

**ONE-DIMENSIONAL SCHRÖDINGER OPERATORS  
WITH SLOWLY DECAYING POTENTIALS:  
SPECTRA AND ASYMPTOTICS  
OR  
*BABY FOURIER ANALYSIS  
MEETS TOY QUANTUM MECHANICS***

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Workshop on Oscillatory Integrals and Dispersive Equations

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These informal notes survey research carried out jointly by the authors over the last few years, and particularly developments since the Fall of 1999. The text heavily emphasizes our own efforts, with limited discussion of the extensive prior literature.<sup>1</sup> Among many possible sources for an introduction to that literature are [43, 50, 18, 54, 8]. The papers cited in the bibliography contain more complete references.

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## 1. INTRODUCTION AND BACKGROUND

The basic object of study is a time-independent Schrödinger operator on the real line

$$(1) \quad H = -\frac{d^2}{dx^2} + V(x).$$

Standing hypotheses throughout these notes (except for the discussion of a few examples in §1) are that  $V$  is real-valued and that  $\int_{|x-y|<1} |V(y)| dy \rightarrow 0$  as  $|x| \rightarrow \infty$ , although Theorem 22 can be formulated more generally. Then  $H$  is self-adjoint on  $L^2(\mathbb{R})$ . Considered instead as an operator on  $L^2[0, \infty)$  with (say) Dirichlet or Neumann boundary conditions, it is likewise self-adjoint.

A quantum-mechanical interpretation is that the free Hamiltonian  $H_0 = -d^2/dx^2$  describes the behavior of a free electron, while  $H_0 + V$  describes one electron interacting with an external electrical field, described by the potential  $V$ . One can sometimes think of  $V$  as representing some disorder. We will be interested in the case of small disorder, where  $V \rightarrow 0$  at  $\infty$  in some sense.

If  $V$  is sufficiently small, then the spectrum of  $H_0 + V$  should resemble that of  $H_0$ . One of the goals of the theory surveyed in these notes is to justify this expectation for certain classes of potentials. In particular, reasonably precise and sharp conditions will be given for the persistence of absolutely continuous spectrum under perturbations.

To any self-adjoint operator  $H$  on a Hilbert space  $\mathcal{H}$  and any vector  $\varphi \in \mathcal{H}$  is associated a spectral measure  $\mu_\varphi$ , satisfying

$$(2) \quad \langle f(H)\varphi, \varphi \rangle = \int_{\mathbb{R}} f(\lambda) d\mu_\varphi(\lambda)$$

for any Borel measurable, bounded function  $f$ .

Any finite measure  $\mu$  decomposes as  $\mu_s + \mu_{ac}$  where the summands are respectively singular with respect to, and mutually absolutely continuous with respect to, Lebesgue measure. The singular component decomposes further as  $\mu_{pp} + \mu_{sc}$  where the last summand contains no atoms, while  $\mu_{pp}$  is a countable linear combination of Dirac masses.  $H$  is said to have (some) absolutely continuous spectrum if there exists  $\varphi \neq 0$  such that  $(\mu_\varphi)_{ac} \neq 0$ , and to have purely absolutely continuous spectrum if  $\mu_\varphi = (\mu_\varphi)_{ac}$  for every  $\varphi$ . We often abbreviate “absolutely continuous” as “ac”. Similarly one speaks of pure point and purely sc spectrum.  $\mathcal{H}_{ac}$  denotes the maximal subspace of  $\mathcal{H}$  on which  $H$  has purely absolutely continuous spectrum.

The point spectrum is dictated by the eigenfunctions, that is, the  $L^2$  solutions of  $Hu = Eu$ . By a *generalized eigenfunction*<sup>2</sup> of  $H = H_0 + V$  we mean a solution  $u$  of  $Hu \equiv Eu$  for some  $E \in \mathbb{R}$ . A generalized eigenfunction has at most exponential growth if  $V$  is uniformly in  $L^1_{loc}$ ; the spectrum is related to those with at most polynomial growth. More precise links between growth rates and the ac spectrum will be discussed in §3.

Suppose that a Borel measure  $\mu$  on  $\mathbb{R}$  is absolutely continuous with respect to Lebesgue measure. By an essential support of  $\mu$  is meant a Borel set  $S$  such that  $\mu(\mathbb{R} \setminus S) = 0$ , and  $\mu(E) > 0$  whenever  $E \subset S$  has positive Lebesgue measure.

To  $\mu_\varphi$  is associated its Borel transform

$$(3) \quad \mathcal{M}_{\mu_\varphi}(z) = \langle (H - z)^{-1}\varphi, \varphi \rangle = \int_{\mathbb{R}} (\lambda - z)^{-1} d\mu_\varphi(\lambda).$$

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<sup>2</sup>This term is often used in with a more specific meaning, but in these notes will always mean any solution of the eigenfunction equation, with no growth restriction.

Since  $\mu_\varphi$  is a finite measure,  $\mathcal{M}(z)$  is well-defined whenever  $\operatorname{Re}(z) > 0$ . Its imaginary part is

$$(4) \quad \operatorname{Im}(\mathcal{M}_{\mu_\varphi}(E + i\varepsilon)) = \int_{\mathbb{R}} \frac{\varepsilon}{(\lambda - E)^2 + \varepsilon^2} d\mu_\varphi(\lambda).$$

Let  $\mu$  be any locally finite positive measure. Define

$$(5) \quad D^\alpha \mu(x) = \limsup_{\varepsilon \rightarrow 0} \frac{\mu(x - \varepsilon, x + \varepsilon)}{(2\varepsilon)^\alpha}.$$

By differentiation theory, in order to prove that a finite measure  $\mu$  has a nonzero absolutely continuous component, it suffices to prove that  $D^1 \mu(x) > 0$  for all  $x$  in some set having positive Lebesgue measure. If  $\limsup_{\varepsilon \rightarrow 0^+} \operatorname{Im}(\mathcal{M}_\mu)(E + i\varepsilon) > 0$ , then  $D^1 \mu(E) > 0$ .

A brief tour of some basic classes of potentials, and the spectral properties of the associated Schrödinger operators:

- The free Hamiltonian  $H_0 = -d^2/dx^2$  has purely ac spectrum.
- If  $V(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$  then the spectrum consists of a discrete sequence of eigenvalues tending to  $+\infty$ .
- If  $V \in L^1(\mathbb{R})$  then  $H_0 + V$  has only point spectrum (aka bound states) in  $\mathbb{R}^-$ , with 0 as its only possible accumulation point. In  $\mathbb{R}^+$  the spectrum is purely absolutely continuous, and an essential support is  $\mathbb{R}^+$  itself. If  $xV(x) \in L^1$  then there are only finitely many bound states.
- (Wigner-von Neumann potential) There exists a potential with asymptotic behavior  $V(x) \sim c \sin(2x)/x$  which has an eigenvalue at  $E = +1$ , embedded in the continuous spectrum.
- In  $\mathbb{R}^n$  for<sup>3</sup>  $n > 1$ , if  $V(x) = O(|x|^{-r})$  for some  $r > 1$ , then there are no positive eigenvalues [26, 53, 1]. Moreover, by the theorem of Agmon-Kato-Kuroda [43],  $\mu_{\text{sc}} = \emptyset$ , and an essential support of  $\mu_{\text{ac}}$  is  $\mathbb{R}^+$ .
- If  $V$  is periodic in  $\mathbb{R}$  then the spectrum is purely absolutely continuous, and consists of a countable sequence of intervals  $[a_j, b_j]$  with  $b_j \leq a_{j+1}$  and  $a_j, b_j \rightarrow +\infty$ .
- The Almost Mathieu operators  $h$  act on  $\ell^2$  by

$$(6) \quad hu(n) = u(n-1) + u(n+1) + v(n)u(n),$$

$$(7) \quad v(n) = \lambda \cos(\pi \alpha n + \theta)$$

where  $\lambda, \alpha, \theta$  are parameters. They exhibit all manner of spectra, including purely absolutely continuous, dense pure point, and purely singular continuous, depending on the magnitude of  $\lambda$  and Diophantine properties of  $\alpha$ .

- ([42], [30]) Let  $h \geq 0$  be continuous and compactly supported, but not  $\equiv 0$ . Consider potentials with large gaps:

$$(8) \quad V(x) = \sum_n a_n h(x - x_n)$$

where  $x_n \rightarrow +\infty$ , and  $a_n \rightarrow 0$ . If  $x_n/x_{n+1} \rightarrow 0$  sufficiently rapidly then the spectrum is purely absolutely continuous if  $a \in \ell^2$ , and is purely singular otherwise.

- Consider instead a family of potentials

$$(9) \quad V_\omega(x) = \sum_{n=-\infty}^{\infty} a_n(\omega) h(x - n)$$

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<sup>3</sup>With rare exceptions, we discuss only operators on  $\mathbb{R}^1$ ; this is one exception.

where  $a \in \ell^\infty$ , and  $\omega \in \Omega$ , a probability space. Suppose that  $\int h = 0$ , and that  $a_n = b_n r_n(\omega)$  where  $b_n \in \mathbb{R}$  and  $\{r_n\}$  are independent, identically distributed random variables, whose distributions are bounded and absolutely continuous with respect to Lebesgue measure. If  $b_n \equiv b$ , a nonzero constant, then for almost every  $\omega$ , the spectrum of  $H_0 + V_\omega$  consists entirely of (a dense in  $\mathbb{R}^+$  set of) eigenvalues; there is no continuous spectrum. The same holds more generally, if  $b_n \sim n^{-r}$  and  $r \leq 1/2$ .

- In the preceding example, if instead  $\{b_n\} \in \ell^2$ , then for almost every  $\omega$ , the spectrum is purely absolutely continuous.
- For any  $r < 1$ , there exist potentials satisfying  $V(x) = O(|x|^{-r})$  for which the set of all eigenvalues is *dense* in  $\mathbb{R}^+$  [38],[55].

When  $V \in L^1$ , there is no point spectrum embedded in the continuous spectrum. When for instance  $V = O(|x|^{-r})$  and  $r < 1$ , it remained an open question until around 1996 whether there was necessarily any continuous spectrum. The first progress, for  $r > 3/4$ , was due to Kiselev [28]; a series of papers, culminating in [9] and Remling [44], established existence of ac spectrum for all  $r > 1/2$ . The situation is highly unstable, in the sense that for  $r < 1$  there is sometimes *dense* point spectrum embedded in the continuous spectrum.

Deift and Killip obtained a definitive result, in some respects, by quite a different method [19].

**Theorem 1** (Deift and Killip). *If  $V \in L^1 + L^2(\mathbb{R})$  then  $H_0 + V$  has nonempty absolutely continuous spectrum; moreover, an essential support equals  $(0, \infty)$ .*

These notes describe a further development of the method of [9], which requires slightly stronger hypotheses on  $V$ , but yields additional information. In particular, this additional information can be used to study the associated Schrödinger group  $\exp(itH)$ .

## 2. TWO (SAMPLE) PRINCIPAL RESULTS

The main purpose of the work outlined in these notes is to better understand Schrödinger operators with rather slowly decaying potentials by

- (1) Analyzing the behavior of the associated generalized eigenfunctions.
- (2) Applying the resulting estimates to analyze the evolution group  $\exp(itH)$ .
- (3) Extending the discussion to wider classes of potentials.

In the course of doing so, we will also

- (4) Extend the range of the much-used WKB approximation, establishing a theory in which WKB asymptotics hold for parametrized families of functions, almost surely but not uniformly with respect to the parameter.
- (5) Develop general machinery concerning multilinear integral operators, and maximal versions thereof.

We now formulate two sample theorems, in order to indicate more concretely where we are heading.

