

GLOBAL REGULARITY FOR THE CRITICAL DISPERSIVE DISSIPATIVE SURFACE QUASI-GEOSTROPHIC EQUATION

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ABSTRACT. We consider surface quasi-geostrophic equation with dispersive forcing and critical dissipation. We prove global existence of smooth solutions given sufficiently smooth initial data. This is done using a maximum principle for the solutions involving conservation of a certain family of moduli of continuity.

1. INTRODUCTION

In this paper, we study the following dispersive dissipative surface quasi-geostrophic (SQG) equation:

$$\theta_t = u \cdot \nabla \theta - (-\Delta)^{1/2} \theta + Au_2, \quad \theta(x, 0) = \theta_0(x). \quad (1.1)$$

Here θ is a scalar real-valued function, A is an amplitude parameter, and the velocity u is given by $u = (-R_2\theta, R_1\theta)$ with R_1, R_2 being the usual Riesz transforms. We will consider (1.1) on a torus \mathbb{T}^2 (or, equivalently, on \mathbb{R}^2 with periodic initial data). The equation (1.1) arises in geophysical fluid dynamics and can be interpreted as the evolution of buoyancy on a 2D surface in the presence of an ambient buoyancy gradient. Physically, the presence of the background gradient gives rise to dispersive waves and hence, the system supports both wave-like and turbulent motions (see Held et al. [4] for additional physical insight into (1.1) and Sukhatme and Smith [8] for its interpretation as part of a broader family).

In the recent years, the SQG equation has been focus of intense mathematical research, initiated by Constantin, Majda and Tabak [2]. The equation is physically motivated, and it is perhaps the simplest equation of fluid dynamics which is still poorly understood. Global regularity for the SQG equation without dispersion is known in the subcritical regime, when the dissipative term is $(-\Delta)^\alpha$, $\alpha > 1/2$ [7]. Global regularity has recently been extended to the critical case $\alpha = 1/2$ in [5] and [1]. The supercritical case $\alpha < 1/2$ in general remains open.

Mathematically, a key ingredient in some regularity proofs are a maximum principle for $\|\theta(x, t)\|_{L^\infty}$ (subcritical case, [7]) and a stronger nonlocal maximum principle for a certain modulus of continuity (critical case, [5]). In this work, we extend the latter technique to the case where dispersion is also present. The main result is

Theorem 1.1. *The dispersive critical surface quasi-geostrophic equation (1.1) with smooth periodic initial data has a unique global smooth solution.*

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Remarks. 1. We note that in the case of stronger dissipation $\alpha > 1/2$ the result also holds true and can be proven in a standard way (once the control of $\|\theta(x, t)\|_{L^\infty}$ is established - which is a part of our proof and which can be extended to the subcritical case in a straightforward manner).

2. The smoothness assumption on the initial data can be relaxed to $\theta_0 \in H^1$. Indeed, local existence of the solution smooth for $t > 0$ starting from such initial data can be proven by standard methods. The linear dispersive part does not present any difficulty in this respect (see e.g. [3] for the SQG case or [6] for an argument in the case of Burgers equation, which can be easily adapted to our situation). Once $t > 0$, one can apply the Theorem above to get global smooth solution.

3. The proof of the uniqueness in the setting of Theorem 1.1 is also standard.

4. The key step in the proof is, like in [5], the derivation of a uniform estimate on $\|\nabla\theta\|_{L^\infty}$ by using a family of moduli of continuity preserved by the evolution. Once one has this estimate, the proof of global existence of regular solution is achieved by well-known approach of using local existence theorem and differential inequalities for the Sobolev norms of the solution. Thus, in what follows we will focus on the essential issue of gaining control of $\|\nabla\theta\|_{L^\infty}$.

2. THE PROOF

Our first observation is that the L^2 norm of the solution over a single period cell is non-increasing.

Lemma 2.1. *The L^2 norm of a smooth solution of (1.1) is non-increasing.*

Proof. Multiplying the equation by $\theta(x, t)$ and integrating we obtain

$$\frac{1}{2}\partial_t\|\theta\|_{L^2}^2 = - \int_{\mathbb{T}^2} \theta(-\Delta)^{1/2}\theta dx + A \int_{\mathbb{T}^2} \theta u_2 dx.$$

The first term on the right hand side is negative, while the second is, up to a constant factor, equal to

$$\sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{k_1}{|k|} |\hat{\theta}(k_1, k_2)|^2.$$

The latter expression is zero since θ is real-valued and so $|\hat{\theta}|$ is even. □

Our next step is gaining control of the L^∞ norm of the solution of (1.1). One can no longer claim it is non-increasing as in the non-dispersive case (it isn't), but it remains uniformly bounded.

Lemma 2.2. *There exists a constant $D = D(A, \theta_0)$ such that the L^∞ norm of a smooth solution $\theta(x, t)$ of (1.1) satisfies*

$$\|\theta(x, t)\|_{L^\infty} \leq D$$

for all times while the solution remains smooth.

