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The Euler Equations and Nonlocal Conservative Riccati Equations

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This note presents an infinite-dimensional family of exact solutions of the incompressible three-dimensional Euler equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \qquad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

The solutions we present have infinite kinetic energy and blow-up in finite time. Blow-up of other similar infinite energy solutions of Euler equations has been proved before in [7], [2]. The particular type of solution we describe was proposed in [4]. The Eulerian-Lagrangian approach we take in [3] is not restricted to this particular case, but exact integration of the equations is. We consider a two-dimensional basic square Q of side L. The particular form of the solutions in [4] is

$$\mathbf{u}(x, y, z, t) = \big(u(x, y, t), z\gamma(x, y, t)\big), \tag{2}$$

where the scalar valued function γ is periodic in both spatial variables with period L, and the two-dimensional vector $u(x, y, t) = (u_1(x, y, t), u_2(x, y, t))$ is also periodic with the same period. The associated two-dimensional curl is

$$\omega(x, y, t) = \frac{\partial u_2(x, y, t)}{\partial x} - \frac{\partial u_1(x, y, t)}{\partial y}. \tag{3}$$

This represents the vertical (third) component of the vorticity $\nabla \times \mathbf{u}$ of the Euler system,

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and, using the ansatz (2), it follows from the familiar three-dimensional vorticity equation that the equation

$$\partial_t \omega + u \cdot \nabla \omega = \gamma \omega \tag{4}$$

should be satisfied for the recipe to succeed. On the other hand, one can easily check that the vertical component of the velocity $z\gamma(x, y, t)$ solves the vertical component of the Euler equations if γ solves the nonlocal Riccati equation

$$\frac{\partial \gamma}{\partial t} + u \cdot \nabla \gamma = -\gamma^2 + I(t), \tag{5}$$

where $I(t)$ is a function that depends on time only. The divergence-free condition for $\mathbf{u}$ becomes

$$\nabla \cdot u = -\gamma. \tag{6}$$

Because of the spatial periodicity of u, we must make sure that

$$\int_Q \gamma(x, t)\, dx = 0 \tag{7}$$

holds throughout the evolution. This can be done provided the function $I(t)$ is given by

$$I(t) = \frac{2}{|Q|} \int_Q \gamma^2(x, t)\, dx, \tag{8}$$

where

$$|Q| = \int_Q dx = L^2.$$

The velocity is determined from ω and γ using a stream function $\psi(x, y, t)$ and a potential $h(x, y, t)$ by

$$u = \nabla^{\perp} \psi + \nabla h, \tag{9}$$

$$-\Delta h = \gamma, \tag{10}$$

$$-\Delta \psi = \omega, \tag{11}$$

with periodic boundary conditions.

The ansatz $\mathbf{u}(x, y, z, t) = (u(x, y, t), z\gamma(x, y, t))$ associates to solutions of the system (4), (5), (8), (9), (10), and (11) in $d = 2$ velocities $\mathbf{u}$ that obey the incompressible three-dimensional Euler equations (see [4], [6]). The divergence condition (6) follows from (9) and (10). The compatibility condition $\int_Q \omega dx = 0$ is maintained throughout the evolution because of (4) and (6). We consider initial data

$$\gamma(x, y, 0) = \gamma_0(x, y), \qquad \omega(x, y, 0) = \omega_0(x, y) \tag{12}$$

that are smooth and have mean zero $\int_Q \gamma_0 dx = \int_Q \omega_0 dx = 0$. The solutions of the system above have local existence, and the velocity is as smooth as the initial data are, as long as $\int_0^T \sup_x |\gamma(x, t)| dt$ is finite (the result can be proved in [5] following the idea of the proof of the well-known result in [1]). We consider the characteristics

$$\frac{dX}{dt} = u(X, t), \tag{13}$$

and denoting $X(a, t)$ the characteristic that starts at $t = 0$ from a, $X(a, 0) = a$, we note that, prior to blow-up, the map $a \mapsto X(a, t)$ is one-to-one and onto as a map from $\mathbf{T}^2 = \mathbf{R}^2/L\mathbf{Z}^2$ to itself. The injectivity follows from the uniqueness of solutions of ordinary differential equations. The surjectivity can be proved by reversing time on characteristics, which can be done as long as the velocity is smooth. Our result is an explicit formula for γ on characteristics

$$\gamma(x, t) = \alpha(\tau(t)) \left\{ \frac{\gamma_0(A(x, t))}{1 + \tau(t)\gamma_0(A(x, t))} - \overline{\phi}(\tau(t)) \right\}, \tag{14}$$

where $A(x, t)$ is the inverse of $X(a, t)$ (the "back-to-labels" map), and the functions $\tau(t)$, $\alpha(\tau)$, and $\overline{\phi}(\tau)$ are computed from the initial datum γ_0. More precisely, we show the following theorem.

