thus, it suffices to show that $|u(x,t) - u(y,t)| < \omega(|x-y|)$ holds for all t close enough to τ and x, y such that $\delta \leq |x-y| \leq \delta^{-1}$ with some $\delta > 0$. But this follows immediately from the inequality for time τ, smoothness of the solution and compactness of the domain. □

Remark. The key point of the above lemma is that we do not have to worry about ω being violated "first" at the diagonal $x = y$, namely $||u'||_{L^\infty} = \omega'(0)$; the modulus of continuity equality must happen first at two distinct points. This knowledge makes the argument below simpler by ruling out the extra case which otherwise would have to be considered.

Proof. [Proof of Theorem 3.1] Assume now that $u(y,t_1) - u(z,t_1) = \omega(|y-z|)$ for some y, z and $|y-z| = \xi > 0$. We will henceforth omit t_1 from notation. The plan now is to show that we have necessarily $\frac{d}{dt}(u(y,t_1) - u(z,t_1)) < 0$. We need to estimate the flow and the dissipative terms entering the Burgers equation. First, note that

$$u(y)u'(y) = \frac{d}{dh} u(y + hu(y)) \Big|_{h=0}$$

and similarly for z. But

$$u(y + hu(y)) - u(z + hu(z)) \leq \omega\left(|y-z| + h|u(y) - u(z)|\right) \leq \omega(\xi + h\omega(\xi)).$$

Since also $u(y) - u(z) = \omega(\xi)$, we conclude that

$$u(y)u'(y) - u(z)u'(z) \le \omega(\xi)\omega'(\xi).$$

Note that we assume differentiability of ω here. The ω that we will construct below is differentiable except at one point, and this special point is handled easily (by using the larger of the one-sided derivatives). Next let us estimate the difference of dissipative terms. Due to translation invariance, it is sufficient to consider $y = \xi/2$ and $z = -\xi/2$. Let us denote by P_h the one dimensional Poisson kernel, $P_h(x) = \dfrac{1}{\pi}\dfrac{h}{x^2+h^2}$. Recall that

$$-(-\Delta)^{1/2}u(x) = \frac{d}{dh}P_h * u(x)\Big|_{h=0}.$$

By the Poisson summation formula, this equality is valid for periodic $u(x)$ of every period. By symmetry and monotonicity of the Poisson kernel,

$$(P_h * u)(y) - (P_h * u)(z) = \int_0^\infty [P_h(\xi/2-\eta) - P_h(-\xi/2-\eta)](u(\eta) - u(-\eta))\,d\eta \le$$
$$\int_0^\infty [P_h(\xi/2-\eta) - P_h(-\xi/2-\eta)]\omega(2\eta)\,d\eta =$$
$$\int_0^\xi P_h(\xi/2-\eta)\omega(2\eta)\,d\eta + \int_0^\infty P_h(\xi/2+\eta)[\omega(2\eta+2\xi) - \omega(2\eta)]\,d\eta.$$

The last formula can also be rewritten as

$$\int_0^{\xi/2} P_h(\eta)[\omega(\xi+2\eta) + \omega(\xi-2\eta)]\,d\eta + \int_{\xi/2}^\infty P_h(\eta)[\omega(2\eta+\xi) - \omega(2\eta-\xi)]\,d\eta.$$

Since $\int_0^\infty P_h(\eta)\,d\eta = 1/2$, we see that the difference $(P_h * u)(y) - (P_h * u)(z) - \omega(\xi)$ can be estimated from above by

$$\int_0^{\xi/2} P_h(\eta)[\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)]\,d\eta + \int_{\xi/2}^\infty P_h(\eta)[\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)]\,d\eta.$$

Dividing by h and passing to the limit as $h \to 0+$, we obtain the following upper bound on the contribution of the dissipative term into the time derivative

$$\frac{1}{\pi}\int_0^{\xi/2} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta + \frac{1}{\pi}\int_{\xi/2}^\infty \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta.$$

Note that due to concavity of ω, both terms are strictly negative. We will denote the first integral by $I_{\omega,1}(\xi)$ and the second integral by $I_{\omega,2}(\xi)$.

Now we are ready to define our modulus of continuity. Let us set $\xi_0 \equiv \left(\frac{K}{4\pi}\right)^2$, where K is to be chosen later. Then ω is given by

$$\omega(\xi) = \begin{cases} \frac{\xi}{1+K\sqrt{\xi}} & \text{for } 0 \le \xi \le \xi_0; \\ C_K \log \xi & \text{for } \xi \ge \xi_0. \end{cases} \tag{31}$$

Here C_K is chosen to provide continuity of ω. A direct computation shows that

$$C_K \sim (\log K)^{-1} \text{ as } K \to \infty. \tag{32}$$

One can check that if K is sufficiently large, then ω is concave, with negative and increasing second order derivative on both intervals in (31) (on the first interval, $\omega''(\xi) = -K(3\xi^{-1/2} + K)/4(1 + K\sqrt{\xi})^3$). The first derivative of ω may jump at ξ_0, but the left derivative at ξ_0 is $\sim K^{-2}$, while the right derivative is $\sim K^{-2}(\log K)^{-1}$. We choose K large enough so that the left derivative is larger than the right derivative assuring concavity. The bound on the flow term is valid at ξ_0 using the value of larger (left) derivative.

It remains to verify that we have

$$\omega(\xi)\omega'(\xi) + I_{1,\omega}(\xi) + I_{2,\omega}(\xi) \le 0$$

for any ξ.

I. *The case* $\xi \le \xi_0$. Using the second order Taylor formula and the fact that ω'' is negative and monotone increasing on $[0, \xi]$, we obtain that

$$\omega(\xi + 2\eta) + \omega(\xi - 2\eta) \le \omega(\xi) + \omega'(\xi)2\eta + \omega(\xi) - \omega'(\xi)2\eta + 2\omega''(\xi)\eta^2.$$

This leads to an estimate

$$I_{1,\omega}(\xi) \le \frac{1}{\pi}\xi\omega''(\xi).$$

From (31), we find that

$$2\omega(\xi)\omega'(\xi) = \frac{\xi^{1/2}(2\xi^{1/2} + K\xi)}{(1 + K\xi^{1/2})^3},$$

while

$$\frac{1}{\pi}\xi\omega''(\xi) = -\frac{K(3\xi^{1/2} + K\xi)}{4\pi(1 + K\xi^{1/2})^3}.$$

Taking into account that $\xi \le \xi_0$, we find

$$2\omega(\xi)\omega'(\xi) + I_{1,\omega}(\xi) \le 0, \tag{33}$$

for any K.

II. *The case* $\xi > \xi_0$. Due to concavity, we have $\omega(\xi + 2\eta) \le \omega(2\eta - \xi) + \omega(2\xi)$, and thus

$$I_{2,\omega}(\xi) \le \frac{1}{\pi}\int_{\xi/2}^{\infty} \frac{\omega(2\xi) - 2\omega(\xi)}{\eta^2}\, d\eta.$$

Clearly we have $\omega(2\xi) \leq \frac{3}{2}\omega(\xi)$ for $\xi \geq \xi_0$ provided that K was chosen large enough. In this case, we obtain $I_{2,\omega}(\xi) \leq -\frac{\omega(\xi)}{\pi\xi}$. Now it follows from (31) that $2\omega(\xi)\omega'(\xi) = 2C_K^2\xi^{-1}\log\xi$, while $\xi^{-1}\omega(\xi) = C_K\xi^{-1}\log\xi$. Given (32), it as clear that

$$2\omega(\xi)\omega'(\xi) + I_{2,\omega}(\xi) \leq 0, \quad \xi \geq \xi_0 \tag{34}$$

if only K was chosen sufficiently large. □

Observe that as a byproduct, the proof also yields uniform in time control of $\|u'\|_{L^\infty}$.