Remark. Observe that in contrast to the non-dispersive case, there is no L^∞ norm maximal principle: the L^∞ norm can grow, and numerical computations suggest it often does [8]. Instead, we just have an upper bound on the L^∞ norm.

Proof. Consider a point x where $\theta(x, t)$ reaches its maximum, M (the case of a minimum is similar). At the point of maximum, we have

$$\partial_t \theta(x, t) = -(-\Delta)^{1/2} \theta(x, t) + AR_1 \theta(x, t).$$

We will use the representation

$$-(-\Delta)^{1/2} \theta(x, t) = \frac{d}{dh} \mathcal{P}_h * \theta|_{h=0} = \lim_{h \rightarrow 0} \frac{1}{h} \int_{\mathbb{R}^2} \frac{h}{(|y|^2 + h^2)^{3/2}} (\theta(x - y) - M) dy, \quad (2.1)$$

where \mathcal{P}_h is the usual Poisson kernel in \mathbb{R}^2 . Observe that we can pass to the limit in (2.1) obtaining the kernel $|y|^{-3}$. On the other hand,

$$AR_1 \theta(x, t) = A \int_{\mathbb{R}^2} \frac{f(\hat{y})}{|y|^2} \theta(x - y) dy, \quad (2.2)$$

where f is a smooth mean zero function on the unit circle and $\hat{y} = y/|y|$. The integral converges in the principal value sense. Because of the mean zero property of f we can replace $\theta(x - y)$ in (2.2) with $\theta(x - y) - M$.

Consider a ball of radius ρ centered at x , $B_\rho(x)$, and the portion of the integrals in (2.1), (2.2) corresponding to that ball:

$$\int_{B_\rho(x)} \left(\frac{1}{|y|^3} + \frac{Af(\hat{y})}{|y|^2} \right) (\theta(x - y) - M) dy. \quad (2.3)$$

We can choose $\rho = \rho(A)$ sufficiently small independently of M so that (2.3) does not exceed

$$\frac{1}{2} \int_{B_\rho(x)} \frac{1}{|y|^3} (\theta(x - y) - M) dy. \quad (2.4)$$

Let us denote by m the Lebesgue measure on \mathbb{T}^2 . Since by Lemma 2.1, $\|\theta(x, t)\|_{L^2(\mathbb{T}^2)} \leq \|\theta_0\|_{L^2(\mathbb{T}^2)}$, we have that

$$m \left(y \in \mathbb{T}^2 \left| |\theta(y, t)| \geq \frac{M}{2} \right. \right) \leq \frac{4\|\theta_0\|_{L^2(\mathbb{T}^2)}^2}{M^2}. \quad (2.5)$$

Assume that ρ is sufficiently small so that B_ρ fits into a single period cell. Since $|y|^{-3}$ is monotone decreasing, the expression in (2.4) is maximal if points where $\theta(y, t)$ is large are concentrated near $y = 0$. In particular, assuming that M is sufficiently large, we see from (2.5) that the expression in (2.4) is less than or equal to

$$-\frac{1}{2} \int_{B_\rho(x) \setminus B_r(x)} \frac{M}{2|y|^3} dy,$$

where $r = 2\pi^{-1/2} \|\theta_0\|_{L^2(\mathbb{T}^2)}^2 M^{-1}$. Therefore, the expression in (2.4) does not exceed

$$-\frac{\pi}{2} \int_r^\rho \frac{M}{|y|^2} d|y| \leq -\frac{\pi^{3/2}}{4\|\theta_0\|_{L^2(\mathbb{T}^2)}} M^2 + C(A)M. \quad (2.6)$$

The integral over the complement of B_ρ in (2.1) is negative, so it remains to control

$$A \int_{\mathbb{R} \setminus B_\rho(x)} \frac{f(\hat{y})}{|y|^2} (\theta(x-y) - M) dy.$$

A straightforward estimate yields a bound by CAM . Therefore, we have

$$\partial_t \theta(x, t) \leq -C(\theta_0)M^2 + C(A)M,$$

which is negative provided that M is large enough; define $\tilde{D}(A, \theta_0)$ so that this is true if $M \geq \tilde{D}(A, \theta_0)$. But then it is clear that $\theta(x, t)$ can never reach such value of M unless $\|\theta_0\|_{L^\infty}$ was already larger - but in this case, $\|\theta\|_{L^\infty}$ will decay until reaching at least $\tilde{D}(A, \theta_0)$. Setting $D(A, \theta_0) = \max\{\tilde{D}(A, \theta_0), \|\theta_0\|_{L^\infty}\}$, we obtain the result of the Lemma. \square

Now we introduce a family of moduli of continuity. This is the same family that was considered in [5] in the case of the critical SQG. Namely, let $\omega(\xi)$ be continuous and defined by

$$\begin{aligned} \omega(\xi) &= \xi - \xi^{3/2}, & 0 \leq \xi \leq \delta; \\ \omega'(\xi) &= \frac{\gamma}{\xi(4 + \log(\xi/\delta))}, & \xi \geq \delta, \end{aligned} \tag{2.7}$$

and set $\omega_B(\xi) = \omega(B\xi)$. Here $0 < \gamma < \delta$ are certain constants defined in [5]; the modulus of continuity ω is increasing, concave and differentiable at every point except $\xi = \delta$.