Theorem 1. Consider the nonlocal conservative Riccati system (see (4), (5), (8), (9), (10), and (11)). There exist smooth, mean zero initial data for which the solution becomes infinite in finite time. Both the maximum and the minimum values of the solution γ diverge to plus infinity and, respectively, to negative infinity at the blow-up time. There is no initial datum for which only the minimum diverges. The general solution is given on characteristics in terms of the initial data $\gamma(x, 0) = \gamma_0(x)$ by (see (24))

$$\gamma(X(a, t), t) = \alpha(\tau(t)) \left(\frac{\gamma_0(a)}{1 + \tau(t)\gamma_0(a)} - \overline{\phi}(\tau(t)) \right),$$

where

$$\overline{\phi}(\tau) = \left\{ \int_Q \frac{\gamma_0(a)}{(1 + \tau\gamma_0(a))^2} da \right\} \left\{ \int_Q \frac{1}{1 + \tau\gamma_0(a)} da \right\}^{-1},$$

$$\alpha(\tau) = \left\{ \frac{1}{|Q|} \int_Q \frac{1}{1 + \tau\gamma_0(a)}\, da \right\}^{-2},$$

and

$$\frac{d\tau}{dt} = \alpha(\tau), \qquad \tau(0) = 0.$$

The function $\tau(t)$ can be obtained from

$$t = \left(\frac{1}{|Q|}\right)^2 \int_Q \int_Q \frac{1}{\gamma_0(a) - \gamma_0(b)} \log\left(\frac{1 + \tau\gamma_0(a)}{1 + \tau\gamma_0(b)}\right) da\, db.$$

The Jacobian $J(a, t) = \mathrm{Det}\{\partial X(a, t)/\partial a\}$ is given by

$$J(a, t) = \frac{1}{1 + \tau(t)\gamma_0(a)} \left\{ \frac{1}{|Q|} \int_Q \frac{da}{1 + \tau(t)\gamma_0(a)} \right\}^{-1}.$$

The moments of γ are given by

$$\int_Q (\gamma(x, t))^p\, dx = (\alpha(\tau))^p \int_Q \left\{ \frac{\gamma_0(a)}{1 + \tau(t)\gamma_0(a)} - \overline{\phi}(\tau(t)) \right\}^p J(a, t)\, da.$$

The blow-up time $t = T_*$ is given by

$$T_* = \frac{1}{|Q|^2} \int\int \frac{1}{\gamma_0(a) - \gamma_0(b)} \log\left(\frac{\gamma_0(a) - m_0}{\gamma_0(b) - m_0}\right) da\, db,$$

where

$$m_0 = \min_Q \gamma_0(a) < 0.$$ □

We note that the curl ω plays a secondary role in this calculation and in the blow-up. Indeed, the same formula and blow-up occurs if $\omega_0 = 0$, or if the curl ω was smooth and computed in a different fashion than via (4).

1 Solving on characteristics

We now solve the nonlocal Riccati equation on characteristics. We start with an auxiliary problem. Let ϕ solve

$$\partial_\tau \phi + v \cdot \nabla\phi = -\phi^2$$

together with

$$\nabla \cdot \nu(x,\tau) = -\phi(x,\tau) + \frac{1}{|Q|}\int_Q \phi(x,\tau)\,dx.$$

We consider initial data that are smooth, periodic, and have zero mean,

$$\int_Q \phi_0(x)\,dx = 0.$$

We also assume that the curl $\zeta = (\partial\nu_2/\partial x) - (\partial\nu_1/\partial y)$ obeys

$$\partial_\tau\zeta + \nu \cdot \nabla\zeta = \left(\phi - \frac{3}{|Q|}\int_Q \phi(x,\tau)\,dx\right)\zeta.$$