Corollary 3.5 *Assume that the initial data $u_0(x)$ is such that $\|u_0'\|_{L^\infty} < \infty$. Then for every time t, the solution $u(x,t)$ of the critical Burgers equation satisfies*

$$\|u'(x,t)\|_{L^\infty} \leq \|u_0'\|_{L^\infty} \exp(C\|u_0\|_{L^\infty}).$$

Proof. Choose B so that $u_0(x)$ has the modulus of continuity ω_B. Given the asymptotic behavior of ω for large ξ, this is guaranteed if

$$C_K \log\left(\frac{B}{\|u_0'\|_{L^\infty}}\right) \geq \|u_0\|_{L^\infty}.$$

The Corollary then follows from (31) and preservation of ω_B by evolution. □

Finally, we prove Theorem 3.2, establishing analyticity of the solution.

Proof. [Proof of Theorem 3.2] We will assume that the initial data $u_0 \in H^2$. Even if we started from $u_0 \in H^s$, $s > 1/2$, Theorem 2.5 implies that we gain the desired smoothness immediately.

Let us rewrite the equation (4) on the Fourier side ($\alpha = 1/2$, without loss of generality assume the period is equal to one):

$$\frac{d\hat{u}^N(k)}{dt} = \pi i \sum_{a+b=k,\,|a|,|b|,|k|\leq N} k\hat{u}^N(a)\hat{u}^N(b) - |k|\hat{u}^N(k).$$

To simplify notation we will henceforth omit the restrictions $|a|, |b|, |k| \leq N$ in the summation, but they are always present in the remainder of the proof. Put $\xi_k^N(t) := \hat{u}^N(k,t)e^{\frac{1}{2}|k|t}$. Observe that since $u(x,t)$ is real, $\overline{\xi}_k^N = \xi_{-k}^N$. We have

$$\frac{d\xi_k^N}{dt} = \pi i \sum_{a+b=k} e^{-\gamma_{a,b,k}t} k\xi_a^N\xi_b^N - \frac{1}{2}|k|\xi_k^N, \tag{35}$$

where $\gamma_{a,b,k} := \frac{1}{2}(|a| + |b| - |k|)$. Note that

$$0 \leq \gamma_{a,b,k} \leq \min\{|a|,\, |b|\}. \tag{36}$$

Consider $Y_N(t) := \sum\limits_k |k|^4 |\xi_k^N(t)|^2$. Then we have

$$\frac{dY_N}{dt} = \Re\left(-2\pi i \sum_{a+b+k=0} e^{-\gamma_{a,b,k}t} k|k|^4 \xi_a^N \xi_b^N \xi_k^N\right) - \sum_k |k|^5 |\xi_k^N|^2$$

$$= \Re\left(-2\pi i \sum_{a+b+k=0} k|k|^4 \xi_a^N \xi_b^N \xi_k^N\right) + \Re\left(-2\pi i \sum_{a+b+k=0} (e^{-\gamma_{a,b,k}t} - 1) k|k|^4 \xi_a^N \xi_b^N \xi_k^N\right) \tag{37}$$

$$- \sum_k |k|^5 |\xi_k^N|^2 =: I_1 + I_2 + I_3.$$

Symmetrizing I_1 over a, b and k we obtain

$$I_1 = \frac{2\pi}{3} \Re\left(-i \sum_{a+b+k=0} (k|k|^4 + a|a|^4 + b|b|^4) \xi_a^N \xi_b^N \xi_k^N\right). \tag{38}$$

Thus

$$|I_1| \leq 4\pi \sum_{a+b+k=0,\ |a|\leq|b|\leq|k|} |k|k|^4 + a|a|^4 + b|b|^4||\xi_a^N||\xi_b^N||\xi_k^N| \leq \tag{39}$$

$$\leq 160\pi \sum_{a+b+k=0,\ |a|\leq|b|\leq|k|} |a||b|^2|k|^2|\xi_a^N||\xi_b^N||\xi_k^N|$$

$$\leq 160\pi Y_N \sum |a||\xi_a^N| \leq C_1 Y_N^{3/2}.$$

Here in the second step we used $a + b + k = 0$ (compare to (7)), and in the last step we used Hölder inequality:

$$\sum_a |a||\xi_a^N| \leq \left(\sum_{a\neq 0} |a|^{-2}\right)^{1/2} Y_N^{1/2}(t). \tag{40}$$

For I_2 we have

$$|I_2| \leq 2\pi \sum_{a+b+k=0} \min(|a|, |b|) t |k|^5 |\xi_a^N||\xi_b^N||\xi_k^N|.$$

Here we used (36). Furthermore,

$$\sum_{a+b+k=0} \min(|a|, |b|)|k|^5|\xi_a^N||\xi_b^N||\xi_k^N| \leq \sum_{a+b+k=0,\ |a|\leq|b|\leq|k|} 3|a||k|^5|\xi_a^N||\xi_b^N||\xi_k^N| +$$

$$\sum_{a+b+k=0,\ |b|\leq|a|\leq|k|} 3|b||k|^5|\xi_a^N||\xi_b^N||\xi_k^N| \leq \sum_{a+b+k=0} 6|a||b|^{5/2}|k|^{5/2}|\xi_a^N||\xi_b^N||\xi_k^N|$$

$$\leq 6\left(\sum_a |a||\xi_a^N|\right)\left(\sum_k |k|^5|\xi_k^N|^2\right).$$

We used Young's inequality for convolution in the last step. Combining all estimates and applying (40), we obtain

$$|I_2| \leq CtY_N^{1/2} \sum_k |k|^5 |\xi_k^N|^2. \tag{41}$$

Combining (37), (39) and (41) we arrive at

$$\frac{dY_N}{dt} \leq C_1 Y_N^{3/2} + (C_2 Y_N^{1/2} t - 1) \sum_k |k|^5 |\xi_k^N|^2. \tag{42}$$

Note that $Y_N(0) = \|u_0^N\|_2^2$. Thus we have a differential inequality for Y_N ensuring upper bound on Y_N uniform in N for a short time interval τ which depends only on $\|u_0\|_2$. Observe that Theorem 2.4 and Corollary 3.5 ensure that the H^2 norm of solution $u(x,t)$ is bounded uniformly on $[0,\infty)$. Thus we can use the above construction to prove for every $t_0 > 0$ uniform in N and $t > t_0$ bound on $\sum_k |\hat{u}^N(k,t)|^2 e^{\delta|k|}$ for some small $\delta(t_0, u_0) > 0$. By construction of u, it must satisfy the same bound. □

4 Blow-up for the supercritical case.

Our main goal here is to prove Theorem 1.4. At first, we are going to produce smooth initial data $u_0(x)$ which leads to blow up in finite time in the case where the period $2L$ is large. After that, we will sketch a simple rescaling argument which gives the blow up for any (and in particular unit) period.

The proof will be by contradiction. We will fix L and the initial data, and assume that by time $T = T(\alpha)$ the blow up does not happen. In particular, this implies that there exists N such that $\|u(x,t)\|_{C^3} \leq N$ for $0 \leq t \leq T$. This will lead to a contradiction. The overall plan of the proof is to reduce the blow up question for front-like data to the study of a system of differential equations on the properly measured steepness and size of the solution. To control the solution, the first tool we need is a time splitting approximation. Namely, consider a time step h, and let $w(x,t)$ solve

$$w_t = ww_x, \;\; w(x,0) = u_0(x), \tag{43}$$

while $v(x,t)$ solves

$$v_t = -(-\Delta)^\alpha v, \;\; v(x,0) = w(x,h). \tag{44}$$

The idea of approximating $u(x,t)$ with time splitting is fairly common and goes back to the Trotter formula in the linear case (see for example [2], page 120, and [17], page 307, for some applications of time splitting in nonlinear setting). The situation in our case is not completely standard, since the Burgers equation generally does blow up, and moreover the control we require is in a rather strong norm.

