

# A SIMPLE ENERGY PUMP FOR THE PERIODIC 2D QGE

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The main purpose of this draft is to show that the identically 0 solution is strongly unstable in  $H^{14}$ , which means that for every  $A > 0$  there exists a simple trigonometric polynomial  $p$  in the first few harmonics (independent of  $A$ ) such that  $\|p\| \leq 1$  (which norm doesn't matter) and such that, for the solution of the non-dissipative quasi-geostrophic equation with the initial data  $\theta(t, \cdot) = p(\cdot)$ , we have  $\sup_{t>0} \|\theta\|_{H^{14}} > A$  (a blow-up is allowed but it clearly cannot occur before the  $H^{14}$  norm becomes large).

We shall view the solutions  $\theta$  as sequences of Fourier coefficients  $\theta_x$ ,  $x \in \mathbb{Z}^2$ . Our trigonometric polynomial  $p$  will be just given by  $\theta_e = \theta_{-e} = 1$ ,  $\theta_g = \theta_{g+e} = \theta_{-g} = \theta_{-g-e} = \tau$  where  $e = (1, 0)$ ,  $g = (0, 2)$ , and  $\tau = \tau(A) > 0$  will be chosen later. Then the solution is an even real-valued function with  $\theta_0 = 0$  for all times. Moreover,  $\theta_x = 0$  whenever  $x_2$  is odd.

We have two conservation laws:  $\sum_x \theta_x^2 = 2 + 4\tau^2$  and  $\sum_x \frac{\theta_x^2}{|x|} = 2 + 2\tau^2 \left( \frac{1}{2} + \frac{1}{\sqrt{5}} \right)$ . It follows from here that  $\theta_e = \theta_{-e} \in \left( \frac{1}{2}, 2 \right)$  and  $\sum_{x \neq \pm e} \theta_x^2 \leq 10\tau^2$  for all times.

Consider the quadratic form

$$\mathcal{J}(\theta) = \sum_{x \in \mathbb{Z}_+^2} \Phi(x) \theta_x \theta_{x+e}.$$

We have

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}(\theta) = & \sum_{x \in \mathbb{Z}_+^2} \Phi(x) \left[ \theta_x \sum_{y+z=x+e, y, z \neq \pm e} (y \wedge z) \left( \frac{1}{|y|} - \frac{1}{|z|} \right) \theta_y \theta_z \right. \\
& \left. + \theta_{x+e} \sum_{y+z=x, y, z \neq \pm e} (y \wedge z) \left( \frac{1}{|y|} - \frac{1}{|z|} \right) \theta_y \theta_z \right] + \\
& \theta_e \sum_{x \in \mathbb{Z}_+^2} (e \wedge x) \Phi(x) \left[ \left( 1 - \frac{1}{|x|} \right) \theta_x^2 - \left( 1 - \frac{1}{|x+2e|} \right) \theta_x \theta_{x+2e} \right. \\
& \left. - \left( 1 - \frac{1}{|x+e|} \right) \theta_{x+e}^2 + \left( 1 - \frac{1}{|x-e|} \right) \theta_{x+e} \theta_{x-e} \right] \\
& = \sigma + \Sigma.
\end{aligned}$$

Since for  $y+z=x$ , we have  $\left| (y \wedge z) \left( \frac{1}{|y|} - \frac{1}{|z|} \right) \right| \leq 2|x|$  and  $|x+e| \asymp |x|$  for  $x \in \mathbb{Z}_+^2$ , we conclude that

$$|\sigma| \leq \left( \sum_{x \in \mathbb{Z}_+^2} |x| |\Phi(x)| |\theta_x| \right) \left( \sum_{y \neq \pm e} \theta_y^2 \right) \leq C\tau^2 \sum_{x \in \mathbb{Z}_+^2} |x| |\Phi(x)| |\theta_x|.$$

Now,  $\Sigma$  can be rewritten as

$$\begin{aligned}
& \sum_{x_2 > 0} x_2 \sum_{x_1 \in \mathbb{Z}} \frac{1}{2} \times \\
& \left[ (\Phi(x-e) - \Phi(x-2e)) \left( 1 - \frac{1}{|x-e|} \right) \theta_{x-e}^2 + (\Phi(x+e) - \Phi(x)) \left( 1 - \frac{1}{|x+e|} \right) \theta_{x+e}^2 + \right. \\
& \left. 2 \left\{ \Phi(x) \left( 1 - \frac{1}{|x-e|} \right) - \Phi(x-e) \left( 1 - \frac{1}{|x+e|} \right) \right\} \theta_{x-e} \theta_{x+e} \right]
\end{aligned}$$

Take now  $\Phi(x) = x_1 + \frac{1}{2}$ . We get the sum of quadratic forms with the coefficients

$$1 - \frac{1}{\sqrt{(x_1-1)^2 + x_2^2}}, \quad 1 - \frac{1}{\sqrt{(x_1+1)^2 + x_2^2}}$$

at the squares and

$$\left( x_1 + \frac{1}{2} \right) \left( 1 - \frac{1}{\sqrt{(x_1-1)^2 + x_2^2}} \right) - \left( x_1 - \frac{1}{2} \right) \left( 1 - \frac{1}{\sqrt{(x_1+1)^2 + x_2^2}} \right)$$

at the double product.

When  $x_1 = 0$ , this form is degenerate and when  $x_1 \neq 0$ , it is strictly positive definite and dominates  $\frac{c}{|x|^3}(\theta_{x-e}^2 + \theta_{x+e}^2)$ .

Thus

$$\Sigma \geq c \sum_{x \in \mathbb{Z}_+^2} \frac{\theta_x^2}{|x|^3}.$$

Now there are several possibilities.

A)  $\sum_{x \in \mathbb{Z}_+^2} |x|^2 |\theta_x| \geq \sum_{x \in \mathbb{Z}_+^2} |x| |\Phi(x)| |\theta_x| \geq \tau^{1/2}$ .

Then, since  $\sum_{x \in \mathbb{Z}_+^2} \theta_x^2 \leq 10\tau^2$ , we get

$$\sum_{x \in \mathbb{Z}_+^2} |x|^2 |\theta_x| \leq \left( \sum_{x \in \mathbb{Z}_+^2} \theta_x^2 \right)^{1/3} \left( \sum_{x \in \mathbb{Z}_+^2} |x|^{21} \theta_x^2 \right)^{1/6} \left( \sum_{x \in \mathbb{Z}_+^2} |x|^{-3} \right)^{1/2}$$

whence the  $H^{11}$ -norm gets large.

B) The case (A) never occurs but  $\sum_{x \in \mathbb{Z}_+^2} |x|^{-3} \theta_x^2$  becomes comparable with  $\tau^{5/2}$ . Note that until this moment  $\mathcal{J}(\theta)$  increases from its initial value about  $\tau^2$ . Also,  $\mathcal{J}(\theta) \leq \sum_{x \in \mathbb{Z}_+^2} |x| \theta_x^2$ .

Thus, in this case,

$$\sum_{x \in \mathbb{Z}_+^2} |x| \theta_x^2 \leq \left( \sum_{x \in \mathbb{Z}_+^2} |x|^{-3} \theta_x^2 \right)^{5/6} \left( \sum_{x \in \mathbb{Z}_+^2} |x|^{21} \theta_x^2 \right)^{1/6}.$$

and, again, the  $H^{11}$ -norm gets large.

At last, if neither (A), nor (B) occur, then  $\mathcal{J}(\theta)$  grows without bound and the  $H^{1/2}$ -norm gets large eventually.