Passing to characteristics

$$\frac{dY}{d\tau} = \nu(Y,\tau), \tag{15}$$

we integrate and obtain

$$\phi\big(Y(a,\tau),\tau\big) = \frac{\phi_0(a)}{1+\tau\phi_0(a)},$$

which is valid as long as

$$\inf_{a\in Q}\big(1+\tau\phi_0(a)\big) > 0.$$

We need to compute

$$\overline{\phi}(\tau) = \frac{1}{|Q|}\int_Q \phi(x,\tau)\,dx.$$

The Jacobian

$$J(a,\tau) = \mathrm{Det}\left\{\frac{\partial Y}{\partial a}\right\}$$

obeys

$$\frac{dJ}{d\tau} = -h(a,\tau)J(a,\tau),$$

where

$$h(a,\tau) = \phi(Y(a,\tau),\tau) - \overline{\phi}(\tau).$$

Initially, the Jacobian equals 1, so

$$J(a,\tau) = e^{-\int_0^\tau h(a,s)\,ds}.$$

Then

$$J(a,\tau) = e^{\int_0^\tau \overline{\phi}(s)\,ds} \exp\left(-\int_0^\tau \frac{d}{ds}\log\left(1+s\phi_0(a)\right)ds\right)$$

and thus

$$J(a,\tau) = e^{\int_0^\tau \overline{\phi}(s)\,ds}\frac{1}{1+\tau\phi_0(a)}.$$

The map $a \mapsto Y(a,\tau)$ is one-to-one and onto. The change of variables formula gives

$$\int_Q \phi(x,\tau)\,dx = \int_Q \phi(Y(a,\tau),t)J(a,\tau)\,da,$$

and, therefore,

$$\overline{\phi}(\tau) = e^{\int_0^\tau \overline{\phi}(s)\,ds}\frac{1}{|Q|}\int_Q \frac{\phi_0(a)}{(1+\tau\phi_0(a))^2}\,da. \tag{16}$$

Consequently,

$$\frac{d}{d\tau}e^{-\int_0^\tau \overline{\phi}(s)\,ds} = \frac{d}{d\tau}\frac{1}{|Q|}\int_Q \frac{1}{1+\tau\phi_0(a)}\,da.$$

Because both sides at $\tau = 0$ equal 1, we have

$$e^{-\int_0^\tau \overline{\phi}(s)\,ds} = \frac{1}{|Q|}\int_Q \frac{1}{1+\tau\phi_0(a)}\,da \tag{17}$$

and, using (16),

$$\overline{\phi}(\tau) = \left\{\int_Q \frac{\phi_0(a)}{(1+\tau\phi_0(a))^2}\,da\right\}\left\{\int_Q \frac{1}{1+\tau\phi_0(a)}\,da\right\}^{-1}. \tag{18}$$

Note that the function $\delta(x,\tau) = \phi(x,\tau) - \overline{\phi}(\tau)$ obeys

$$\frac{\partial\delta}{\partial\tau} + v\cdot\nabla\delta = -\delta^2 + 2\frac{1}{|Q|}\int_Q \delta^2\,dx - 2\overline{\phi}\delta.$$

We now consider the function

$$\sigma(x,\tau) = e^{2\int_0^\tau \overline{\Phi}(s)\,ds}\,\delta(x,\tau)$$

and the velocity

$$U(x,\tau) = e^{2\int_0^\tau \overline{\Phi}(s)\,ds}\,v(x,\tau).$$

Multiplying the equation of δ by $e^{4\int_0^\tau \overline{\Phi}(s)ds}$, we obtain

$$e^{2\int_0^\tau \overline{\Phi}(s)ds}\,\frac{\partial\sigma}{\partial\tau} + U\cdot\nabla\sigma = -\sigma^2 + \frac{2}{|Q|}\int \sigma^2\,dx.$$

Note that

$$\nabla\cdot U = -\sigma.$$

Now we change the time scale. We define a new time t by the equation

$$\frac{dt}{d\tau} = e^{-2\int_0^\tau \overline{\Phi}(s)\,ds}, \tag{19}$$

$t(0) = 0$, and new variables

$$\gamma(x,t) = \sigma(x,\tau)$$

and

$$u(x,t) = U(x,\tau).$$

Now γ solves the nonlocal conservative Riccati equation

$$\frac{\partial\gamma}{\partial t} + u\cdot\nabla\gamma = -\gamma^2 + \frac{2}{|Q|}\int \gamma^2\,dx \tag{20}$$

with periodic boundary conditions,

$$u = (-\Delta)^{-1}\left[\nabla^\perp\omega + \nabla\gamma\right] \tag{21}$$

and

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \gamma\omega. \tag{22}$$

The initial data are given by

$$\gamma_0(x) = \delta_0(x) = \phi_0(x).$$

Using (17) and integrating equation (19), we see that the time change is given by the formula

$$t = \left(\frac{1}{|Q|}\right)^2 \int_Q \int_Q \frac{1}{\phi_0(a) - \phi_0(b)} \log \frac{1 + \tau\phi_0(a)}{1 + \tau\phi_0(b)} \, da \, db. \tag{23}$$

Note that the characteristic system

$$\frac{dX}{dt} = u(X, t)$$

is solved by

$$X(a, t) = Y(a, \tau),$$

where Y solves the system (15). This implies the formula

$$\gamma(X(a, t), t) = \alpha(\tau) \left(\frac{\phi_0(a)}{1 + \tau\phi_0(a)} - \overline{\phi}(\tau) \right) \tag{24}$$

with

$$\alpha(\tau) = e^{2\int_0^\tau \overline{\phi}(s)\, ds}. \tag{25}$$

In view of (17), (18), and (23), we have obtained a complete description of the general solution in terms of the initial data.