**Theorem 2.** [10] *Let  $V \in L^1 + L^p(\mathbb{R})$  for some  $1 < p < 2$ . Then an essential support for the absolute spectrum of  $H = -\partial_x^2 + V$  is  $\mathbb{R}^+$ . For almost every  $\lambda \in \mathbb{R}$  there exists a generalized eigenfunction satisfying*

$$(10) \quad u(x, \lambda) - e^{i\phi(x, \lambda)} \rightarrow 0 \text{ as } x \rightarrow +\infty,$$

where<sup>4</sup>  $\phi(x, \lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V$ .

$du/dx$  has corresponding asymptotics  $i\lambda \cdot e^{i\phi(x, \lambda)}$ .

Lack of smoothness of the potential is not the issue here; assuming that  $\partial^k V / \partial^k x \in L^p + L^1$  for all  $k$  would not change the conclusions, nor would it make the theorem any easier to prove.<sup>5</sup> Indeed, the examples (9) are smooth in this sense. This is as one might expect, from the uncertainty principle; spectral properties of  $H$  at energies in any fixed compact subinterval of  $\mathbb{R}$  should not depend strongly on behavior of the potential on scales  $\lesssim 1$ . The situation is quite different if  $V$  satisfies symbol-type hypotheses, ensuring that successive derivatives of  $V$  decay successively more rapidly [24].

This theorem captures a certain tradeoff: it has weaker hypotheses than  $V \in L^1$ , but offers a (necessarily) weaker conclusion. The improvement from hypothesizing merely that  $V \in L^p$ , rather than power decay  $V = O(|x|^{-r})$ , has the physical interpretation that long gaps in which the potential vanishes identically do not affect the ac spectrum (so long as  $V$  is sufficiently small; gaps are the essential feature of the examples of Pearson described above).

Our second sample result concerns long-time asymptotics for the associated evolution  $e^{itH}$ . For definitions of wave and scattering operators see §12.

**Theorem 3.** [14] *Let  $H = H_0 + V$  on  $L^2(\mathbb{R}^+)$  with Dirichlet boundary condition at the origin. Suppose that  $V \in L^p + L^1$  for some  $1 < p < 2$ . Suppose further that*

$$(11) \quad \lim_{x \rightarrow +\infty} \int_0^x V(y) dy \quad \text{exists.}$$

*Then for each  $f \in L^2(\mathbb{R}^+)$ , the wave operators  $\Omega_{\pm}$  exist in  $L^2$  norm as  $t \rightarrow \mp\infty$ . Moreover,  $\Omega^{\pm}$  are bijective isometries from  $\mathcal{H} = L^2(\mathbb{R}^+)$  to  $\mathcal{H}_{ac}$ .*

For the physical interpretation, see §12.

Moreover, the scattering operator  $(\Omega^+)^{-1} \circ \Omega^-$  can be identified as a “Fourier multiplier” operator, which can be explicitly described in terms of the asymptotics of the phase  $\phi(x, \lambda)$ . See Theorem 27.

### 3. A CRITERION FOR AC SPECTRUM

How can one get a grip on the spectral measure for a selfadjoint operator  $H$ ? A criterion of Weyl characterizes points  $E$  of the essential<sup>6</sup> spectrum by the existence of sequences of unit vectors  $\varphi_n$  tending weakly to zero, for which  $\|(H - E)\varphi_n\|_{\mathcal{H}} \rightarrow 0$ . For Schrödinger operators (in any dimension, satisfying mild hypotheses not stated here), an extension by Simon of a theorem of Sch’nol [47] states that the spectrum, as a set<sup>7</sup>, coincides with the closure of the set of energies for which there is a polynomially bounded<sup>8</sup> generalized eigenfunction; see also [54], page 501.

Several devices are potentially available for studying the ac spectrum:

<sup>4</sup>A perhaps more familiar form for the exponent is  $i \int_0^x \sqrt{\lambda^2 - V(y)} dy$ . If  $V \in L^2$ , this is equivalent to  $i\phi(x, \lambda)$ .

<sup>5</sup>Throughout these notes, the class  $L^p + L^1$  can be replaced by the Birman-Solomjak space  $\ell^p(L^1)$ , which is the Banach space of all functions satisfying  $\sum_n (f_n^{n+1} |V|)^p < \infty$ .

<sup>6</sup>The essential spectrum is the spectrum minus all isolated eigenvalues of finite multiplicity.

<sup>7</sup>More precisely, for almost every energy with respect to any fixed  $\mu_{\varphi}$ , there exists a polynomially bounded generalized eigenfunction.

<sup>8</sup>Subexponential growth suffices.

- (1) (Perhaps the best-known strategy for this IPAM workshop audience.) Analyze the associated group  $e^{itH}$  ( $e^{it\sqrt{H}}$  being problematic because  $H$  need not be positive), and recover  $\mu_\varphi$  by Fourier inversion from the formula  $\langle e^{itH}\varphi, \varphi \rangle = \int e^{it\lambda} d\mu_\varphi(\lambda)$ .
- (2) Estimate resolvents  $(H - \lambda^2)^{-1}$ , and apply Stone's formula via the Weyl  $m$ -function.
- (3) Apply the subordinacy theory developed by Gilbert and Pearson.
- (4) Apply a criterion [15] relating the spectral measure to the growth properties of approximate eigenfunctions. See Proposition 4 and Corollary 5, below.

We next discuss these criteria in slightly greater detail. While the time-dependent first strategy has been enormously successful in other aspects of spectral theory, it does not appear promising here. One seeks to recover the absolutely continuous component of some measure, which may have also have a singular component, from the asymptotics of its Fourier transform, a daunting prospect.

The Weyl  $m$ -function is  $m(\lambda) = u_\lambda^+{}'(0)/u_\lambda^+(0)$ ; it equals  $\partial_{x,y}^x G_{\lambda^2}(x, y)$  evaluated at  $(0, 0)$  where  $G_\zeta$  is Green's function, that is, the kernel associated to  $(H - \zeta)^{-1}$ . A formula of Stone reads

$$(12) \quad \pi^{-1} \operatorname{Im} (m(E + i\varepsilon)) dE \rightarrow d\mu(E) \quad \text{as } \varepsilon \rightarrow 0^+$$

in the sense of weak limits, where  $\mu$  is a positive measure on  $\mathbb{R}$  such that  $H$  is unitarily equivalent to multiplication by  $E$  on<sup>9</sup>  $L^2(\mathbb{R}, \mu)$ .

$m$  can be related to other quantities, specifically to the reciprocals  $a, b$  of transmission and reflection coefficients (for the definitions see (34) below), by

$$(13) \quad m(\lambda^2) = i\lambda \frac{a(\lambda) - b(\lambda)}{a(\lambda) + b(\lambda)}.$$

From this, it follows that

$$(14) \quad \operatorname{Im} (m(\lambda)) \geq 4\lambda^{-1}|a(\lambda)|^{-2}.$$

In this way, upper bounds for  $a$  lead to lower bounds for  $\operatorname{Im} (m)$ , thence to the presence of absolutely continuous spectrum by (12) plus a limiting argument. Later we will discuss in detail upper bounds for  $a$  and for related quantities.

A generalized eigenfunction  $u(x)$  with energy  $E$ , for a one-dimensional Schrödinger operator, is said to be *subordinate* at  $+\infty$  if

$$(15) \quad \lim_{y \rightarrow +\infty} \frac{\int_0^y |u(x)|^2 dx}{\int_0^y |v(x)|^2 dx} = 0$$

for any linearly independent generalized eigenfunction  $v$  with the same energy. Subordinacy at  $-\infty$  is defined analogously. Then [22] the singular spectrum is supported on the set of all  $E$  for which there exists a generalized eigenfunction that is subordinate at both  $+\infty$  and  $-\infty$ , and the absolutely continuous spectrum on the set of all  $E$  for which there exists no such generalized eigenfunction.

A consequence of the subordinacy theory is [58] that if  $V$  is uniformly in  $L_{\text{loc}}^1$ , and if all generalized eigenfunctions are globally bounded on  $\mathbb{R}$  for all  $E$  in some set  $\Lambda$ , then there is ac spectrum everywhere on  $\Lambda$ , and no singular spectrum. See also [52] for a proof and discussion, including further references to the general subordinacy theory. For further development see [25].

This consequence of subordinacy theory is all that is required to deduce spectral implications from properties of generalized eigenfunctions which will be established in these

<sup>9</sup>Functions in  $L^2(\mathbb{R}, \mu)$  taking values in an appropriate auxiliary Hilbert space.

notes. However, before proceeding to that analysis, we discuss here an alternative approach, which may offer potential advantages for other problems, in particular, for higher dimensions. This may turn out to be of some importance, because the subordinacy theory and  $m$ -function approaches are special to dimension one. (Note that the space of all generalized eigenfunctions with given energy is two-dimensional for  $\mathbb{R}^1$ , but infinite-dimensional in higher dimensions.)

This new criterion relies on a notion of approximate eigenfunctions.

**Proposition 4.** [15] *For any spectral measure  $\mu_\varphi$  associated to a self-adjoint operator  $H$  on a Hilbert space  $\mathcal{H}$ , any  $E \in \mathbb{R}$ , and any  $\varepsilon > 0$ ,*

$$(16) \quad \operatorname{Im} \mathcal{M}_{\mu_\varphi}(E + i\varepsilon) \geq c_0 \varepsilon^{-1} \sup_{\psi} |\langle \varphi, \psi \rangle|^2$$

where the supremum is taken over all  $\psi \in \mathcal{H}$  satisfying

$$(17) \quad \|\psi\| = 1 \text{ and } \|(H - E)\psi\| \leq \varepsilon.$$

Thus in order to prove that the absolutely continuous component of  $\mu_\varphi$  charges a set  $S$ , it suffices to show that for almost every  $E \in S$  there exists a sequence  $\psi_j \in \mathcal{H}$  such that  $\|(H - E)\psi_j\|/\|\psi_j\| \rightarrow 0$  and

$$(18) \quad \frac{|\langle \varphi, \psi_j \rangle|^2}{\|\psi_j\| \cdot \|(H - E)\psi_j\|} \geq c > 0,$$

uniformly as  $j \rightarrow \infty$ . Indeed, normalizing to make each  $\psi_j$  a unit vector and setting  $\varepsilon_j = \|(H - E)\psi_j\|$ , we deduce that  $\limsup_{\varepsilon \rightarrow 0} \operatorname{Im} \mathcal{M}(E + i\varepsilon)$  is strictly positive. Hence  $D^1\mu_\varphi(E)$  is likewise strictly positive, by the standard majorization of the maximal Poisson integral by the Hardy-Littlewood maximal function.

Proposition 4 is quite easy to prove. By the spectral theorem, we may assume that  $H$  is multiplication by the coordinate  $\lambda$  on  $L^2(\mathbb{R}^+, d\nu)$  for some positive measure  $\nu$ ; then  $d\mu_\varphi = |\varphi|^2 d\nu$ . Now

$$(19) \quad \begin{aligned} & \left| \int \varphi \psi d\nu(\lambda) \right|^2 \\ & \leq \left[ \int |\varphi(\lambda)|^2 \frac{\varepsilon}{(\lambda - E)^2 + \varepsilon^2} d\nu(\lambda) \right] \left[ \int |\psi(\lambda)|^2 \frac{(\lambda - E)^2 + \varepsilon^2}{\varepsilon} d\nu(\lambda) \right] \\ & \leq |\operatorname{Im} \mathcal{M}_{\mu_\varphi}(E + i\varepsilon)| \varepsilon^{-1} (\|(H - E)\psi\|^2 + \varepsilon^2 \|\psi\|^2), \end{aligned}$$

from which the conclusion follows by invoking the conditions  $\|\psi\| = 1$  and  $\|(H - E)\psi\| \leq \varepsilon$ .  $\square$

A more concrete criterion for the existence of absolutely continuous spectrum is as follows.

**Corollary 5.** *Suppose that  $V \in L^1_{loc}(\mathbb{R})$ , uniformly on unit intervals. Let  $S \subset \mathbb{R}$  and suppose that for each  $E \in S$ , there exists a generalized eigenfunction  $u_E$  of  $H_0 + V$  satisfying the growth restriction*

$$(20) \quad \liminf_{R \rightarrow \infty} R^{-1} \int_{|x| \leq R} |u_E(x)|^2 dx < \infty.$$

Then for any compact subset  $S' \subset S$  of positive Lebesgue measure, there exists  $\varphi$  such that  $\mu_\varphi(S') > 0$ .