The solution of the problem (44) with the initial data $v_0(x)$ is given by the convolution

$$v(x,t) = \int_{\mathbb{R}} \Phi_t(x-y)v_0(y)\,dy \;(= e^{-(-\Delta)^\alpha t}v_0(x)), \tag{45}$$

where

$$\Phi_t(x) = t^{-1/2\alpha}\Phi(t^{-1/2\alpha}x), \;\; \Phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(ix\xi - |\xi|^{2\alpha})\,d\xi. \tag{46}$$

It is evident that $\Phi(x)$ is even and $\int \Phi(x)\,dx = 1$. We will need the following further properties of the function Φ :

$$\Phi(x) > 0; \;\; x\Phi'(x) \le 0, \;\; \Phi(x) \le \frac{K(\alpha)}{1+|x|^{1+2\alpha}}, \;\; |\Phi'(x)| \le \frac{K(\alpha)}{1+|x|^{2+2\alpha}}. \tag{47}$$

These properties are not difficult to prove; see e.g. [11] for some results, in particular positivity (Theorem XIII.6.1). We need the following lemma.

Lemma 4.1 *For every $f \in C^{n+1}$, $\;n \ge 0$,*

$$\|(e^{-(-\Delta)^\alpha t} - 1)f\|_{C^n} \le C(\alpha)t\|f\|_{C^{n+1}}. \tag{48}$$

Proof. Obviously, it is sufficient to prove the Lemma for $n = 0$. We have (see (45), (46), (47))

$$\begin{gathered}\left|(e^{-(-\Delta)^\alpha t} - 1)f\right| = \left|\int_{-\infty}^{\infty} \Phi_t(y)(f(x-y) - f(x))dy\right| \le \\ \left|\int_{-1}^{1} \Phi_t(y)(f(x-y) - f(x))dy\right| + \left|\int_{|y|\ge 1} \Phi_t(y)(f(x-y) - f(x))dy\right| \le \\ 2\|f\|_{C^1} t^{\frac{1}{2\alpha}} \int_0^{t^{-\frac{1}{2\alpha}}} y\Phi(y)dy + 4\|f\|_C \int_{t^{-\frac{1}{2\alpha}}}^{\infty} \Phi(y)dy \le C\frac{K(\alpha)}{\alpha}\|f\|_{C^1}t.\end{gathered} \tag{49}$$

□

Next lemma provides local solvability for our splitting system.

Lemma 4.2 *Assume $\|u_0(x)\|_{C^3} \le N$. Then for all h small enough, $v(x,h)$ is C^3 and is uniquely defined by* (43), (44)*. Moreover, it suffices to assume $h \le CN^{-1}$ to ensure*

$$\|w(x,t)\|_{C^3}, \|v(x,t)\|_{C^3} \le 2N \tag{50}$$

for $0 \le t \le h$.

Proof. Using the characteristics one can explicitly solve equation (43). We have $w(t, y) = u_0(x)$, where $x = x(y)$ is such that

$$y = x - u_0(x)t. \tag{51}$$

Now, implicit function theorem and direct computations show that $\|w(t, \cdot)\|_{C^3} \leq 2\|u_0\|_{C^3}$ provided that $\|u_0\|_{C^3} t \leq c$ for some small constant $c > 0$. This proves the statement of the Lemma for w. To prove it for v we just notice that v is a convolution of the $w(h, x)$ with $\Phi_t(x)$. Since $\|\Phi_t\|_{L^1} = 1$ we obtain that $\|v(t, \cdot)\|_{C^3} \leq \|w(h, \cdot)\|_{C^3}$. □

The main time splitting result we require is the following

Proposition 4.3 *Assume that $\|u_0(x)\|_{C^3} \leq N$ for $0 \leq t \leq T$. Define $v(x, t)$ by (43), (44) with time step h. Then for all h small enough, we have*

$$\|u(x, h) - v(x, h)\|_{C^1} \leq C(\alpha, N)h^2.$$

Proof. Since $\|u_0(x)\|_{C^3} \leq N$, let us choose h as in Lemma 4.2. Notice that by Duhamel's principle,

$$u(x, h) = e^{-(-\Delta)^\alpha h} u_0(x) + \int_0^h e^{-(-\Delta)^\alpha (h-s)} (u(x, s)u_x(x, s))\, ds,$$

while

$$v(x, h) = e^{-(-\Delta)^\alpha h} u_0(x) + \int_0^h e^{-(-\Delta)^\alpha h} (w(x, s)w_x(x, s))\, ds.$$

Then it follows from (48) that

$$\|u(x, h) - v(x, h)\|_{C^1} \leq \int_0^h \|e^{-(-\Delta)^\alpha (h-s)} (u(x, s)u_x(x, s)) - e^{-(-\Delta)^\alpha h} (w(x, s)w_x(x, s))\|_{C^1}\, ds \leq$$

$$\int_0^h \|u(x, s)u_x(x, s) - w(x, s)w_x(x, s)\|_{C^1}\, ds + \int_0^h \| \left(e^{-(-\Delta)^\alpha (h-s)} - 1\right) u(x, s)u_x(x, s)\|_{C^1} +$$

$$\int_0^h \| \left(e^{-(-\Delta)^\alpha h} - 1\right) w(x, s)w_x(x, s)\|_{C^1}\, ds \leq \int_0^h \|u(x, s)u_x(x, s) - w(x, s)w_x(x, s)\|_{C^1}\, ds +$$

$$C(\alpha)h \int_0^h \left(\|u(x, s)u_x(x, s)\|_{C^2} + \|w(x, s)w_x(x, s)\|_{C^2}\right)\, ds. \tag{52}$$

From (50), it follows that the last integral does not exceed $C(\alpha)N^2h^2$. To estimate the remaining integral, we need the following

Lemma 4.4 *For every* $0 \le s \le h$, *we have* $\|u(x,s) - w(x,s)\|_{C^2} \le C(\alpha)N^2 h$.

Proof. Observe that $g(x,s) \equiv u(x,s) - w(x,s)$ solves

$$g_t = gu_x + wg_x - (-\Delta)^\alpha u, \;\; g(x,0) = 0.$$

Thus

$$g(x,t) = \int_0^t (gu_x + wg_x - (-\Delta)^\alpha u)\, ds.$$

Because of (50) and the assumption on u, we have $\|gu_x\|_{C^2}, \|wg_x\|_{C^2} \le CN^2$, and $\|(-\Delta)^\alpha u\|_{C^2} \le CN$. Therefore, we can estimate that $\|g(x,t)\|_{C^2} \le C(\alpha)tN^2$, for every $0 \le t \le h$. □

From Lemma 4.4 it follows that

$$\int_0^h \|uu_x - ww_x\|_{C^1}\, ds \le \int_0^h (\|(u-w)u_x\|_{C^1} + \|w(u_x - w_x)\|_{C^1})\; ds \le C(\alpha)N^3h^2.$$

This completes the proof of Proposition 4.3. □

The next stage is to investigate carefully a single time splitting step. The initial data $u_0(x)$ will be smooth, $2L$-periodic, odd, and satisfy $u_0(L) = 0$. It is not hard to see that all these assumptions are preserved by the evolution. We will assume a certain lower bound on $u_0(x)$ for $0 \le x \le L$, and derive a lower bound that must hold after the small time step. The lower bound will be given by the following piecewise linear functions on $[0, L]$:

$$\varphi(\kappa, H, a, x) = \begin{cases} \kappa x, & 0 \le x \le \delta \equiv H/\kappa \\ H, & \delta \le x \le L - a \\ \frac{H}{a}(L - x), & L - a \le x \le L. \end{cases}$$

Here L, κ, H and a may depend only on α and will be specified later. We will set $a \le L/4$, $\delta \le L/4$ and will later verify that this condition is preserved throughout the construction. We assume that blow up does not happen until time T (to be determined later). Let $N = \sup_t \|u(x,t)\|_{C^3}$.