2 Blow-up

Consider an initial smooth function $\gamma_0(a) = \phi_0(a)$ and assume that it has mean zero and that its minimum is $m_0 < 0$. As it is evident from the explicit formula, the blow-up time for $\phi(Y(a, \tau), \tau)$ is

$$\tau_* = -\frac{1}{m_0}.$$

It is also clear that $\phi(Y(a, \tau), \tau)$ diverges to negative infinity for some a, and not at all for others. This of course does not necessarily mean that γ blows up in the same fashion. Let us discuss the simplest case, in which the minimum is attained at a finite number of locations a_0, and near these locations, the function ϕ_0 has nonvanishing second derivatives, so that locally

$$\phi_0(a) \geq m_0 + C|a - a_0|^2$$

for $0 \leq |a - a_0| \leq r$. Then it follows that the integral

$$\frac{1}{|Q|}\int \frac{da}{\epsilon^2 + \phi_0(a) - m_0}$$

behaves like

$$\frac{1}{|Q|}\int \frac{da}{\epsilon^2 + \phi_0(a) - m_0} \sim \log\left\{\sqrt{1 + \left(\frac{Cr}{\epsilon}\right)^2}\right\},$$

for small ϵ. Taking

$$\epsilon^2 = \frac{1}{\tau} - \frac{1}{\tau_*},$$

we deduce that, for these kinds of initial data,

$$e^{-\int_0^\tau \overline{\phi}(s)\,ds} \sim \log\left\{\sqrt{1 + \frac{C}{\tau_* - \tau}}\right\}.$$

For the same kind of functions and small $(\tau_* - \tau)$, the integral

$$\frac{1}{|Q|}\int_Q \frac{\phi_0(a)}{(1 + \tau\phi_0(a))^2}\,da \sim -\frac{C}{\tau_* - \tau}$$

and $t(\tau)$ has a finite limit $t \to T_*$ as $\tau \to \tau_*$. The average $\overline{\phi}(\tau)$ diverges to negative infinity,

$$\overline{\phi}(\tau) \sim -\frac{C}{\tau_* - \tau}\left[\log\left\{\sqrt{1 + \frac{C}{\tau_* - \tau}}\right\}\right]^{-1}.$$

The prefactor α becomes vanishingly small

$$\alpha(\tau) \sim \big(\log(\tau_* - \tau)\big)^{-2},$$

and (24) becomes

$$\gamma(X(a,t),t) \sim (\log(\tau_* - \tau))^{-2}\left(\frac{\phi_0(a)}{1 + \tau\phi_0(a)} - \overline{\phi}(\tau)\right).$$

If the label is chosen so that $\phi_0(a) > 0$, then the first term in the brackets does not blow-up and γ diverges to plus infinity. If the label is chosen at the minimum, or nearby, then the first term in the brackets dominates and the blow-up is to negative infinity, as expected from the ordinary differential equation. From equation (19)

$$(\alpha(\tau))^{-1}\, d\tau = dt,$$

it follows that

$$T_* - t \sim (\tau_* - \tau)\left(1 + \log\left(\frac{1}{\tau_* - \tau}\right)\right)^2,$$

and the asymptotic behavior of the blow-up in t follows from the one in τ. We end by addressing a question that was at some point raised by numerical simulations: Can there be a one-sided blow-up? From the representation in (24) of the solution, it follows that

$$M(t) \geq -\overline{\phi}(\tau) e^{2\int_0^\tau \overline{\phi}(s)\, ds}$$

holds for $M(t) = \sup_x \gamma(x, t)$. If we assume that, up to the putative blow-up

$$M(t) \leq C$$

with some fixed constant C, then it follows that

$$-\frac{d}{d\tau} e^{2\int_0^\tau \overline{\phi}(s)\, ds} \leq 2C,$$

and, integrating between τ and τ_*, that

$$e^{2\int_0^\tau \overline{\phi}(s)\, ds} \leq 2C(\tau_* - \tau).$$

This in turn would imply that $T_* = \infty$, and therefore, no blow-up for γ can occur in finite t. So the answer is that for no initial datum can there exist a one-sided blow-up for γ.

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