In particular, (20) is satisfied if  $u_E \in L^\infty$ .

This criterion may seem paradoxical, for it asserts in particular that the existence for each  $E$  in some interval of a genuine eigenfunction (that is, a solution  $u_E \in L^2(\mathbb{R})$  of  $(H - E)u_E =$

0) implies ac spectrum; the paradox is that genuine eigenfunctions correspond not to ac spectrum, but to point spectrum. However, such a situation is impossible; eigenfunctions can occur only for a countable set of values of  $E$ . A key point here is that approximate eigenfunctions satisfying the relevant growth bound are required to exist for a set of  $E$  having positive Lebesgue measure.

In order to deduce the Corollary from Proposition 4, one must produce a sequence of approximate eigenfunctions. This is done by multiplying generalized eigenfunctions by sequences of cutoff functions. Fix  $h \in C_0^2(\mathbb{R})$ , with  $h \equiv 1$  in a neighborhood of the origin. Set  $u_E^{(R)}(x) = h(x/R)u(x)$ . Then

$$(21) \quad (H_0 + V - E)(u_E^{(R)}) \equiv -R^{-2}h''(x/R)u_E(x) - 2R^{-1}h'(x/R)u_E'(x).$$

Fix any nonnegative  $\varphi$  supported in a sufficiently small interval near 0 but in  $[0, \infty)$ ; then it is easy to see that  $|\langle u_E, \varphi \rangle| = |\langle u_E^{(R)}, \varphi \rangle| \neq 0$  for all  $E \in S'$  for which  $|u_E(0)/u_E'(0)|$  is sufficiently large; other  $E$  may be handled similarly by a different choice of  $\varphi$ . One-dimensional elliptic regularity theory applied to the equation  $Hu = Eu$ , in conjunction with (20), reveals that  $u_E'$  likewise satisfies (20). From (21), we then deduce that  $\|(H - E)u_E^{(R)}\| \cdot \|u_E^{(R)}\| = O(R^{-1/2}) \cdot O(R^{1/2})$ , so the criterion (18) holds.  $\square$

This corollary is rather general. The criterion (20) applies in Euclidean space of any dimension. It likewise applies to  $-L + V$  on any Lie group, where  $L$  is a self-adjoint left-invariant (subelliptic, not necessarily elliptic) sub-Laplacian and  $V \in L^\infty$ , with  $\{|x| \leq R\}$  replaced by the ball of radius  $R$ , with fixed center, with respect to the Carnot-Carathéodory metric associated to  $L$ .

#### 4. EXPANSIONS FOR GENERALIZED EIGENFUNCTIONS

To begin to analyze the generalized eigenfunctions, suppose that  $V \in L^1$ . The equation  $-u'' - \lambda^2 u + Vu = 0$  may be written formally as  $u = (\partial_x^2 + \lambda^2)^{-1}Vu$ , modulo adding an element of the nullspace of  $\partial_x^2 + \lambda^2$ . One of several inverse operators is

$$(22) \quad (\partial_x^2 + \lambda^2)^{-1}g(x) = (2i\lambda)^{-1} \int_{y>x} [e^{i\lambda(x-y)} - e^{-i\lambda(x-y)}]g(y) dy .$$

Seeking a solution asymptotic to  $e^{i\lambda x}$  as  $x \rightarrow +\infty$ , we arrive at the integral equation

$$(23) \quad u(x) = e^{i\lambda x} + (2i\lambda)^{-1} \int_x^\infty [e^{i\lambda(x-y)} - e^{-i\lambda(x-y)}]V(y)u(y) dy.$$

If  $V \in L^1$  then the usual contraction mapping argument yields for every  $\lambda \neq 0$  the existence of a solution satisfying  $u_\lambda(x) - e^{i\lambda x} \rightarrow 0$  as  $x \rightarrow +\infty$ .

Alternatively, one can iterate the equation, at least formally, to arrive at

$$(24) \quad u(x) = e^{i\lambda x} + (2i\lambda)^{-1} \int_x^\infty [e^{i\lambda(x-y)} - e^{-i\lambda(x-y)}]V(y)e^{i\lambda y} dy \\ + (2i\lambda)^{-2} \iint_{x \leq y_1 \leq y_2} [e^{i\lambda(x-y_1)} - e^{-i\lambda(x-y_1)}]V(y_1) \\ \cdot [e^{i\lambda(y_1-y_2)} - e^{-i\lambda(y_1-y_2)}]V(y_2)u(y_2) dy_1 dy_2 .$$

The first line has no unknown  $u$ . The terms involving  $V$  are scalar multiples of

$$(25) \quad e^{-i\lambda x} \int_x^\infty e^{i2\lambda y} V(y) dy$$

$$(26) \quad e^{i\lambda x} \int_x^\infty V(y) dy$$

(25) is  $e^{-i\lambda x} \widehat{V}_x(-2\lambda)$ , where  $V_x(y) = V(y) \cdot \chi_{[x, \infty)}(y)$ . At this point we recall one formulation of the theorem of Carleson on almost everywhere convergence of Fourier series and integrals: If  $f \in L^2$  then  $\int_{-\infty}^s e^{i\lambda \xi} \hat{f}(\xi) d\xi$  converges as  $s \rightarrow +\infty$ , for almost every  $\lambda$ ; as a function of  $(\lambda, s)$ , the indefinite integral belongs to the space  $L_\lambda^2 L_s^\infty$ . If  $V \in L^2$  then by Plancherel's theorem we may write  $V = \hat{f}$ , so (25) is bounded in  $s$  for almost every  $\lambda$ .

Even for fixed  $s$ , say  $s = +\infty$ , one merely has square integrability in  $\lambda$ , rather than locally uniform bounds. This lack of uniformity is related to the possible presence of dense point spectrum. The connection with Fourier integrals is an indication of the natural role played by  $L^2$  in the analysis of the generalized eigenfunctions.

(26) behaves quite differently; if  $V \notin L^1$  then it may have no reasonable interpretation. One could rearrange matters to replace the interval of integration by  $[0, x]$ , so that it would be finite for fixed  $x$ . But we seek  $L^\infty(dx)$  estimates (or at least the  $L^2$  analogues  $\int_0^R |u|^2 dx \leq CR$  needed to apply Corollary 5, and no such estimates would hold uniformly in  $x$ ).

If this iteration process is carried out to infinite order, one obtains a power series expansion for  $u(x, \lambda)$  in terms of  $V$ . A sample quadratic term is

$$(27) \quad e^{i\lambda x} \iint_{x \leq y_1 \leq y_2} e^{-i2\lambda y_1} e^{i2\lambda y_2} V(y_1) V(y_2) dy_1 dy_2,$$

and then higher-order terms. Each term defines a function of  $(x, \lambda)$  by applying a multilinear operator to  $m$  copies of  $V$ ,  $m = 1, 2, 3, \dots$ . One can hope that this last sample term is not much worse behaved than the expression  $e^{i\lambda x} \hat{V}(2\lambda) \hat{V}(-2\lambda)$  obtained by integrating over all  $y_1, y_2 \in \mathbb{R}$ . On the other hand, infinitely many terms arise which share the defect of (26). We thus face three difficulties: (i) justifying the hope just expressed, (ii) summing bounds over  $m = 1, 2, 3, \dots$ , and (iii) dealing with summands that fail to satisfy the bounds sought for the sum itself.

This last difficulty is familiar; individual terms of the Maclaurin series for the bounded function  $\exp(ix)$  are unbounded. We will see in §7 how the power series expansion for  $u_\lambda$  can be reorganized by grouping certain terms together, so that no obviously unbounded terms remain. In fact, this grouping process amounts to nothing more than summation of the Maclaurin series for the imaginary exponential function.

## 5. WKB APPROXIMATION

Suppose temporarily that  $V$  satisfies symbol-type hypotheses:

$$(28) \quad |\partial_x^k V(x)| \leq C_k |x|^{-\delta - k\rho}$$

for some  $\delta, \rho > 0$ , for all  $k$ , for large  $x$ . We seek now a formal asymptotic approximation to a generalized eigenfunction  $u_\lambda(x)$  for  $H = H_0 + V$ , as  $x \rightarrow +\infty$ , for fixed  $\lambda \neq 0$ . Set

$$(29) \quad u_\lambda(x) \sim e^{i\psi(x)},$$

expand  $\psi \sim \sum_{n=0}^{\infty} \psi_n$ , and set  $\psi_0 = \lambda x$ . We seek a solution of symbol type, with each  $\psi_{k+1}$  decaying more rapidly than  $\psi_k$ .  $\psi$  is to satisfy

$$(30) \quad (\psi')^2 - i\psi'' = \lambda^2 - V.$$

Thus  $(\lambda + \psi_1')^2 - i\psi_1'' \approx \lambda^2 - V$ . Dropping the terms  $\psi_1''$  and  $(\psi_1')^2$  because we expect them to decay more rapidly than  $\psi_1'$  itself, we find that

$$(31) \quad \psi_1' = -(2\lambda)^{-1}V.$$

Thus  $\psi_1' = O(|x|^{-\delta})$ , while  $\psi_1'' = O(|x|^{-\delta-\rho})$  and  $(\psi_1')^2 = O(|x|^{-2\delta})$ . This procedure may be repeated to obtain  $\psi_n'$  for every  $n$ , satisfying symbol-type estimates with gain of  $|x|^{-\min(\rho, \delta)}$  at each iteration.

In the case where  $V$  does not satisfy symbol-type estimates, we will seek generalized eigenfunctions of the form

$$(32) \quad u_\lambda(x) = e^{i\phi(x, \lambda)} + o(1) \text{ as } x \rightarrow +\infty, \text{ with}$$

$$(33) \quad \phi(x, \lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V(y) dy.$$

The lower limit of integration may equally well be chosen differently.

Observe two things. Firstly, the WKB phase shift  $-(2\lambda)^{-1} \int^x V$  is in general unbounded, if  $V \notin L^1$ . Secondly, we may try to measure the quality of an approximate solution  $\tilde{u}$  by the remainder  $-\tilde{u}'' + V\tilde{u} - \lambda^2\tilde{u}$ . For  $\tilde{u} = \exp(i\lambda x)$ , this remainder has modulus  $|V(x)|$ . For  $\tilde{u} = \exp(i\phi(x, \lambda))$ , it has modulus  $|c_1V^2(x) + c_2V'(x)|$  for certain constants  $c_j$  depending on  $\lambda$ . The term  $V^2$  is on the average smaller than  $V$ . However,  $V'$  is in general only defined in the sense of distributions, and in general decays no more rapidly than  $V$  itself, even with a liberal measure of its size, for instance in a Sobolev space  $H_{\text{loc}}^s$  with  $s = -1$ . Nonetheless our main results show that for  $V \in L^1 + L^p$ ,  $1 < p < 2$ , without any differentiability hypothesis, the approximation (32), (33) is accurate for Lebesgue-almost every  $\lambda$ .

See [3, 32, 62] for original papers on the WKB approximation.

## 6. TRANSMISSION AND REFLECTION COEFFICIENTS

Suppose temporarily that  $V$  has compact support. Fix  $\lambda \in \mathbb{R}$ . There exists a unique generalized eigenfunction  $u_\lambda^\dagger$  that is  $\equiv \exp(i\lambda x)$  for  $x$  near  $+\infty$ . Near  $-\infty$ ,

$$(34) \quad u_\lambda^\dagger \equiv a(\lambda)e^{i\lambda x} + b(\lambda)e^{-i\lambda x}$$

for certain coefficients  $a, b$ . The quantities

$$(35) \quad t(\lambda) = 1/a(\lambda), \quad r(\lambda) = b(\lambda)/a(\lambda).$$

are called respectively the transmission and reflection coefficients. Their interpretation is that an incoming wave  $e^{i\lambda x}$  from  $-\infty$  interacts with the potential and splits into a reflected wave  $r(\lambda)e^{-i\lambda x}$  plus a transmitted wave  $t(\lambda)e^{i\lambda x}$ . For our purposes,  $a, b$  are more fundamental than  $t, r$ .