Lemma 4.5 *Assume that the initial data* $u_0(x)$ *for the equation* (43) *satisfies the above assumptions. Then for every* h *small enough (*$h \le CN^{-1}$ *is sufficient), we have*

$$w(x,h) \ge \varphi\left(\frac{\kappa}{1 - \kappa h}, H, a + \|u_0\|_{L^\infty} h, x\right), \quad 0 \le x \le L.$$

Proof. The Burgers equation can be solved explicitly using characteristics. The existence of C^3 solution $w(x,t)$ for $t \le h$ is assured by the assumption on the initial data and h. □

Now we consider the effect of the viscosity time step. Suppose that the initial data $v_0(x)$ for (44) satisfies the same conditions as stated for $u_0(x)$ above: periodic, odd, $v_0(L) = 0$. Then we have

Lemma 4.6 *Assume that for $0 \le x \le L$, $v_0(x) \ge \varphi(\kappa, H, a, x)$. Moreover, assume that*

$$H\kappa^{-1} \le a, \quad L \ge 4a, \quad L^{-2\alpha}\|v_0\|_{L^\infty} \le 4Ha^{-2\alpha}. \tag{53}$$

Then for every sufficiently small h, we have

$$v(x,h) \ge \varphi(\kappa(1 - C(\alpha)hH^{-2\alpha}\kappa^{2\alpha}), H(1 - C(\alpha)hH^{-2\alpha}\kappa^{2\alpha}), a, x), \quad 0 \le x \le L.$$

Proof. Let us compute

$$v(x,h) = \int_{-\delta}^{\delta} \Phi_h(x-y)v_0(y)\,dy + \int_{\delta}^{L-a} (\Phi_h(y-x) - \Phi_h(x+y))v_0(y)\,dy + \\ \int_{L-a}^{\infty} (\Phi_h(y-x) - \Phi_h(x+y))v_0(y)\,dy. \tag{54}$$

In the last integral in (54), we estimate by Mean Value Theorem $|\Phi_h(y-x) - \Phi_h(y+x)| \le 2x\,|\Phi_h'(\tilde{y})|$, where $\tilde{y} \in (y-x, y+x)$. Using (47), we see that the last integral in (54) is controlled by $C(\alpha)hxL^{-1-2\alpha}\|v_0\|_{L^\infty}$. The second integral on the right hand side of (54) can be estimated from below by

$$H\int_{\delta}^{\infty} (\Phi_h(y-x) - \Phi_h(x+y))\,dy \tag{55}$$

with an error, which, by the previous computation, does not exceed $C(\alpha)hxL^{-1-2\alpha}\|v_0\|_{L^\infty}$. The expression in (55) is equal to

$$H\int_{\delta-x}^{\delta+x} \Phi_h(z)\,dz. \tag{56}$$

For the first integral in (54) we have

$$\int_{-\delta}^{\delta} \Phi_h(x-y)v_0(y)dy = \int_{0}^{\delta}(\Phi_h(x-y) - \Phi_h(x+y))v_0(y)dy \ge \\ \int_{0}^{\delta}(\Phi_h(x-y) - \Phi_h(x+y))ky\,dy = \\ \int_{-\delta}^{\delta} \Phi_h(x-y)\kappa y\,dy = \kappa\int_{-\delta-x}^{\delta-x} \Phi_h(z)(x+z)\,dz. \tag{57}$$

Combining (56) and (57), we obtain

$$v(x,h) \ge \kappa x\int_{-\delta-x}^{\delta-x} \Phi_h(z)\,dz + H\int_{\delta-x}^{\delta+x} \Phi_h(z)\,dz + \kappa\int_{-\delta-x}^{\delta-x} z\Phi_h(z)\,dz - \\ C(\alpha)hxL^{-1-2\alpha}\|v_0\|_{L^\infty}. \tag{58}$$

Now we split the proof into several parts according to the regions being considered.
I. Estimate for $0 \le x \le \delta = H/\kappa$. Observe that for $0 \le x \le \delta/2$, the contribution of the second and third integrals in (58) is positive. Indeed, it is equal to

$$\int_{\delta-x}^{\delta+x} (H - \kappa z)\Phi_h(z)\, dz,$$

which is positive due to monotonicity of $\Phi_h(z)$ and equality $H = \kappa\delta$. Thus, in this interval of x we simply estimate v by dropping the combined contribution of the second and the third integrals:

$$v(x,h) \ge \kappa x \int_{-\delta}^{\delta/2} \Phi_h(z)\, dz - C(\alpha)hxL^{-1-2\alpha}\|v_0\|_{L^\infty} \ge \\ \kappa x(1 - C(\alpha)h\delta^{-2\alpha}) - C(\alpha)xhL^{-1-2\alpha}\|v_0\|_{L^\infty} \ge \kappa x(1 - C(\alpha)h\delta^{-2\alpha}).$$

Here we also decreased the interval of integration and used (47).

For $\delta/2 \le x \le \delta$ we combine together the first and the second integral and notice that $H = \kappa\delta \ge \kappa x$:

$$v(x,h) \ge \kappa x \int_{-\delta-x}^{\delta+x} \Phi_h(z)\, dz + \kappa \int_{-\delta-x}^{\delta-x} z\Phi_h(z)\, dz - C(\alpha)hxL^{-1-2\alpha}\|v_0\|_{L^\infty} \ge \\ \kappa x \int_{-\delta}^{\delta} \Phi_h(z)\, dz - \kappa \int_0^{2\delta} z\Phi_h(z)\, dz - C(\alpha)hxL^{-1-2\alpha}\|v_0\|_{L^\infty} \ge \\ \kappa x(1 - C(\alpha)h\delta^{-2\alpha}) - C(\alpha)\kappa h\delta^{1-2\alpha} - C(\alpha)xhL^{-1-2\alpha}\|v_0\|_{L^\infty} \ge \kappa x(1 - C(\alpha)h\delta^{-2\alpha}).$$

Here we again used (47) and (53). Combining the estimates together we have

$$v(x,h) \ge \kappa x(1 - C(\alpha)h\delta^{-2\alpha}) \tag{59}$$

for $0 \le x \le \delta$.

II. Estimate for $L - a \le x \le L$ case. The estimate is virtually identical to the first case due to symmetry; δ has to be replaced by a. Thus, (we recall that $\delta = H\kappa^{-1} \le a$ by assumption of the lemma)

$$v(x,h) \ge \frac{H}{a}(L - x)(1 - C(\alpha)ha^{-2\alpha}) \ge \frac{H}{a}(L - x)(1 - C(\alpha)h\delta^{-2\alpha}), \tag{60}$$

for $L - a \le x \le L$.

III. Estimate for $\delta \le x \le L/2$. Here estimates are similar to the first case. In the last term in (58) we will just estimate x by L. Furthermore, observe that it follows from (57) and monotonicity property of Φ that the sum of the first and the third integrals in (58) is positive. For $2\delta \le x \le L/2$ we ignore the positive combined contribution of the first and the third integrals:

$$v(x,h) \ge H \int_{-\delta}^{\delta} \Phi_h(z)\, dz - C(\alpha)hL^{-2\alpha}\|v_0\|_{L^\infty} \ge$$

$$H(1 - C(\alpha)h\delta^{-2\alpha}) - C(\alpha)hL^{-2\alpha}\|v_0\|_{L^\infty} \geq H(1 - C(\alpha)h\delta^{-2\alpha}).$$

For $\delta \leq x \leq 2\delta$ we combine the first and the second integrals and take into account that $\kappa x \geq \kappa\delta = H$:

$$\begin{gathered} v(x,h) \geq H\int_{-\delta-x}^{\delta+x} \Phi_h(z)\,dz + \kappa\int_{-\delta-x}^{\delta-x} z\Phi_h(z)\,dz - C(\alpha)hL^{-2\alpha}\|v_0\|_{L^\infty} \geq \\ H\int_{-\delta}^{\delta} \Phi_h(z)\,dz - \kappa\int_0^{3\delta} z\Phi_h(z)\,dz - C(\alpha)hL^{-2\alpha}\|v_0\|_{L^\infty} \geq \\ H(1 - C(\alpha)h\delta^{-2\alpha}) - C(\alpha)\kappa h\delta^{1-2\alpha} - C(\alpha)hL^{-2\alpha}\|v_0\|_{L^\infty} \geq H(1 - C(\alpha)h\delta^{-2\alpha}). \end{gathered}$$