We will need the following lemma from [5]:

Lemma 2.3. *If the function θ has modulus of continuity ω_B , then $u = (-R_2\theta, R_1\theta)$ has modulus of continuity*

$$\Omega_B(\xi) = C \left(\int_0^\xi \frac{\omega_B(\eta)}{\eta} d\eta + \xi \int_\xi^\infty \frac{\omega_B(\eta)}{\eta^2} d\eta \right)$$

with some universal constant $C > 0$.

Observe that by a simple change of coordinates and definition of ω_B , $\Omega_B(\xi) = \Omega(B\xi)$.

Our next lemma can be proven exactly as in [5], using that $\omega''(0) = -\infty$:

Lemma 2.4. *Assume that a smooth solution of (1.1) $\theta(x, t)$ has modulus of continuity ω_B at some time t_0 . The only way this modulus of continuity may be violated is if there exists $t_1 \geq t_0$ and $y, z, y \neq z$, such that $\theta(y, t_1) - \theta(z, t_1) = \omega_B(|y - z|)$, while for all $t < t_1$, the solution has modulus of continuity ω .*

Next, consider two points y, z and time t_1 as in Lemma 2.4. Observe that

$$\begin{aligned} \partial_t(\theta(y, t) - \theta(z, t))|_{t=t_1} &= u \cdot \nabla \theta(y, t_1) - u \cdot \nabla \theta(z, t_1) - (-\Delta)^{1/2} \theta(y, t_1) \\ &\quad + (-\Delta)^{1/2} \theta(z, t_1) + Au_2(y, t_1) - Au_2(z, t_1). \end{aligned} \tag{2.8}$$

Let us denote $|y - z| = \xi$. We have the following

Lemma 2.5. *For y, z and t_1 as in Lemma 2.4, we have*

$$|u \cdot \nabla \theta(y, t_1) - u \cdot \nabla \theta(z, t_1)| \leq \omega'_B(\xi) \Omega_B(\xi) \quad (2.9)$$

and

$$|Au_2(y, t_1) - Au_2(z, t_1)| \leq A\Omega_B(\xi). \quad (2.10)$$

Moreover, $\delta > \gamma > 0$ can be chosen so that

$$-(-\Delta)^{1/2} \theta(y, t_1) + (-\Delta)^{1/2} \theta(z, t_1) \leq -2\omega'_B(\xi) \Omega_B(\xi). \quad (2.11)$$

Proof. The inequality (2.10) follows immediately from Lemma 2.3. The proof of the inequality (2.9) is identical to that provided in [5]. The proof of (2.11) is also the same as the treatment of the dissipative term given in [5]. Although the result is not stated in [5] in the same form, it follows immediately from the arguments provided there. In the estimates above at the point $x = \delta$ one should use the larger value out of the one-sided derivatives (which is the left derivative). \square

Now we are ready to prove our main technical result, from which Theorem 1.1 follows as explained in the introduction.

Theorem 2.6. *Assume that the initial data $\theta_0(x)$ is smooth and periodic. Then there exists a constant $B(A, \theta_0)$ such that while the solution of (1.1) $\theta(x, t)$ remains smooth, it satisfies*

$$\|\nabla \theta(\cdot, t)\|_{L^\infty} \leq B. \quad (2.12)$$

Proof. Consider B_0 large enough so that $\theta_0(x)$ has ω_B for any $B > B_0$. Suppose the solution $\theta(x, t)$ loses ω_B , then by Lemma 2.4 we can find y, z and t_1 so that $\theta(y, t_1) - \theta(z, t_1) = \omega_B(|y - z|)$ and $\theta(x, t)$ has ω_B for all $t \leq t_1$. By Lemma 2.5, we have

$$\partial_t(\theta(y, t) - \theta(z, t))|_{t=t_1} \leq -\omega'_B(\xi) \Omega_B(\xi) + A\Omega_B(\xi). \quad (2.13)$$

Moreover, by Lemma 2.2, $\|\theta(\cdot, t)\|_{L^\infty} \leq D(A, \theta_0)$ and so

$$\omega_B(\xi) = \theta(y, t) - \theta(z, t) \leq 2D(A, \theta_0).$$

Since $\omega_B(\xi) = \omega(B\xi)$, it follows that $B\xi \leq \omega^{-1}(2D(A, \theta_0))$. But then since ω' is decreasing,

$$\omega'_B(\xi) = B\omega'(B\xi) \geq B\omega'(\omega^{-1}(2D(A, \theta_0))).$$

In particular, the right hand side in (2.13) is strictly negative if $B \geq A/\omega'(\omega^{-1}(2D(A, \theta_0)))$. This gives a contradiction with the definition of t_1 since by smoothness the modulus of continuity should have been violated at an earlier time. Thus moduli of continuity corresponding to sufficiently large B are preserved by evolution, as claimed by the Theorem. \square

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