The constancy of the Wronskian of  $u^\dagger, \overline{u^\dagger}$  can be equivalently rephrased as<sup>10</sup>

$$(36) \quad |a(\lambda)|^2 = |b(\lambda)|^2 + 1 \text{ for all } \lambda \in \mathbb{R}.$$

<sup>10</sup>In the time-dependent picture, this translates to an infinitesimal form of conservation of energy; it says that the energy of the incoming wave equals the combined energies of the transmitted and reflected waves. In the time-independent framework under discussion here, it can be interpreted as a conservation of probability densities.

Temporarily allowing  $\lambda$  to be complex, we find that  $E = \lambda^2$  is an eigenvalue if  $\lambda \in i\mathbb{R}$  with negative imaginary part, and  $a(\lambda) = 0$ . A small computation shows that  $|a| + |b|$ , and hence  $|a|$  alone, control the magnitude of the vector  $(u^+(x), du^+(x)/dx)$  for  $x$  to the left of the support of  $V$ .

For compactly supported  $V \in L^2$ , the following remarkable trace identity<sup>11</sup> [4, 20] holds:

$$(37) \quad \int_{\mathbb{R}} \log |a(\lambda)| \lambda^2 d\lambda + \frac{2\pi}{3} \sum_k |\lambda_k|^3 = \frac{\pi}{8} \int_{\mathbb{R}} V^2 dx$$

where  $\{\lambda_k\}$  is the collection of all eigenvalues of  $-\partial_x^2 + V$ .<sup>12</sup> This set is necessarily finite, and each  $\lambda_k$  is negative. An outline of a simple, direct proof may be found in [19]; it involves a deformation of the contour of integration into the upper half space in the complex plane.<sup>13</sup> The second term on the right arises from any poles coming from zeros of  $a$ , while the first arises in the limit as the contour is pushed to infinity.

This has the following consequence. Let  $\Lambda \subset \mathbb{R} \setminus \{0\}$  be a compact interval. Denote by  $u_\lambda(x)$  the unique generalized eigenfunctions with  $(u_\lambda(0), u'_\lambda(0))$  equal either to  $(1, 0)$ , or to  $(0, 1)$ . Then if  $V \in L^2[0, x]$ ,

$$(38) \quad \int_{\Lambda} \log(1 + |u_\lambda(x)|) d\lambda \leq C + C \int_{[0, x]} V^2,$$

where  $C < \infty$  depends only on  $\Lambda$ . In particular, if  $V \in L^2(\mathbb{R})$ , then the left-hand side is bounded, uniformly in  $x$ . This bound may seem extraordinarily weak;  $u$  is only logarithmically integrable in  $\lambda$ . But no more can be expected. If one thinks of  $V$  as  $\sum_j V_j$  with each  $V_j$  supported in  $[j, j+1]$ , then the map sending  $(u(j), u'(j))$  to  $(u(j+1), u'(j+1))$  is multiplication by a certain matrix whose entries depend on  $V_j$ ; these matrices are multiplied together in sequence to yield the asymptotic behavior as  $x \rightarrow +\infty$ ; taking a logarithm converts this back to an additive process.

(38) does not seem sufficient for a direct application of the approximate eigenfunction criterion of Proposition 4, but suffices for analysis of the Weyl  $m$ -function, in conjunction with a limiting argument. This is how Deift and Killip [19] proved the existence of ac spectrum for potentials in  $L^2$ . Adding an  $L^1$  perturbation is harmless, either by functional analysis (a relative trace class perturbation does not change the ac spectrum), or because the generalized eigenfunctions for  $H_0 + V_1$  can be used to construct the associated resolvents, whence the generalized eigenfunctions for  $H_0 + V_1 + V_2$  can be analyzed by solving an integral equation  $u = -(H_0 - V_1 + \lambda^2)^{-1} V_2 u$  modulo an element of the nullspace of  $H_0 - V_1 + \lambda^2$  as above; this always works if  $V_2 \in L^1$  and  $H_0 + V_1$  has (for the set of energies in question) bounded generalized eigenfunctions.

<sup>11</sup>This identity is related to the inverse scattering theory for the Korteweg-de Vries equation; the right-hand side is one of the basic quantities invariant under the KdV flow. Dropping the first term on the left, one obtains a fundamental bound of Lieb-Thirring type for the negative eigenvalues.

<sup>12</sup>The exponent 3 is not a typographical error.

<sup>13</sup> $a(\lambda)$  is an entire function. Since  $a(-\lambda) \equiv \overline{a(\lambda)}$  for  $\lambda \in \mathbb{R}$ ,  $\log |a|$  may be replaced by  $\log(a)$  in the integral. The integral equation (23) can be used to obtain an asymptotic expansion for  $a$  as  $|\lambda| \rightarrow \infty$  in the upper half plane.  $a(\lambda) = 0$  if and only if  $\lambda^2$  is an eigenvalue of  $H_0 + V$ ; deforming the contour, taking zeros into account, and invoking an identity which amounts to Plancherel's theorem to control the integral over the limiting contour yields the identity. (It is a bit strange that the series arising from the integral equation (23) works here, since many individual terms of that series fail to satisfy the conclusion desired for the sum.)

## 7. REDUCTION AND EXPANSION

In this section we write the generalized eigenfunction equation  $-u'' + Vu = \lambda^2 u$  as a first-order system, in a way that incorporates the WKB approximation. We then iterate the resulting equation in a fashion parallel to that of §4, to obtain a modified power series expansion for the nonlinear operator mapping  $V$  to  $u_+(x, \lambda)$  where  $u_+$  is, formally<sup>14</sup>, the unique generalized eigenfunction asymptotic to  $\exp(i\phi(x, \lambda))$  as  $x \rightarrow +\infty$ .

$y = \begin{pmatrix} u \\ u' \end{pmatrix}$  satisfies

$$(39) \quad y' = \begin{pmatrix} 0 & 1 \\ V - \lambda^2 & 0 \end{pmatrix} y .$$

Writing  $\phi = \phi(x, \lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V$ , we substitute

$$(40) \quad y = \begin{pmatrix} e^{i\phi} & e^{-i\phi} \\ i\lambda e^{i\phi} & -i\lambda e^{-i\phi} \end{pmatrix} w .$$

Thus boundedness of  $w$  (as a function of  $x$  for given  $\lambda$ ) is equivalent to boundedness of both  $u$  and  $u'$ . And if  $w \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  as  $x \rightarrow +\infty$  then  $u - \exp(i\phi) \rightarrow 0$ . The new unknown  $w$  satisfies

$$(41) \quad w' = \frac{i}{2\lambda} \begin{pmatrix} 0 & -V(x)e^{-2i\lambda x + \frac{i}{\lambda} \int_0^x V(t) dt} \\ V(x)e^{2i\lambda x - \frac{i}{\lambda} \int_0^x V(t) dt} & 0 \end{pmatrix} w .$$

$w$  is directly linked to certain reflection/transmission coefficients. Define  $a(x, \lambda), b(x, \lambda)$  by

$$(42) \quad \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \begin{pmatrix} a(x, \lambda)e^{i\lambda x} + b(x, \lambda)e^{-i\lambda x} \\ i\lambda a(x, \lambda)e^{i\lambda x} - i\lambda b(x, \lambda)e^{-i\lambda x} \end{pmatrix}$$

Then

$$(43) \quad \begin{pmatrix} a(x, \lambda) \\ b(x, \lambda) \end{pmatrix} = \begin{pmatrix} e^{-i(2\lambda)^{-1} \int_0^x V} & 0 \\ 0 & e^{i(2\lambda)^{-1} \int_0^x V} \end{pmatrix} w .$$

In particular, the two components of  $w(x) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$  have the same magnitudes as  $a, b$ , respectively. The conservation identity  $|a|^2 \equiv 1 + |b|^2$  thus is equivalent to

$$(44) \quad |w_1(x)|^2 \equiv 1 + |w_2(x)|^2 .$$

Introduce

$$(45) \quad (Tf)(\lambda) = \int_{\mathbb{R}} e^{2i\lambda x - i\lambda^{-1} \int_0^x V(t) dt} f(x) dx ,$$

defined initially on integrable functions of compact support. We also introduce multilinear operators

$$(46) \quad T_n(f_1, \dots, f_n)(x, \lambda) = \left( \frac{i}{2\lambda} \right)^n \int_x^\infty \int_{t_1}^\infty \dots \int_{t_{n-1}}^\infty \prod_{j=1}^n \left[ e^{(-1)^{n-j} 2\phi(t_j, \lambda)} f_j(t_j) dt_j \right] .$$

<sup>14</sup>Since existence is not yet proved.

Iterating system (41) starting from the vector  $(1, 0)$  we obtain a formal “Taylor series” expansion for the putative generalized eigenfunction  $u_+(x, \lambda)$  with the desired asymptotic  $\exp(i\phi)$  as  $x \rightarrow +\infty$ :

$$(47) \quad \begin{aligned} u_+(x, \lambda) = & e^{i\lambda x - \frac{i}{2\lambda} \int_0^x V(t) dt} \sum_{n=0}^{\infty} (-1)^n T_{2n}(V, \dots, V)(x, \lambda) \\ & + e^{-i\lambda x + \frac{i}{2\lambda} \int_0^x V(t) dt} \sum_{n=1}^{\infty} (-1)^n T_{2n-1}(V, \dots, V)(x, \lambda) \end{aligned}$$

We have set  $T_0(V)(x, \lambda) \equiv 1$ . This is not exactly a Taylor series, since  $V$  still appears in a nonlinear fashion in the exponents.

In a sense, our approach is a part of the program of Calderón [5] and of Coifman and Meyer [16, 17] of analyzing nonlinear operators (in the present case, mapping  $V$  to the collection of all generalized eigenfunctions) via power series expansions in terms of multilinear operators.

## 8. MAXIMAL OPERATORS

We seek to analyze multilinear operators (46). If we simplify by discounting the WKB phase correction for the present, these are built up out of a well-understood operator, the Fourier transform, by two processes. Firstly, integration over  $\mathbb{R}$  is replaced by integration over the nested family of sets  $(-\infty, x]$ , and a supremum estimate in  $x$  is sought. Secondly, multilinear operators are generated by iterated integrals over such sets. In this section we develop a robust, though crude, method for analyzing the first of these two processes. The theory to be developed here depends heavily on the order structure of  $\mathbb{R}$ , as of course does the definition of the multilinear operators (54).

Denote by  $\chi_E$  the characteristic function of a set  $E$ , and by  $\|T\|$  the operator norm of  $T : L^p(\mathbb{R}) \mapsto L^q(\mathbb{R})$ . To any operator  $T$  defined on  $L^p(\mathbb{R})$  can be associated a maximal operator

$$(48) \quad T^* f(x) = \sup_s |T(f \cdot \chi_{(-\infty, s]})(x)|.$$

**Theorem 6.** [11, 10] *Let  $1 \leq p, q \leq \infty$ , and suppose that  $T : L^p(\mathbb{R}) \mapsto L^q(\mathbb{R})$  is a bounded linear operator. Then  $T^*$  is likewise bounded from  $L^p(\mathbb{R})$  to  $L^q(\mathbb{R})$ , provided that  $p < q$ . Moreover,  $\|T^*\|$  is bounded by an absolute constant, depending only on  $p, q$ , times  $\|T\|$ .*

More generally, there is an analogue for any linear operator from  $L^p(Y)$  to  $L^q(X)$ , for arbitrary measure spaces, provided that the sets  $(-\infty, s]$  are replaced by an arbitrary nested family of sets  $Y_n \subset Y$  (that is,  $Y_n \subset Y_{n+1}$ ).