Combining the estimates together we obtain

$$v(x,h) \geq H(1 - C(\alpha)h\delta^{-2\alpha}). \tag{61}$$

IV. Estimate for $L/2 \leq x \leq L - a$. By symmetry we obtain

$$v(x,h) \geq H(1 - C(\alpha)ha^{-2\alpha}) \geq H(1 - C(\alpha)h\delta^{-2\alpha}). \tag{62}$$

Together, (62), (61), (60) and (59) complete the proof. □

Combining Proposition 4.3 and Lemmas 4.5 and 4.6, we obtain

Theorem 4.7 *Assume that the initial data $u_0(x)$ is $2L$-periodic, odd, $u_0(L) = 0$, and $u_0(x) \geq \varphi(\kappa, H, a, x)$. Suppose that (53) holds with v_0 replaced by u_0. Assume also that the solution $u(x,t)$ of the equation (1) with initial data $u_0(x)$ satisfies $\|u(x,t)\|_{C^3} \leq N$ for $0 \leq t \leq T$. Then for every $h \leq h_0(\alpha, N)$ small enough, we have for $0 \leq x \leq L$*

$$u(x,h) \geq \varphi(\tilde{\kappa}, \tilde{H}, a + h\|u_0\|_{L^\infty}, x), \tag{63}$$

where

$$\tilde{\kappa} = \kappa(1 - C(\alpha)\kappa^{2\alpha}H^{-2\alpha}h)(1 - \kappa h)^{-1} - C(\alpha, N)h^2 \tag{64}$$

and

$$\tilde{H} = H(1 - C(\alpha)\kappa^{2\alpha}H^{-2\alpha}h) - C(\alpha, N)h^2. \tag{65}$$

Proof. We can clearly assume that $\kappa h \leq 1/2$; in view of our assumptions on u_0, $h \leq 1/2N$ is sufficient for that. Then Lemmas 4.5 and 4.6 together ensure that the time splitting solution $v(x,h)$ of (43) and (44) satisfies for $0 \leq x \leq L$

$$v(x,h) \geq \varphi(\kappa(1-C(\alpha)\kappa^{2\alpha}H^{-2\alpha}h)(1-\kappa h)^{-1}, H(1-C(\alpha)\kappa^{2\alpha}H^{-2\alpha}h), a+\|u_0\|_{L^\infty}h, x). \tag{66}$$

Furthermore, Proposition 4.3 allows us to pass from the lower bound on $v(x,h)$ to lower bound on $u(x,h)$, leading to (63), (64), (65). □

From Theorem 4.7, we immediately infer

Corollary 4.8 *Under assumptions of the previous theorem and the additional assumption stated below, for all h small enough we have for $0 \leq x \leq L$ and $0 \leq nh \leq T$*

$$u(x, nh) \geq \varphi(\kappa_n, H_n, a_n, x). \tag{67}$$

Here

$$\kappa_n = \kappa_{n-1}(1 - C(\alpha)\kappa_{n-1}^{2\alpha} H_{n-1}^{-2\alpha} h)(1 - \kappa_{n-1}h)^{-1} - C(\alpha, N)h^2, \tag{68}$$

$$H_n = H_{n-1}(1 - C(\alpha)\kappa_{n-1}^{2\alpha} H_{n-1}^{-2\alpha} h) - C(\alpha, N)h^2, \tag{69}$$

and

$$a_n = a + nh\|u_0\|_{L^\infty}. \tag{70}$$

The corollary only holds assuming that for every n, we have

$$H_n\kappa_n^{-1} \leq a_n, \quad L \geq 4a_n, \quad L^{-2\alpha}\|u_0\|_{L^\infty} \leq 4H_n a_n^{-2\alpha}. \tag{71}$$

To study (68) and (69), we introduce the following system of differential equations:

$$\kappa' = \kappa^2 - C(\alpha)\kappa^{1+2\alpha}H^{-2\alpha}; \;\; H' = -C(\alpha)\kappa^{2\alpha}H^{1-2\alpha}. \tag{72}$$

Lemma 4.9 *Assume that $[0, T]$ is an interval on which the solutions of the system (72) satisfy $|\kappa(t)| \leq 2N$, $0 < H_1(\alpha) \leq H(t) \leq H_0(\alpha)$. Then for every $\epsilon > 0$, there exists $h_0(\alpha, N, \epsilon) > 0$ such that if $h < h_0$, then κ_n and H_n defined by (68) and (69) satisfy $|\kappa_n - \kappa(nh)| < \epsilon$, $|H_n - H(nh)| < \epsilon$ for every $n \leq [T/h]$.*

Proof. This is a standard result on approximation of differential equations by a finite difference scheme. Observe that the assumptions on $\kappa(t)$ and $H(t)$ also imply upper bounds on $\kappa'(t)$, $\kappa''(t)$, $H'(t)$ and $H''(t)$ by a certain constant depending only on N and α. The result can be proved comparing the solutions step-by-step inductively. Each step produces an error not exceeding $C_1(\alpha, N)h^2$, and the total error over $[T/h]$ steps is estimated by $C_1(\alpha, N)h$. Choosing $h_0(\alpha, N, \epsilon)$ sufficiently small completes the proof. □

The final ingredient we need is the following lemma on the behavior of solutions of the system (72).

Lemma 4.10 *Assume that the initial data for the system (72) satisfy*

$$H_0^{2\alpha}\kappa_0^{1-2\alpha} \geq C(\alpha)/(1 - 2\alpha). \tag{73}$$

Then on every interval $[0, T]$ on which the solution makes sense (that is, $\kappa(t)$ bounded), the function $H(t)^{2\alpha}\kappa^{1-2\alpha}(t)$ is non-decreasing.

Proof. A direct computation shows that

$$\left(H(t)^{2\alpha}\kappa^{1-2\alpha}(t)\right)' = (1 - 2\alpha)\kappa(t)\left(H(t)^{2\alpha}\kappa(t)^{1-2\alpha} - \frac{C(\alpha)}{1 - 2\alpha}\right).$$

□

Now we are ready to complete the blow up construction.

Proof. [Proof of Theorem 1.4] Set κ_0 to be large enough, in particular

$$\kappa_0 = \left(\frac{3C(\alpha)}{1-2\alpha}\right)^{\frac{1}{1-2\alpha}} \tag{74}$$

will do. Set $H_0 = 1$, $a = \kappa_0^{-1}$, $T(\alpha) = \frac{3}{2\kappa_0}$. Choose L so that

$$L \geq 16a. \tag{75}$$

The initial data $u_0(x)$ will be a smooth, odd, $2L-$periodic function satisfying $u_0(L) = 0$ and $u_0(x) \geq \varphi(\kappa_0, H_0, a, x)$. We will also assume $\|u_0\|_{L^\infty} \leq 2H_0$. Observe that H_0 and κ_0 are chosen so that in particular the condition (73) is satisfied. From (72) and Lemma 4.10 it follows that

$$\kappa' = \kappa^2 - C(\alpha)\kappa^{1+2\alpha}H^{-2\alpha} \geq \frac{2}{3}\kappa^2. \tag{76}$$

This implies $\kappa(t) \geq \frac{1}{\kappa_0^{-1} - \frac{2}{3}t}$. In particular, there exists $t_0 < T(\alpha)$ such that $\kappa(t_0) = 2N$ for the first time. Note that due to (76), for $0 \leq t \leq t_0$ we have

$$\kappa(t) \leq \frac{1}{\frac{2}{3}(t_0 - t) + \frac{1}{2N}} \leq \frac{3/2}{(t_0 - t)}. \tag{77}$$

Rewrite the equation for $H(t)$ as

$$(H^{2\alpha})' = -2C(\alpha)\alpha\kappa^{2\alpha}. \tag{78}$$

Using the estimate (77) in (78), we get that for any $0 \leq t \leq t_0$,

$$H^{2\alpha}(t) \geq H_0^{2\alpha} - 2C(\alpha)\alpha \int_0^{t_0} \kappa^{2\alpha}(s)\, ds \geq H_0^{2\alpha}(1-\alpha).$$

We used the fact that $H_0 = 1$, $t_0 < T(\alpha) = \frac{3}{2\kappa_0}$ and (74). Now we can apply Lemma 4.9 on the interval $[0, t_0]$. Choosing ϵ and h sufficiently small, we find that for $0 \leq nh \leq t_0$, $\kappa_n \geq 1$ and $H_n \geq (1-\alpha)^{1/2\alpha}H_0 \geq H_0/2$. Also, evidently, $a_n \leq a + 2H_0T(\alpha) = 4a$. This allows us to check that the conditions (71) hold on each step due to the choice of L (75), justifying control of the true PDE dynamics by the system (72).