Suppose that  $T$  is represented as an integral operator  $Tf(x) = \int_{\mathbb{R}} K(x, y) f(y) dy$ .

**Corollary 7.** *If  $p < q$  and if  $T$  is bounded from  $L^p$  to  $L^q$  then*

$$(49) \quad \tilde{T}f(x) = \int_{y < x} K(x, y) f(y) dy$$

*is likewise bounded from  $L^p$  to  $L^q$ .*

The hypothesis  $p < q$  is in general necessary, except in the trivial cases  $p = 1$  or  $q = \infty$ , even in the corollary. Consider for example the Hilbert transform, for which  $K(x, y) = (x - y)^{-1}$ ; the associated operator with kernel  $(x - y)^{-1} \chi_{y < x}$  is unbounded on all  $L^p$ .

These results apply as well to functions taking values in Banach spaces. Tao [59] pointed out its applicability to Strichartz-type estimates (see Section 13 below). This generalization

of has been applied by Smith and Sogge [56] to the obstacle problem, and has also been applied by Takaoka and Tzvetkov, by Colliander and Kenig, and perhaps by others to nonlinear evolution equations.

*Outline of proof.* Let  $1 \leq p < \infty$  and let  $0 \neq f \in L^p(\mathbb{R})$  be fixed. Construct a collection  $\{E_j^m\}$  of intervals, indexed by  $m \in \{0, 1, 2, \dots\}$  and  $1 \leq j \leq 2^m$ , satisfying

- For each  $m$ ,  $\{E_j^m : 1 \leq j \leq 2^m\}$  is a partition of  $\mathbb{R}$  into disjoint intervals.
- $E_j^m$  lies to the left of  $E_{j+1}^m$  for all  $m, j$ .
- Each  $E_j^m = E_{2j-1}^{m+1} \cup E_{2j}^{m+1}$ .
- $\int_{E_j^m} |f|^p = 2^{-m} \int_{\mathbb{R}} |f|^p$  for all  $m, j$ .

Let  $\chi_j^m$  denote the characteristic function of  $E_j^m$ , and set  $f_j^m = f \cdot \chi_j^m$ . For any  $s \in \mathbb{R}$ , the interval  $(-\infty, s]$  can be partitioned, modulo a set on which  $f$  vanishes almost everywhere, as  $\cup_\nu E_{j_\nu}^{m_\nu}$  for some sequences such that  $m_1 < m_2 < m_3 \dots$ , and each  $E_{j_\nu}^{m_\nu}$  lies to the left of  $E_{j_{\nu+1}}^{m_{\nu+1}}$ . Thus

$$(50) \quad T^* f(x) \leq \sum_{m=0}^{\infty} \sup_{1 \leq j \leq 2^m} |T(f_j^m)(x)| \leq G_{T,r} f(x)$$

where the last quantity is defined by

$$(51) \quad G_{T,r} f(x) = \sum_m \left( \sum_j |T(f_j^m)(x)|^r \right)^{1/r}$$

for any positive exponent  $r$ . Choosing  $r = q$ , we have

$$\begin{aligned} \|Gf\|_q &\leq \sum_m \left( \int \sum_j |T(f_j^m)|^q \right)^{1/q} \leq \|T\| \sum_m \left( \sum_j \|f_j^m\|_p^q \right)^{1/q} \\ &\leq \|T\| \sum_m (2^m 2^{-mq/p} \|f\|_p^q)^{1/q} = C \|T\| \cdot \|f\|_p \end{aligned}$$

where  $C < \infty$  by the hypothesis  $q > p$ . To justify the final inequality we have used the bound  $\|f_j^m\|_p \leq 2^{-m/p} \|f\|_p$   $\square$

The following two theorems, dating roughly from the 1930's, are immediate corollaries.

**Theorem 8.** *Let  $1 \leq p < 2$ . For any  $f \in L^p(\mathbb{R}^1)$ ,*

$$\lim_{y \rightarrow \infty} \int_0^y e^{-i\lambda x} f(x) dx$$

*exists for almost every  $\lambda$ . Moreover*

$$\sup_y \left| \int_0^y e^{-i\lambda x} f(x) dx \right| \in L^q(\mathbb{R}, d\lambda), \quad \text{where } q = p/(p-1).$$

**Theorem 9.** [35] *Let  $1 \leq p < 2$ . For any<sup>15</sup> orthonormal family  $\{\phi_n\}$  of functions in  $L^2$  of any measure space, and for any sequence  $c_n \in \ell^p$ , the series  $\sum_n c_n \phi_n(x)$  converges for almost every  $x$ .*

<sup>15</sup>The version stated in [63], Theorem (10.1) of chapter XIII, is a refinement by Paley of Menshov's original theorem. It, like the original, requires uniform boundedness of  $\{\phi_n\}$ . It applies to the same class of coefficients  $c$ , but uses a different scale of function spaces and thus its conclusion involves boundedness of an associated maximal operator in a different norm than we obtain.

The first result was obtained in various versions in separate papers by Menshov, Paley, and Zygmund. The former result continues to hold for  $p = 2$  and then is essentially a restatement of Carleson's almost everywhere convergence theorem.<sup>16</sup>

Better known is a closely related theorem of Menshov. If  $\sum_n |c_n|^2 \log(n) < \infty$ , then  $\sum_n c_n \phi_n$  converges almost everywhere<sup>17</sup>; in particular, if  $\int |\hat{f}(\xi)|^2 \log(2 + |\xi|) d\xi < \infty$  then  $(2\pi)^{-1} \int_{-N}^N e^{ix\xi} \hat{f}(\xi) d\xi$  converges almost everywhere to  $f(x)$ , as  $N \rightarrow \infty$ . See [63], chapter XIII.

Theorem 9 is due to Menshov [35], and is false for  $p = 2$ . To fit it into our framework, regard  $c \mapsto \sum_n \phi_n$  as a map from  $L^p(\mathbb{Z})$  to  $L^2$ . This map is bounded for all  $1 \leq p \leq 2$ . Partial summation is integration over  $(-\infty, N]$  for some  $N$ ; these sets are nested; so Theorem 6 implies boundedness of the maximal partial sum operator, and hence almost everywhere convergence.

Our third application is to the variants arising in our generalized eigenfunction analysis. We need a preliminary lemma. We say that  $V \rightarrow 0$  in  $L^1_{\text{loc}}$  if  $\int_{|y-x|<1} |V(y)| dy \rightarrow 0$  as  $|x| \rightarrow \infty$ . As usual,  $\phi(x, \lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V$ .

**Lemma 10.** *Suppose that  $V \rightarrow 0$  in  $L^1_{\text{loc}}(\mathbb{R})$ . Then for any compact subset  $\Lambda$  of  $\mathbb{R} \setminus \{0\}$ , the mapping*

$$(52) \quad f \mapsto \int_0^\infty f(y) e^{i\phi(y, \lambda)} dy$$

maps  $L^p(\mathbb{R})$  boundedly to  $L^q(\Lambda, d\lambda)$  for all  $1 \leq p \leq 2$ , where  $q = p' = p/(p-1)$ .

For  $L^2$  this is proved by dualizing, then integrating by parts. Since the  $L^1 \mapsto L^\infty$  estimate is trivial, the general conclusion follows by interpolation. By combining Theorem 6 with the lemma, we deduce a variant of the Hausdorff-Young inequality.

**Corollary 11.** *For all  $1 \leq p < 2$ , the sublinear operator*

$$(53) \quad f \mapsto \sup_{x \in \mathbb{R}} \left| \int_0^x e^{i\phi(y, \lambda)} f(y) dy \right|$$

maps  $L^p(\mathbb{R})$  boundedly to  $L^{p'}(\Lambda)$ , for every compact subset  $\Lambda$  of  $\mathbb{R} \setminus \{0\}$ .

## 9. MULTILINEAR OPERATORS AND MAXIMAL VARIANTS

A multilinear variant of Theorem 6 is as follows. Let  $T_j : L^p(\mathbb{R}, dx) \mapsto L^q(\Lambda, d\lambda)$  be bounded linear operators with locally integrable distribution kernels  $K_j(\lambda, x)$ . Define

$$(54) \quad \mathcal{M}_n^*(f_1, f_2, \dots, f_n)(\lambda) = \sup_{y \leq y' \in \mathbb{R}} \left| \int \cdots \int_{y \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq y'} \prod_{i=1}^n (K_i(\lambda, x_i) f_i(x_i) dx_i) \right|.$$

If the factors in the integrand are all nonnegative, then this is dominated by the corresponding integral over  $[y, y']^n$ , thus by a simple product  $\prod T_j(f_j \cdot \chi_{[y, y']})$ . The whole difficulty for us is that our integrals are oscillatory, and taking absolute values renders them hopelessly divergent.

**Theorem 12.** [11] *Suppose that  $p < q$ . Then for every  $n \geq 1$ ,  $(f_1, \dots, f_n) \mapsto \mathcal{M}_n^*(f_1, \dots, f_n)$  maps  $\otimes^n L^p(\mathbb{R})$  boundedly to  $L^{q/n}(\Lambda)$ , with operator norm  $\leq B^n \prod_{j=1}^n \|T_j\|_{p, q}$ .*

<sup>16</sup>Carleson's proof does not seem to yield the strong type  $L^2$  estimate.

<sup>17</sup>Our main results have similar extensions to the case where  $(\log |x|)^c f(x) \in L^2$  for a certain constant  $c$ .

Here  $B$  is a finite universal constant. The exponent  $q/n$  is natural; the product mapping  $(f_1, \dots, f_n) \mapsto \prod_j T_j(f_j)$  maps into  $L^{q/n}$  by Hölder's inequality, and we don't expect the iterated integrals to do better. This result is stated in [11] only for  $q \geq 2$ , but that assumption can be eliminated by replacing  $r = 2$  by  $r = q$  in the definition (51) of the auxiliary functional  $G$ .

Our next variant demonstrates a substantial improvement, in the special case when all the functions  $f_i$  are taken to be the same.

**Theorem 13.** [11] *Suppose that  $p < q$  and that  $2 \leq q$ . Then for every  $n \geq 1$  and every  $f \in L^p(\mathbb{R})$ ,*

$$(55) \quad \|\mathcal{M}_n^*(f, f, \dots, f)\|_{L^{q/n}(\Lambda)} \leq \frac{B^n \|T\|_{p,q}^n \|f\|_{L^p}^n}{\sqrt{n!}}.$$

Our applications require a slight generalization, which follows from the same proof. Namely, a factor  $1/\sqrt{n!}$  is still gained, if both the functions  $f_j$  and the operators  $T_j$  are drawn from sets of cardinality  $\leq k$ , independent of  $n$ . The constant  $B$  then depends on  $k, p, q$ , but not on any other quantities. The right-hand side of the conclusion should of course be modified by replacing powers of norms by products.

This bound improves that of the preceding theorem by the factor  $1/\sqrt{n!}$ . No such factor arises in Theorem 12; modulo the factor of  $B^n$ , the bound stated cannot be improved. It is easy to see why there might be some improvement in this “diagonal” case:

$$\int_{y \leq t_1 \leq \dots \leq t_n \leq y'} \prod K(\lambda, t_i) f(t_i) dt_i \equiv \left[ \int_y^{y'} K(\lambda, t) f(t) dt \right]^n / n!.$$

In our application, however,  $K_i(\lambda, t)$  will essentially be  $\exp(\pm 2i\lambda t)$ , with the  $\pm$  signs alternating. Then there is no obvious majorization of the left-hand side of the preceding inequality by the right.<sup>18</sup>

I believe that the second theorem remains valid without the assumption  $q \geq 2$ , with  $(n!)^{-1/2}$  replaced by an appropriate modified power, but have not worked out all the details of the proof. The version stated here is easier than the general case, and is precisely what is most relevant for our applications.