From Lemma 4.9 and $\kappa(t_0) = 2N$, we also see that, given that h is sufficiently small, $\kappa_{n_0} \geq 3N/2$ for some n_0 such that $n_0h \leq t_0 < T(\alpha)$. Thus Corollary 4.8 provides us with a lower bound $u(0, n_0h) = 0$, $u(x, n_0h) \geq 3Nx/2$ for small enough x. This contradicts our assumption that $\|u(x,t)\|_{C^3} \leq N$ for $0 \leq t \leq T(\alpha)$, thus completing the proof. □

We obtained blow up in the case where period $2L$ was sufficiently large (depending only on α). However, examples of blow up with arbitrary periodic data follow immediately from a scaling argument. Indeed, assume $u(x,t)$ is a $2L-$periodic solution of (1). Then $u_1(x,t) = L^{-1+2\alpha}u(Lx, L^{2\alpha}t)$ is a $2-$periodic solution of the same equation. Thus a scaling procedure allows to build blow up examples for any period.

Remark. Formally we proved the blow up only in C^3 class. But since global regularity in H^s class for $s > \frac{3}{2} - 2\alpha$ provides global regularity in C^∞ (see Theorem 1.2), we can conclude that we constructed a blow up in H^s class for every $s > \frac{3}{2} - 2\alpha$.

5 Global existence and regularity for rough initial data for the case $\alpha = 1/2$

In this section we present some results on existence of regular solution for $\alpha = 1/2$ and rough initial data. More precisely, we prove that the solution becomes smooth starting from any initial data of the class L^p, $p > 1$. It is natural that the result can be obtained for the case $\alpha > 1/2$ by more traditional means. In the present section we consider the case $\alpha = 1/2$ only.

Consider the equation

$$u_t = uu_x - (-\Delta)^{1/2}u, \quad u(x,0) = u_0(x), \tag{79}$$

with $u_0 \in L^p$ for some $p > 1$. Let us look first at the approximating equation

$$u^N_t = u^N u^N_x - (-\Delta)^{1/2}u^N, \quad u^N(x,0) = u^N_0(x), \tag{80}$$

where $u^N_0 \in C^\infty$ and $\|u^N_0 - u_0\|_{L^p} \to 0$ as $N \to \infty$. We need the following fact.

Lemma 5.1 *Assume that a smooth function $w(x,t)$ satisfies the equation* (79) *with smooth initial data $w_0(x)$. Then for every $1 < p \le \infty$ and every t, we have $\|w(x,t)\|_{L^p} \le \|w_0(x)\|_{L^p}$.*

This Lemma can be proven in the same way as a corresponding result for the quasi-geostrophic equation, using the positivity of $\int |w|^{p-2}w(-\Delta)^\alpha w\, dx$. See [16] or [10] for more details.

We divide our proof of regularity into three steps.

Step I. Here we prove uniform (in N) estimates for the L^∞ norm. Put

$$M_N(t) := \|u^N(\cdot,t)\|_{L^\infty}.$$

Fix $t \ge 0$. Consider any point x_0 where $|u^N(x_0,t)| = M_N$. Without loss of generality, we may assume that $x_0 = 0$ and $u^N(0,t) = M_N$. Then

$$u^N_t(0,t) = (-(-\Delta)^{1/2}u^N)(0,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u^N(y,t) - M_N}{y^2}dy. \tag{81}$$

Denote Lebesgue measure of a measurable set S by $m(S)$. Since by Lemma 5.1 we have

$$\|u^N\|_{L^p}^p \leq C,$$

we obtain that

$$m\left(x|\,|u^N(x,t)| \geq M_N/2\right) \leq C2^p M_N^{-p}.$$

Then the right hand side of (81) does not exceed

$$-M_N \int\limits_{L\geq|y|\geq C2^{p-1}/M_N^p} y^{-2}dy.$$

Here $2L$ is the period. Then

$$u_t^N(0,t) < -C_1 M_N^{p+1} + C_2 M_N. \tag{82}$$

The same bound holds for any point x_0 where M_N is attained and by continuity in some neighborhoods of such points. So, we have (82) in some open set U_N. Due to smoothness of the approximating solution, away from U_N we have

$$\max_{x\notin U_N} |u^N(x,\tau)| < M_N(\tau)$$

for every τ during some period of time $[t, t+\tau_N]$, $\tau_N > 0$. Thus we obtain that

$$\frac{d}{dt}M_N < -C_1 M_N^{p+1} + C_2 M_N. \tag{83}$$

Solving equation (83), we get the uniform estimate

$$M_N^p(t) \leq \frac{e^{C_2pt}}{M_N^{-p}(0) + \frac{C_1}{C_2}(e^{C_2pt}-1)} \leq \frac{C_2}{C_1(1-e^{-C_2pt})}.$$

In particular,

$$t^{1/p}\|u^N\|_{L^\infty} \leq C, \quad t \leq 1. \tag{84}$$

Step II. Here we obtain uniform in N estimates on the approximations u^N that will imply smoothness of the solution. We will use the construction similar to the one appearing in the proof of Theorem 3.1.

Clearly, it is sufficient to work with $t \leq 1$. Let us define

$$G(t) = \inf\nolimits_{0\leq\omega(x)\leq Ct^{-1/p}} \frac{\omega(x)}{x},$$

where C is as in (84). Observe that, since ω is concave and increasing, the function $G(t)$ is equal to $Ct^{-1/p}/\omega^{-1}(Ct^{-1/p})$. Define also

$$F(t) = \left(\int\limits_0^t G(s)\,ds\right)^{-1}. \tag{85}$$

We claim that solution $u^N(x,t)$ has modulus of continuity $\omega_{F(t)}$ for every $t>0$ and every N. Here ω is defined by (31). Let us fix an arbitrary $N>0$. Since u_0^N and u^N are both smooth and $F(t)\to\infty$ as $t\to 0$, it follows that $u^N(x,t)$ has $\omega_{F(t)}$ for all $t<t_0(N)$, $t_0(N)>0$. By the argument completely parallel to that of Lemma 3.4, we can show that if the modulus of continuity $\omega_{F(t)}$ is ever violated, then there must exist $t_1>0$ and $x\neq y$ such that

$$u^N(x,t_1)-u^N(y,t_1)=\omega(F(t_1)|x-y|)$$

and $u^N(x,t)$ has $\omega_{F(t)}$ for any $t\le t_1$. Let us denote $|x-y|$ by ξ. Now consider

$$\frac{\partial}{\partial t}\left[\frac{u^N(x,t)-u^N(y,t)}{\omega(F(t)\xi)}\right]\Bigg|_{t=t_1} = \frac{\partial_t(u^N(x,t)-u^N(y,t))|_{t=t_1}\omega(F(t_1)\xi)}{\omega(F(t_1)\xi)^2} - \frac{\omega(F(t_1)\xi)F'(t_1)\xi\omega'(F(t_1)\xi)}{\omega(F(t_1)\xi)^2}. \tag{86}$$

It follows from the proof of Theorem 3.1 (see (33), (34)) that

$$\frac{d}{dt}\left(u^N(x,t)-u^N(y,t)\right)\Big|_{t=t_1} < -\omega(F(t_1)\xi)\frac{d}{d\xi}\omega(F(t_1)\xi).$$

Thus the numerator on the right hand side of (86) is smaller than

$$-\omega(F(t_1)\xi)^2\omega'(F(t_1)\xi)F(t_1)-\omega(F(t_1)\xi)F'(t_1)\omega'(F(t_1)\xi)\xi.$$

The numerator is strictly negative as far as

$$-\frac{F'(t_1)}{F(t_1)^2}\le\frac{\omega(F(t_1)\xi)}{F(t_1)\xi}. \tag{87}$$

Notice that by (84), we have

$$\omega(F(t_1)\xi)=u^N(x,t_1)-u^N(y,t_1)\le 2Ct_1^{-1/p}.$$

Using the definition of the function $G(t)$, we obtain the estimate

$$\frac{\omega(F(t_1)\xi)}{F(t_1)\xi}\ge G(t_1).$$

Thus (87) is satisfied if

$$\left(\frac{1}{F}\right)'\le G(t),$$

which is correct by definition of F. Therefore we obtain

$$\frac{\partial}{\partial t}\left[\frac{u^N(x,t)-u^N(y,t)}{\omega(F(t)\xi)}\right]\Bigg|_{t=t_1}<0.$$

Since N was arbitrary, it follows that $u(x,t)$ has the modulus of continuity $\omega_{F(t)}$ for any $t>0$, and thus

$$F(t)\|u^N\|_{W^1_\infty} \le C, \quad t \le 1. \tag{88}$$

To obtain higher order regularity of the solution we apply arguments from the proof of Theorem 2.4. We start with $s=1,\ q=1$. We can repeat the proof step by step. The only difference is that now we will use the estimate (88) instead of uniform bound for the norm $\|u^N\|_1$. Finally, we obtain the estimates

$$F_n(t)\|u^N(\cdot,t)\|_{1+\frac{n}{2}} \le C_n, \quad n \ge 1, \quad t \le 1, \tag{89}$$

with some functions F_n which can be calculated inductively. Now, we can choose a subsequence N_j (cf. proof of Theorem 2.5) such that $u^{N_j} \to u$ as $N_j \to \infty$ and function u satisfies differential equation (79) as well as the bounds (84), (88), (89) on $(0,1]$.

Step III. Here we prove that the function u can be chosen to satisfy the initial condition.

Lemma 5.2 *Assume that* $p \in (1,\infty)$*. Then* $\|u(\cdot,t)-u_0(\cdot)\|_{L^p} \to 0$ *as* $t \to 0$.

Proof. Let $\varphi(x)$ be an arbitrary C^∞ function. Put

$$g^N(t,\varphi) := (u^N,\varphi) = \int u^N(x,t)\varphi(x)dx.$$

Obviously, $g^N(\cdot,\varphi) \in C([0,1])$. We will use the estimate (18):

$$\int_0^1 |g_t^N|^{1+\delta}\,dt \le C\left(\int_0^1 \|u^N\|_{L^2}^{2+2\delta}\|\varphi\|_{W^1_\infty}^{1+\delta}\,dt + \int_0^1 \|u^N\|_{L^2}^{1+\delta}\|\varphi\|_{2\alpha}^{1+\delta}\,dt\right), \tag{90}$$

which holds for any $\delta>0$. Put $\delta := 1$ if $p \ge 2$ and $\delta := (p-1)/(2-p)$ if $1<p<2$. Due to (84) we obtain

$$\|u^N\|_{L^2}^2 \le \|u^N\|_{L^\infty}^{2-p}\|u^N\|_{L^p}^p \le Ct^{-\frac{2-p}{p}}, \quad t \le 1. \tag{91}$$

Substituting (91) into (90) we see that $\|g_t^N(\cdot,\varphi)\|_{L^{1+\delta}} \le C(\varphi)$. By the same argument as used in the proof of Theorem 2.5 we conclude that there exists a subsequence u^{N_j} such that for any $\varphi \in L^{p'}$ the sequence (u^{N_j},φ) tends to (u,φ) uniformly on $[0,1]$.

Next,

$$|(u-u_0,\varphi)| \le |(u-u^{N_j},\varphi)| + |(u^{N_j}-u_0^{N_j},\varphi)| + |(u_0^{N_j}-u_0,\varphi)|. \tag{92}$$

The first and the third terms in the right hand side of the (92) can be made small uniformly on $[0,1]$ by choosing sufficiently large N_j. The second term tends to zero as $t \to 0$ for every fixed N_j. Thus $u(\cdot,t)$ converges to $u_0(\cdot)$ as $t \to 0$ weakly in L^p. In particular,

$$\|u_0(\cdot)\|_{L^p} \le \liminf_{t\to 0}\|u(\cdot,t)\|_{L^p}. \tag{93}$$

Due to monotonicity property of Lemma 5.1 we have

$$\|u_0(\cdot)\|_{L^p} \geq \limsup_{t\to 0} \|u(\cdot,t)\|_{L^p}. \tag{94}$$

Thus $\|u_0(\cdot)\|_{L^p} = \lim_{t\to 0} \|u(\cdot,t)\|_{L^p}$. Now, it follows from uniform concavity of the space L^p, $p \in (1,\infty)$, that weak convergence and convergence of norms imply convergence in the norm sense. □

Let us combine the results in the following theorem.

Theorem 5.3 *Let $u_0 \in L^p$ for some $p \in (1,\infty)$. Then there exists a solution $u(x,t)$ of the equation* (79) *such that u is real analytic for $t > 0$,*

$$\|u(\cdot,t) - u_0(\cdot)\|_{L^p} \to 0 \quad \textit{as} \;\; t \to 0; \tag{95}$$

$$t^{1/p}\|u(\cdot,t)\|_{L^\infty} \leq C(\|u_0\|_{L^p}), \quad 0 < t \leq 1; \tag{96}$$

$$F(t)^{-1}\|u(\cdot,t)\|_{W^1_\infty} \leq C(\|u_0\|_{L^p}), \quad 0 < t \leq 1; \tag{97}$$

Here F is defined in (85).

Remark 1. If $u_0 \in H^s$ for some $s > 1/2$ then u converges to u_0 in the H^s norm as well. However, the question whether we have convergence in H^r norm if $u_0 \in H^r$, $0 < r < 1/2$, is still open. For the case $r = 1/2$ the answer is positive (see Section 6).

Remark 2. Another interesting open question is the uniqueness of the solution from Theorem 5.3. Due to the highly singular nature of estimates as t approaches zero, the usual uniqueness argument based on some sort of Gronwall inequality does not seem to go through.

6 The critical Sobolev space

Here we show that the results of Theorems 1.1, 1.2, 1.3, 1.4, 2.5, 3.1 and 3.2 hold for $s = 3/2 - 2\alpha$, as well.