The factor  $1/\sqrt{n!}$  plays a twofold role in our analysis. Firstly, it is used to deduce convergence of the “Taylor series” (47), for almost every  $\lambda$ . Secondly, it leads to a bound for the generalized eigenfunctions:

**Proposition 14.** *Let  $1 < p < 2$ , and let  $V \in L^1 + L^p(\mathbb{R})$ . Denote by  $u(x, \lambda)$  a generalized eigenfunction for  $H_0 + V$ , normalized so that  $(u(0), u'(0))$  equals either  $(1, 0)$  or  $(0, 1)$ . Then for any compact interval  $\Lambda \subset \mathbb{R} \setminus \{0\}$ ,*

$$(56) \quad \int_{\Lambda} \log \sup_{x \in \mathbb{R}} (1 + |u(x, \lambda)|) d\lambda < \infty$$

<sup>18</sup>One could try to explain the factor of  $\sqrt{n!}$  by assuming  $n$  to be even, and letting the symmetric group  $S_{n/2}$  act on the coordinates  $(t_1, \dots, t_n)$  by permuting them in blocks of two, grouping  $t_{2j-1}$  with  $t_{2j}$  for each  $j$ . Alternatively, one could argue for a larger denominator  $n!$  by considering the action of  $S_{n/2} \times S_{n/2}$  defined by letting one factor permute  $\{t_{2j-1}\}$ , and the other permute  $\{t_{2j}\}$ . There are two objections, besides the fact that these two heuristics yield different conclusions. Firstly, neither argument is rigorous, since the region of integration is not a fundamental domain for the action of either group on the full space  $\mathbb{R}^n$  of coordinates, and since cancellation plays an essential role here. Secondly, the conclusion holds with the same factor  $\sqrt{n!}$ , when  $K(\lambda, t_j) f(t_j)$  is replaced by  $K_j(\lambda, t_j) f_j(t_j)$ , with the kernels  $K_j$  and functions  $f_j$  drawn from finite sets; in that more general case the symmetry is broken.

This bound has the same general form as the inequality deduced from the trace identity (37), except that the supremum over  $x$  is inside the integral. The factor  $1/\sqrt{n!}$  turns out to be exactly what we need in order to obtain local integrability of the first power of the logarithm.

Nonetheless, it is perhaps worth understanding that our main conclusions, almost sure boundedness and WKB asymptotics for generalized eigenfunctions and presence of absolutely continuous spectrum everywhere in  $\mathbb{R}^+$ , could be deduced instead without the improved numerical factor, roughly as follows: Fix a compact interval  $\Lambda$ . If  $V$  has sufficiently small norm in  $L^p + L^1$ , then the factor of  $\|V\|^n$  on the right-hand side more than compensates for  $B^n$ . Combining this with the pointwise bound of Lemma 15 below (in the weaker form without the factor  $1/\sqrt{n!}$ ), one can deduce convergence and uniform boundedness for a subset of  $\Lambda$  having positive Lebesgue measure; moreover, the measure of the set where convergence and boundedness are not obtained approaches zero as  $\|V\|$  does. Since the norm of the restriction to  $[x, \infty)$  of  $V$  tends to zero as  $x \rightarrow +\infty$ , the exceptional set of energies has measure zero.

There is one flaw in this scheme: the bounds for  $\mathcal{M}_n^*$  are in  $L^{q/n}$ , and  $q/n \rightarrow 0$ , so no triangle inequality is available to sum the infinite series. This can be dealt with in two ways. The first way is to apply Chebyshev's inequality to obtain bounds for the distribution functions of  $\mathcal{M}_n^*$ , then to show almost everywhere finiteness of the sum by a bare hands computation; the factor of  $1/\sqrt{n!}$  is essential here. The second, and preferable, route, which could be used without these favorable numerical factors, is to exploit stronger pointwise versions of the above two theorems, which we now discuss. We state only the analogue of Theorem 13.

Suppose we are given a collection of intervals  $E_j^m \subset \mathbb{R}$  satisfying all the properties listed in §8. We call such a collection of sets a martingale structure. Define the functionals

$$(57) \quad \tilde{\mathfrak{g}}(f) = \sum_{m=0}^{\infty} \left( \sum_{j=1}^{2^m} \left| \int f \cdot \chi_j^m \right|^2 \right)^{1/2},$$

$$(58) \quad \mathfrak{g}(f) = \sum_{m=0}^{\infty} m \left( \sum_{j=1}^{2^m} \left| \int f \cdot \chi_j^m \right|^2 \right)^{1/2},$$

These operators depend on  $\{E_j^m\}$ . They are essentially linear operations, being norms in Banach spaces like  $\ell^1(\ell^2)$  of a linear operator applied to  $f$ . But in our final application, the martingale structure will itself depend on  $f$ , so they will become rather nonlinear.

Consider integrals

$$(59) \quad M_n(f)(y, y') = \int \cdots \int_{y \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq n'} \prod_{k=1}^n f(t_k) dt_k$$

$$(60) \quad M_n^*(f) = \sup_{y, y' \in \mathbb{R}} |M_n(f)(y, y')|.$$

**Lemma 15.** *There exists a finite constant  $B$  such that for any martingale structure, any  $n$ , and any  $f \in L^1$ ,*

$$(61) \quad |M_n(y, y')| \leq B^n \tilde{\mathfrak{g}}(f)^n / \sqrt{n!}$$

$$(62) \quad |M_n^*(y, y')| \leq B^n \mathfrak{g}(f)^n / \sqrt{n!}.$$

From this lemma there follows a stronger form of Theorem 13. Let  $f \in L^p(\mathbb{R})$  be given. Let  $\{E_j^m\}$  be a martingale structure, constructed so as to be compatible with  $f$  in the sense that all  $f_j^m = f \cdot \chi_j^m$  satisfy  $\|f_j^m\|_p^p = 2^{-m}\|f\|_p^p$ . Let  $K(\lambda, x)$  be the kernel function associated to a linear operator  $T$ . Define

$$(63) \quad G(f)(\lambda) = \sum_{m=0}^{\infty} m \left( \sum_{j=1}^{2^m} |T(f_j^m)(\lambda)|^2 \right)^{1/2} .$$

**Corollary 16.** [11] *There exists a constant  $B < \infty$  such that for any  $f, T, n, \lambda$ ,*

$$(64) \quad \mathcal{M}_n^*(f, \dots, f)(\lambda) \leq \frac{B^n G(f)(\lambda)^n}{\sqrt{n!}} .$$

For our application to generalized eigenfunctions, this corollary expresses a sort of conspiracy; heuristically it says that the terms of the ‘‘Taylor’’ series tend to be simultaneously all good or simultaneously all bad, in the weak sense that a single functional controls them all.

This implies Theorem 13 by

**Lemma 17.** *Suppose that  $p < q$  and  $2 \leq q$ . Then there exists  $C < \infty$  such that for any linear operator  $T$  bounded from  $L^p$  to  $L^q$ , for any  $f \in L^p$ ,*

$$(65) \quad \|G(f)\|_q \leq C \|T\|_{p,q} \cdot \|f\|_p$$

Here  $\|T\|_{p,q}$  denotes the operator norm. It is essential that  $G(f)$  be defined via a martingale structure compatible with  $f$ , in the sense described above. This lemma is a simple consequence of the triangle inequality, as in §8.

An immediate application of the corollary and the ‘‘Taylor series’’ representation of generalized eigenfunctions is the formal estimate

$$(66) \quad \sup_{x \in \mathbb{R}} |u(x, \lambda)| \leq C \exp(CG(V)(\lambda)^2),$$

obtained by majorizing the sum of the series by  $\sum_{n=0}^{\infty} B^n G(V)(\lambda)^n / \sqrt{n!}$ . Here the relevant operator  $T$  has kernel  $K(\lambda, x) = \exp(i\lambda x - i(2\lambda)^{-1} \int_0^x V)$ . From (66) we conclude that  $\log \sup_x |u(x, \lambda)|$  is locally integrable.

To conclude this section, we outline the proof of Lemma 15. To majorize  $M_n(f)$  (The analysis of  $M_n^*$  requires a small additional step, which we omit here.) we first replace  $y, y'$  by  $-\infty, +\infty$  respectively, and note the recursion

$$(67) \quad |M_n(f)| \leq |M_n(f_1^1)| + \left| \int_{E_2^1} f \cdot M_{n-1}(f_1^1) \right. \\ \left. + \sum_{j=2}^{n-2} |M_{n-j}(f_1^1)| \cdot |M_j(f_2^1)| + \left| \int_{E_1^1} f \cdot M_{n-1}(f_2^1) + |M_n(f_2^1)| \right| .$$

This is obtained by decomposing the region  $t_1 \leq \dots \leq t_n$  of integration into subregions, depending on which subset of the variables  $t_j$  belong to  $E_1^1$ , and which belong to  $E_2^1$ . Each case gives rise to one term in (67).

The next step rests on a variant of the binomial identity.

**Lemma 18.** [11] *There exists  $\gamma \in \mathbb{R}^+$  such that the numbers  $c_k$  defined by*

$$(68) \quad \beta_k = k^{-k/2} k^{-\gamma} \quad \text{for all } k \geq 2$$

satisfy for every  $k \geq 6$  the inequalities

$$(69) \quad y^k + \sum_{j=2}^{k-2} \frac{\beta_j \beta_{k-j}}{\beta_k} x^j y^{k-j} + x^k \leq (x^2 + y^2)^{k/2} \quad \text{for all } x, y \geq 0.$$

The ratios  $\beta_j \beta_{k-j} / \beta_k$  behave roughly binomial coefficients  $\binom{k/2}{j/2}$ . The only role of the factor  $k^{-\gamma}$  and assumption  $k \geq 6$  is to make the proof work. Because the lemma is to be used inside a recursive argument, it is essential that the right-hand side of the inequality be exactly  $(x^2 + y^2)^{1/2}$ , rather than a constant multiple.

The proof of Lemma 18 uses Cauchy-Schwarz and term-by-term comparison with the binomial series for  $(x^2 + y^2)^{k/2}$  (taking into account that our series has twice as many terms) in the case where  $k$  is even, with appropriate modifications in the odd case.

To deduce the desired majorization for  $M_n(f)$ , we combine Lemma 18 with (67), and argue by induction on the generation number  $m = 1, 2, 3, \dots$ . Thus  $M_n(f_j^m)$  can be expressed in terms of  $\{M_k(f_i^{m+1}) : k < n, i \leq 2^k\}$ . The terms  $|\int_{E_2^1} f| \cdot M_{n-1}(f_1^1)$  and  $|\int_{E_1^1} f| \cdot M_{n-1}(f_2^1)$  cannot be handled in this way, essentially because  $1+x$  cannot be dominated by  $(1+Cx^2)^{1/2}$  for small  $x$ , so an extra step is required to incorporate them. See [11].

A final step is needed to handle the supremum over  $y, y'$ ; it is similar to the argument for the linear case, Theorem 6.

Lemma 18 is the source of the factor  $1/\sqrt{n!}$ . It is a lemma about nonnegative numbers; the square root does not come about through any orthogonality.

In the above discussion we have omitted one aspect of the problem. The validity of WKB-type asymptotics is a kind of almost-everywhere convergence problem; one wants  $\exp(-i\phi(x, \lambda))u(x, \lambda) \rightarrow 1$  as  $x \rightarrow +\infty$ , for almost every  $\lambda$ . The usual strategy for proving such a result is to first prove a maximal function inequality in some  $L^q$  norm, then to observe the (usually obvious) fact that the convergence holds (usually in a rather strong sense) for some appropriate dense class of functions. The almost everywhere convergence follows by combining these.

Because we have multilinear operators rather than linear ones, this last step is a bit more complicated. One must compare  $T_m(V, V, \dots, V)$  with  $T_m(W, W, \dots, W)$  where  $W$  has compact support, and  $V - W$  has small  $L^p$  norm. This is of course done in part by analyzing expressions  $T_m(V, V, \dots, V, W, \dots, W)$ . For details see [10].