Assume that $u_0 \in H^s$, $s \geq q = 3/2 - 2\alpha$, $1 > \alpha > 0$. We introduce the following Hilbert spaces of periodic functions. Let $\varphi : [0,\infty) \to [1,\infty)$ be an unbounded increasing function. Then $H^{s,\varphi}$ consists of periodic functions $f \in L^2$ such that its Fourier coefficients satisfy

$$\|f\|^2_{H^{s,\varphi}} := \sum_n |n|^{2s}(\varphi(|n|))^2|\hat{f}(n)|^2 < \infty. \tag{98}$$

Note that $u_0 \in H^{s,\varphi}$ for some function φ. Without loss of generality we may assume, in addition, that $\varphi \in C^\infty$ and

$$\varphi'(x) \leq Cx^{-1}\varphi(x) \tag{99}$$

for some constant C. It follows from (99) that

$$\varphi(2x) \leq 2^C \varphi(x). \tag{100}$$

We start from Galerkin approximations. Consider the sum arising from the nonlinear term when estimating the H^s norm of the solution:

$$S := \sum_{a+b+k=0,|a|,|b|,|k|\leq N} k|k|^{2s}(\varphi(|k|))^2 \hat{u}^N(a)\hat{u}^N(b)\hat{u}^N(k).$$

In what follows, for the sake of brevity, we will omit mentioning restrictions $|a|, |b|, |k| \leq N$ in notation for the sums; all sums will be taken with this restriction. Observe that (cf. (6))

$$|S| \leq 6 \sum_{k+a+b=0,|a|\leq|b|\leq|k|} |k|k|^{2s}(\varphi(|k|))^2 + a|a|^{2s}(\varphi(|a|))^2 + b|b|^{2s}(\varphi(|b|))^2||\hat{u}^N(k)\hat{u}^N(a)\hat{u}^N(b)|. \tag{101}$$

Recall that under conditions $|a| \leq |b| \leq |k|,\ \ a+b+k=0$, we have $|a| \leq |k|/2,\ |b| \geq |k|/2$. Next, due to (99) and (100) we estimate

$$\begin{aligned} &|k|k|^{2s}(\varphi(|k|))^2 + a|a|^{2s}(\varphi(|a|))^2 + b|b|^{2s}(\varphi(|b|))^2| = \\ &|b(|b|^{2s}(\varphi(|b|))^2 - |b+a|^{2s}(\varphi(|b+a|))^2) + a(|a|^{2s}(\varphi(|a|))^2 - |k|^{2s}(\varphi(|k|))^2| \leq \\ &C|a||k|^{2s}(\varphi(|k|))^2 \leq C|a||b|^s\varphi(|b|)|k|^s\varphi(|k|). \end{aligned} \tag{102}$$

Fix $M > 0$ to be specified later. Notice that sum over $|k| \leq M$ in (101) can be bounded by a constant $C(M)$. Splitting summation in a over dyadic shells scaled with $|k|$, define

$$S_1(l) = \sum_{k+a+b=0,|b|\leq|k|,|k|\geq M,|a|\in[2^{-l-1}|k|,2^{-l}|k|]} |a|^{1-2\alpha}|b|^{s+\alpha}\varphi(|b|)|k|^{s+\alpha}\varphi(|k|)|\hat{u}^N(k)\hat{u}^N(a)\hat{u}^N(b)|.$$

Then due to (102) and the relationship between a, b and k in the summation for S we have

$$|S| \leq C\sum_{l=1}^{\infty} 2^{-2l\alpha} S_1(l) + C(M). \tag{103}$$

Think of $S_1(l)$ as a quadratic form in $\hat{u}^N(k)$ and $\hat{u}^N(b)$. Then applying Schur test to each $S_1(l)$ we obtain

$$\begin{aligned} &S_1(l) \leq \|u^N\|^2_{H^{s+\alpha,\varphi}} \cdot \sup_{|k|\geq M} \sum_{|a|\in[2^{-l-1}|k|,2^{-l}|k|]} |a|^{1-2\alpha}|\hat{u}^N(a)| \leq \\ &C\|u^N\|^2_{H^{s+\alpha,\varphi}}\|u^N\|_{H^{q,\varphi}}(\varphi(2^{-l}M))^{-1}. \end{aligned} \tag{104}$$

Next, note that

$$\sum_{l=1}^{\infty} 2^{-2l\alpha} S_1(l) = \sum_{l=1}^{l_0} 2^{-2l\alpha} S_1(l) + \sum_{l=l_0}^{\infty} 2^{-2l\alpha} S_1(l) \leq$$
$$C\|u^N\|^2_{H^{s+\alpha,\varphi}} \|u^N\|_{H^{q,\varphi}} \left(\frac{1}{1-2^{-2\alpha}} (\varphi(2^{-l_0}M))^{-1} + \frac{2^{-2l_0\alpha}}{1-2^{-2\alpha}} \right). \quad (105)$$

Given $\epsilon > 0$, we can choose, first, sufficiently large l_0 and then sufficiently large M to obtain from (103), (105) and unboundedness of φ

$$|S| \leq C\epsilon \|u^N\|^2_{H^{s+\alpha,\varphi}} \|u^N\|_{H^{q,\varphi}} + C(M(\epsilon)). \quad (106)$$

It follows from (4) and (106) that

$$\frac{d}{dt}\|u^N\|^2_{H^{s,\varphi}} \leq (C\epsilon \|u^N\|_{H^{q,\varphi}} - 1)\|u^N\|^2_{H^{s+\alpha,\varphi}} + C(\epsilon), \quad s \geq q = 3/2 - 2\alpha, \ \alpha > 0. \quad (107)$$

Using this estimate and the same arguments as before we can extend the results of Theorems 1.1, 1.2, 1.3, 1.4, 2.5, 3.1 and 3.2 to the case $s = 3/2 - 2\alpha$, $\alpha > 0$. Here we formulate them for convenience of future references.

Theorem 6.1 *Assume that $\alpha > 1/2$, and the initial data $u_0(x) \in H^s$, $s \geq 3/2 - 2\alpha$, $s \geq 0$. Then there exists a global solution of the equation (1) $u(x,t)$ which belongs to $C([0,\infty), H^s)$ and is real analytic in x for $t > 0$.*

Theorem 6.2 *Assume $\alpha = 1/2$, and $u_0 \in H^s$, $s \geq 1/2$. Then there exists a global solution $u(x,t)$ of (1) which belongs to $C([0,\infty), H^s)$ and is real analytic in x for $t > 0$. If v is another weak solution of (1) with initial data u_0 such that $v \in C([0,T], L^2) \cap L^{3/2\delta}([0,T], H^\delta)$ with some $\delta \in (1/2, 1]$, then v coincides with u on $[0,T]$.*

Theorem 6.3 *Assume that $0 < \alpha < 1/2$, and the initial data $u_0(x) \in H^s$, $s \geq 3/2 - 2\alpha$. Then there exists $T = T(\alpha, u_0) > 0$ such that there exists a weak solution of the equation (1) $u(x,t) \in C([0,T], H^s) \cap L^2([0,T], H^{s+\alpha})$. Moreover, $u(x,t) \in C^\infty$ for any $0 < t < T$. If v is another weak solution of (1) with initial data u_0 such that $v \in C([0,T], L^2) \cap L^{3/2\delta}([0,T], H^\delta)$ with some $\delta \in (1/2, 1]$, then v coincides with u.*

Theorem 6.4 *Assume that $0 < \alpha < 1/2$. Then there exists smooth periodic initial data $u_0(x)$ such that the solution $u(x,t)$ of (1) blows up in H^s for each $s \geq \frac{3}{2} - 2\alpha$ in a finite time.*

Theorem 6.5 *Assume that $s \geq 3/2 - 2\alpha$, $s \geq 0$, $\alpha > 0$, and $u_0 \in H^s$. Then there exists $T = T(\alpha, u_0) > 0$ and a solution $u(x,t)$ of (1) such that*

$$u \in L^2([0,T], H^{s+\alpha}) \cap C([0,T], H^s); \quad (108)$$

$$t^{n/2} u \in C((0,T], H^{s+n\alpha}) \cap L^\infty([0,T], H^{s+n\alpha}) \quad (109)$$

for every $n > 0$.

Remark. If $s > 3/2 - 2\alpha$ then $T(\alpha, u_0) = T(\alpha, \|u_0\|_s)$. If $s = 3/2 - 2\alpha$ then $u_0 \in H^{s,\varphi}$ for some function φ described at the beginning of the section and $T(\alpha, u_0) = T(\alpha, \varphi, \|u_0\|_{H^{s,\varphi}})$.

Acknowledgement. AK and RS have been supported in part by NSF grant DMS-0314129. FN has been supported in part by NSF grant DMS-0501067. AK thanks Igor Popov for useful discussions.

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