**Remark.** The method applies to more general multiple integrals with variables which are partially, rather than linearly, ordered, such as

$$\int_{\Omega} K(\lambda, t)K(\lambda, s_1)K(\lambda, s_2)f_0(t)f_1(s_1)f_2(s_2) dt ds_1 ds_2$$

where  $\Omega = \{(t, s_1, s_2) : t \leq s_1 \text{ and } t < s_2\}$ . Such expressions, with a branching factor of 2 at each level, arose in our analysis [9] of the power-decaying case  $V = O(|x|^{-r})$ , because we used a different expansion for the generalized eigenfunctions.

**Remark.** If  $V = O(|x|^{-r})$  for some  $r > 1/2$  then the issue of summation of an infinite series can be avoided altogether; see [9].

**Remark.** Our theory admits a limited generalization to higher dimensions.<sup>19</sup> Consider a bounded linear operator  $T : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n)$ , with distribution-kernel  $K(x, y)$ ; we assume

<sup>19</sup>I am grateful to Alan McIntosh for posing this question during the IPAM OIDE conference.

for simplicity that  $K \in L^1_{\text{loc}}$ . Write  $y \gtrsim z$  to mean that  $y_j \geq z_j$  for all  $1 \leq j \leq n$ , where  $y = (y_1, \dots, y_n)$  and likewise for  $z$ . Consider the associated maximal operators

$$(70) \quad T^* f(x) = \sup_{z \in \mathbb{R}^n} \left| \int_{y \gtrsim z} K(x, y) f(y) dy \right|.$$

Then  $T^*$  is likewise bounded from  $L^p$  to  $L^q$ , provided that  $q > p$ .

In spirit, this generalization bears the same relation to the one-dimensional case that the Marcinkiewicz multiplier theorem, and the strong maximal function of Jessen, Marcinkiewicz, and Zygmund, bear to their one-dimensional counterparts: it is controlled by an iteration of one-dimensional maximal operators. For simplicity, consider the representative case  $n = 2$ . The operator

$$(71) \quad \mathcal{T}^*(f)(x) = \sup_{z_2} \left| \int_{y_2 \geq z_2} K(x, y) f(y) dy \right|$$

is bounded, from  $L^p$  to  $L^q$ , by the one-dimensional theory; it suffices to view  $T$  as an operator from  $L^p(\mathbb{R}^1, B_1)$  to  $L^q(\mathbb{R}^1, B_2)$  where the Banach spaces  $B_i$  are respectively  $L^p(\mathbb{R}^1)$ ,  $L^q(\mathbb{R}^1)$ . Then  $\mathcal{T}^*$  is merely sublinear, rather than linear, but Theorem 6 applies equally well to sublinear operators, with the same proof as outlined above. We have

$$(72) \quad T^* f(x) \leq \sup_{z_1} \left| \int_{y_1 \geq z_1} K(x, y) \chi_{y_2 > z_2} f(y) dy \right|,$$

so that the proof of Theorem 6 reduces matters to the boundedness of  $\mathcal{T}^*$ .  $\square$

## 10. PERTURBATIONS OF STARK OPERATORS

A single electron in a uniform external electrical field (independent of space and time) is modeled quantum mechanically by the Stark Hamiltonian  $H(u) = -u'' - xu$ , the factor  $x$  representing the electrical potential. We consider perturbations

$$(73) \quad H_q(u) = -u'' - xu + qu,$$

where  $q$  represents some perturbing electrical potential. Physical intuition suggests, and earlier results in the literature confirm, that weaker hypotheses on  $q$  suffice to guarantee the presence of absolutely continuous spectrum than are needed without the background field; the force exerted by the field tends to push everything off to infinity, making it more difficult for bound states to exist. The following theorems refine various earlier results, which required faster decay or more smoothness of the perturbation.

For convenience we assume always that  $q$  is uniformly in  $L^1_{\text{loc}}$  as  $x \rightarrow -\infty$ ; much weaker hypotheses would suffice there because the external potential  $-x$  is so large.

**Theorem 19.** [13] *Consider a Stark operator  $H_q$  on  $\mathbb{R}^1$ . Assume that the potential  $q(x)$  admits a decomposition  $q = q_1 + q_2$ , where both  $q_1(x^2)$  and  $x^{-1}q'_2(x^2)$  belong to  $(L^1 + L^p)(\mathbb{R}, dx)$  for some  $1 < p < 2$ . Assume further that there exists  $\zeta < 1$  such that  $|q_2(x)| \leq \zeta|x|$  for sufficiently large  $|x|$ . Then for almost every energy  $E \in \mathbb{R}$  there exists a generalized eigenfunction  $u_+(x, E)$  satisfying  $H_q u_+ = E u_+$ , with asymptotic behavior*

$$(74) \quad u_+(x, E) = (x - q_2(x) + E)^{-1/4} e^{i\phi(x, E)} (1 + o(1))$$

as  $x \rightarrow +\infty$ , where

$$\phi(x, E) = \int_0^x \left[ \sqrt{t - q_2(t) + E} - \frac{q_1(t)}{2\sqrt{x - q_2(t) + E}} \right] dt$$

An essential support for the absolutely continuous spectrum of  $H_q$  is the entire line  $\mathbb{R}$ .

**Corollary 20.** *If  $q$  is Hölder continuous of order  $\alpha > 1/2$ , or if  $q(x) = O(|x|^{-\delta})$  for some  $\delta > 1/4$ , then  $\mathbb{R}$  is an essential support for the absolutely continuous spectrum of  $H_q$ . For almost every  $E \in \mathbb{R}$ , all generalized eigenfunctions satisfy  $u(x) = O(|x|^{-1/4})$  and  $u'(x) = O(|x|^{+1/4})$  as  $x \rightarrow +\infty$ .*

The corollary is deduced from the theorem by verifying that any function Hölder continuous of order  $> 1/2$  can be decomposed as a sum of two functions satisfying the hypotheses of Theorem 19. The endpoint case  $p = 2$  of the theorem remains open, but otherwise the result is rather sharp:

**Theorem 21.** [13] *There exists a potential  $q$  which is  $O(|x|^{-1/4})$  and is also Hölder continuous of order  $1/2$ , for which the spectrum of  $H_q$  is purely singular.*

What is actually shown is that the spectrum is almost surely purely singular, for a certain family of random potentials satisfying both these restrictions. The analysis is based closely on a similar result of Kiselev, Last, and Simon [30].

The method of proof of Theorem 19 is in outline the same as that for perturbations of the vacuum. We convert to a first-order system, and diagonalize it modulo small errors. Then we reformulate as an integral equation and iterate to obtain an expansion of the generalized eigenfunctions in WKB phase-modified power series in  $q$ . After making the change of variables  $x \mapsto \sqrt{x}$  for  $x \gg 1$ , we invoke the multilinear maximal operator machinery.

We will not give the relevant formulae in detail. A caricature for the “linear” term in the “Taylor” expansion for the generalized eigenfunctions is

$$(75) \quad \int_{x^{1/2}}^{\infty} e^{i\lambda s + is^3} q(s^2) ds;$$

the higher-order multilinear operators may be similarly caricatured. Numerous simplifications have been made to arrive here. From (75) one sees the relevance of the hypothesis  $q(x^2) \in L^p(dx)$ . This also indicates why hypotheses such as Hölder continuity, or  $x^{-1}q'(x^2) \in L^p$ , are relevant: integration by parts allows one to exploit the term  $s^3$  in the exponent, for large  $s$ , to substantial advantage.

Theorem 21 is a straightforward adaptation of the analysis by Kiselev, Last, and Simon of  $-\partial_x^2 + V_\omega(x)$ , where  $V_\omega$  is defined by (9). For the Stark case, we modify the perturbing potentials, as follows. Fix  $f \in C_0^\infty((0, 1))$ , not identically zero, and let  $a_n(\omega)$  be independent, identically distributed random variables with uniform distribution in  $[0, 2\pi]$ . Define

$$(76) \quad q_\omega(x) = \sum_{n=1}^{\infty} n^{-1/2} f(\sqrt{x} - n) \sin\left(\frac{4}{3}x^{\frac{3}{2}} + a_n(\omega)\right).$$

Then [13] for almost every  $\omega$ , the spectrum of the corresponding perturbed Stark operator  $-\partial_x^2 - x + q_\omega$  is purely singular on the whole real line.

## 11. SLOWLY VARYING AND POWER-DECAYING POTENTIALS

The Fourier transform has the following properties. (i) If  $\partial_x^k f \in L^p$  for some  $1 \leq p \leq 2$ , then  $\hat{f}$  is almost everywhere finite. (ii) If  $\hat{f}, \hat{g}$  are both almost everywhere finite, then so is  $\widehat{f+g}$ .

We regard the mapping  $V \mapsto u(x, \lambda)$ , from the potential to the unique generalized eigenfunction with appropriate asymptotics at  $+\infty$ , as a nonlinear variant of the Fourier transform. Thus it is natural to ask whether basic properties of the ordinary Fourier transform are shared. The above two properties are of interest in idealized quantum mechanics;

for instance, a potential could easily arise as the sum of contributions from different types of effects, so we would like to handle sums of potentials. This is potentially troublesome in a nonlinear situation, if different arguments are required for different classes of potentials.

Throughout this section, we assume the following conditions. Let  $n \geq 0$  be a non-negative integer, and let  $p \in [1, 2)$  be an exponent. Let  $V$  be a measurable, real-valued function defined on the real line  $\mathbb{R}$ . We assume<sup>20</sup> that  $V \rightarrow 0$  in  $L^1_{\text{loc}}$  at  $\pm\infty$ , that is, that  $\int_{|y-x| \leq 1} |V| \rightarrow 0$  as  $x \rightarrow \pm\infty$ . Suppose that  $V$  admits a decomposition  $V = V_0 + V_n$  where<sup>21</sup>  $V_0 \in L^p + L^1$ ,  $V_n$  is continuous and tends to zero, and  $d^n V_n / dx^n \in L^p + L^1$ , in the sense of distributions. Note that under these hypotheses,  $V$  can tend to zero arbitrarily slowly in  $L^1_{\text{loc}}$ . We continue to write  $H = H_0 + V = -\partial_x^2 + V$ .

A classical theorem of Weidmann [61] asserts that if  $V = V_0 + V_1$  with  $V_0$  and  $dV_1/dx \in L^1(\mathbb{R})$ , and if  $V_1(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , then  $\mathbb{R}^+$  is an essential support of the absolutely continuous spectrum (moreover, at positive energies,  $H_0 + V$  is unitarily equivalent to  $H_0$ ). For higher derivatives,  $L^1$  results were obtained by Behncke [2] and Stolz [57]. We extend this to  $L^p$ ,  $p < 2$ , with a (necessarily) weaker form of the conclusion.

**Theorem 22.** [12] *Under the above hypotheses, for almost every  $\lambda \in \mathbb{R}$ , each solution of the generalized eigenfunction equation  $Hu = \lambda^2 u$  is a bounded function of  $x \in \mathbb{R}$ . An essential support for the absolutely continuous spectrum of  $H$  is  $\mathbb{R}^+$ .*

Moreover, suitably generalized WKB asymptotics are valid for almost every  $\lambda$ ; there exists a solution satisfying  $u(x, \lambda) = \exp(i\Psi(x, \lambda)) + o(1)$  as  $x \rightarrow +\infty$ , where  $\Psi$  (which depends in a much more complicated way on  $n, V$ ) has bounded imaginary part and may in principle be computed in terms of  $V$  by a recipe described below.

A result of Molchanov, Novitskii and Vainberg [36], in the spirit of the work of Deift and Killip based on trace identities, asserts existence of absolutely continuous spectrum for potentials satisfying  $d^n V / dx^n \in L^2$ , under the supplementary hypothesis that  $V \in L^{n+1}$ .

For potentials with more rapidly decaying derivatives, our conclusions can be strengthened. Define  $p' = p/(p-1)$ .

**Theorem 23.** [12] *Suppose that  $n \geq 0$ ,  $1 \leq p \leq 2$ ,  $0 < \gamma$ , and  $\gamma p' \leq 1$ . Let  $V$  be a measurable, real-valued function defined on  $\mathbb{R}$ . Suppose that  $V = V_0 + V_n$  where  $V_n$  is bounded and continuous, and both  $(1+|x|)^\gamma V_0$  and  $(1+|x|)^\gamma d^n V_n / dx^n$  belong to  $L^p + L^1$ . Then every solution of  $Hu = Eu$  is a bounded function of  $x \in \mathbb{R}$ , for all  $E > 0$ , except for a set of values of  $E$  having Hausdorff dimension  $\leq 1 - \gamma p'$ .*

For  $n = 0$  with stronger power decay hypotheses  $V(x) = O(|x|^{-r})$  for  $r > 1/2$ , this result is due to Remling [45].

Again, generalized WKB asymptotics hold on the complement of the lower-dimensional exceptional set. In the case  $n = 0$ , Remling and Kriecherbauer [33, 46] have constructed examples demonstrating that WKB asymptotics can indeed fail to hold on sets of the stated dimension. The question of behavior for the exceptional energies is of considerable interest, firstly because it determines to what extent these energies contribute to the spectrum, and in particular whether singular continuous spectrum can arise, and secondly because it is connected with asymptotic completeness for the associated time-dependent Schrödinger evolution; see §12 below.

<sup>20</sup>For the generalization to the case where  $V$  need not tend in any sense to zero, but is merely uniformly in  $L^1_{\text{loc}}$ , see [12].

<sup>21</sup>This includes any potential decomposable as  $\sum_{k=0}^n V_k$  where  $d^k V_k / dx^k \in (L^p + L^1)(\mathbb{R})$  for each  $k \geq 0$ , and where  $\sum_{k=1}^n V_k \rightarrow 0$  in  $L^1_{\text{loc}}$ .

To see how to control the Hausdorff dimension of the exceptional set, let us first see how to do so for the Fourier transform itself.

**Observation 24.** *If  $1 \leq p \leq 2$ ,  $\gamma > 0$ , and  $(1 + |x|)^\gamma f(x) \in L^p(\mathbb{R})$  then*

$$(77) \quad \lim_{x \rightarrow +\infty} \int_0^x e^{-i\lambda y} f(y) dy$$

*exists for all  $\lambda \in \mathbb{R} \setminus S$ , where  $S$  has Hausdorff dimension  $\leq 1 - \gamma p'$ .*

For the proof, let  $B$  be the Banach space consisting of all doubly indexed sequences  $\{a_{m,j}\}$  for which  $\sum_{m \geq 0} m (\sum_j |a_{m,j}|^2)^{1/2}$  is finite. Consider the linear operator mapping  $f$  to  $\{\int_{\mathbb{R}} e^{-i\lambda y} f_j^m(y) dy\}$ , a function  $g(f)(\lambda)$  taking values in  $B$ . The hypothesis  $|x|^\gamma f \in L^p$  implies that  $\hat{f}$  belongs to the Sobolev space of functions possessing  $\gamma$  derivatives in  $L^{p'}$ , and as is well known, a simple potential-theoretic argument shows that such a Sobolev function is well defined outside a set of the desired dimension. The same reasoning, coupled with the analysis outlined in earlier sections of these notes, shows that  $g(f)$  is (on compact subsets of  $\mathbb{R} \setminus \{0\}$ ) a  $B$ -valued function in this same Sobolev space. The potential-theoretic argument then applies as before.

This analysis can be adapted to the ‘‘Taylor series’’ representation of generalized eigenfunctions, by following the arguments outlined for the case  $\gamma = 0$  in preceding sections of these notes.

The principal change needed to adapt our machinery to the slowly varying case is a substantially modified WKB approximation. To analyze the Fourier transform of a function possessing some smoothness, one typically integrates by parts; in our formalism, this integration by parts is implicitly incorporated when the modified WKB approximation is inserted into the analysis of the first-order system  $y' = \begin{pmatrix} 0 & 1 \\ V - \lambda^2 & 0 \end{pmatrix} y$ .

To begin, we decompose<sup>22</sup>  $V = W + \tilde{V}$  via a partition of unity on the Fourier transform side;  $W$  is the low-frequency part of  $V$  in the sense that  $\hat{V}(\xi) \equiv \hat{W}(\xi)$  in a neighborhood of  $\xi = 0$ , and  $\tilde{W}$  has compact support.

In step 2, we seek an approximation  $\exp(i\Psi(x, \lambda))$  to a generalized eigenfunction  $u(x, \lambda)$ . Replacing  $V$  by  $W$  and  $\Psi'$  by an unknown  $\Phi$ , the equation  $(-\partial_x^2 + W - \lambda^2) \exp(i \int \Phi) \approx 0$  becomes the eikonal equation

$$(78) \quad \Phi^2 - i\Phi' + W - \lambda^2 \approx 0 .$$

We solve the recursion

$$(79) \quad \Phi_{k+1} = \sqrt{\lambda^2 - W + i\Phi'_k}$$

by induction on  $k$ , with  $\Phi_0 \equiv \lambda$ . Derivatives of  $W$  up to order  $k - 1$  appear in  $\Phi_k$ ; this is why we are led to decompose  $V = W + \tilde{V}$  with  $W \in C^\infty$ , and to omit the nonsmooth part,  $\tilde{V}$ , in this WKB part of the analysis. Since  $W \rightarrow 0$  as  $|x| \rightarrow \infty$ , together with all its derivatives, there is no difficulty in carrying out this recursion for all sufficiently large  $x$ .

The error

$$(80) \quad E_k = \Phi_k^2 - i\Phi'_k + W - \lambda^2$$

<sup>22</sup>Observe that for the WKB approximation  $\phi(x, \lambda) = \lambda x - (2\lambda)^{-1} \int_0^x V$ , replacing  $V$  by  $W$  makes no essential difference since  $\int_0^x \tilde{V} \rightarrow 0$  as  $x \rightarrow \infty$ .

satisfies the useful recursion

$$(81) \quad E_{k+1} = i \frac{d}{dx} \frac{E_k}{\Phi_k + \sqrt{\Phi_k^2 - E_k}},$$

so that needed properties of  $\Phi_k, E_k$  can be deduced by induction. Set  $\Phi = \Phi_n$  where  $n$  is the index in the hypothesis of the theorem, and set

$$(82) \quad \Psi(x, \lambda) = \int_0^x \left( \Phi_n - \frac{\tilde{V} - E_n}{2 \operatorname{Re} \Phi_n} \right) (y, \lambda) dy.$$

Finally, set

$$(83) \quad \mathcal{E}(x, \lambda) = -E_n - \tilde{V}.$$

The recursions for  $\Phi_k, E_k$ , along with standard Sobolev embedding estimates, can be used to show that  $\mathcal{E}(x, \lambda) \in L^1 + L^p(\mathbb{R}, dx)$ , and the same holds for all its derivatives with respect to  $\lambda$ .

In step 3, to solve the first-order system  $y' = \begin{pmatrix} 0 & 1 \\ V - \lambda^2 & 0 \end{pmatrix} y$ , we set<sup>23</sup>

$$(84) \quad y = \begin{pmatrix} e^{i\Psi} & e^{-i\tilde{\Psi}} \\ i\Phi e^{i\Psi} & -i\tilde{\Phi} e^{-i\tilde{\Psi}} \end{pmatrix} z;$$

$\Psi$  is not in general real-valued. Under our hypotheses, it can be shown to have bounded real part, which need not tend to a limit as  $x \rightarrow +\infty$  and hence is not negligible in the asymptotics. The upshot of all these algebraic manipulations is a simplified first-order evolution:

$$(85) \quad z' = \begin{pmatrix} 0 & \frac{i\bar{\mathcal{E}}}{2 \operatorname{Re} \Phi'} e^{-i\psi} \\ -\frac{i\mathcal{E}}{2 \operatorname{Re} \Phi'} e^{i\psi} & 0 \end{pmatrix} z.$$

where

$$(86) \quad \psi = 2 \operatorname{Re} \Psi.$$

This is like the system in (41), with the potential replaced by  $-i\mathcal{E}/2 \operatorname{Re} \Phi'$ . The denominator  $\operatorname{Re} \Phi'$  turns out to be relatively harmless; the main new complication is that the “effective potential”  $\mathcal{E}/\operatorname{Re} \Phi$  depends strongly, though smoothly, on  $\lambda$ . The method applies, after relatively minor modifications.

Here is a typical result concerning energy-dependent potentials.<sup>24</sup> Its proof, rather than the result itself, is what is required to complete the proof of Theorem 22.

**Theorem 25.** [10] *Let  $J$  be a compact subinterval of  $\mathbb{R} \setminus \{0\}$ . Suppose that  $p < 2$ , that  $W(x, \lambda)$  is real-valued, and that*

$$\partial^j W(x, \lambda) / \partial \lambda^j \in L^p(\mathbb{R})$$

*uniformly in  $\lambda \in J$  for  $j = 0, 1$ . Suppose further that the derivatives  $\partial^j W(x, \lambda) / \partial \lambda^j \rightarrow 0$  as  $|x| \rightarrow \infty$ , uniformly in  $\lambda \in J$ , for  $j = 2, 3$ . Then for almost every  $\lambda \in J$ , there exist linearly independent, bounded solutions  $u_{\pm}(x, \lambda)$  of*

$$-u'' + W(x, \lambda)u = \lambda^2 u$$

<sup>23</sup>The presence of  $\Phi$  in the second row of the coefficient matrix, where one might expect to see instead  $\Psi'$ , is not a typo.

<sup>24</sup>One could try to eliminate the WKB phase correction factor,  $\exp(-i(2\lambda)^{-1} \int_0^x V)$  in the case  $n = 0$ , by incorporating it into the potential as well, but that would not work because its derivative with respect to  $\lambda$  is in general unbounded.

with WKB asymptotic behavior as  $x \rightarrow +\infty$ .

The number of derivatives hypothesized here may not be optimal.

The main idea in the proof is quite standard. To estimate for example  $\int_{\mathbb{R}} e^{-i\lambda x} V(x, \lambda) dx$  for  $\lambda$  in some compact interval, consider more generally  $g(\lambda, \rho) = \int_{\mathbb{R}} e^{-i\lambda x} V(x, \rho) dx$ . If  $\partial^k V / \partial \lambda^k \in L^p$  for  $k = 0, 1$ , for some  $1 \leq p \leq 2$ , then  $\partial^k g / \partial \rho^k \in L^{p'}(d\lambda)$ , uniformly in  $\rho$  in an interval. The Sobolev embedding theorem then controls the restriction of  $g$  to  $\rho = \lambda$ .

## 12. WAVE OPERATORS AND SCATTERING

If we aspire to at least a caricature of quantum mechanics, we ought to study the Schrödinger group  $\exp(itH)$ , and in particular, its long term dynamics, including scattering. To the IPAM workshop audience for whom these notes are intended, the question of Strichartz estimates may leap to mind, but caution is required. For the class of potentials under discussion, the point spectrum can be nonempty, and indeed dense in  $\mathbb{R}^+$ . Bound states evolve without dispersion, so no Strichartz estimates can hold for arbitrary initial data.

A second difficulty is the distinction between short and long range forces. A scattered particle cannot be expected to behave asymptotically like a free particle, if the forces acting on it are of sufficiently long range, as is the case for a slowly decaying potential  $V$ , even one of “symbol type” whose derivatives decay faster than  $V$  itself. This effect is already seen in the phase correction in our WKB asymptotics:  $u_\lambda(x) \sim \exp(i\lambda x - i(2\lambda)^{-1} \int_0^x V)$ . The correction term indicates heuristically that particles with energy  $\lambda^2$  propagate with velocities slightly different from  $\pm\lambda$ .

In principle, good control over all generalized eigenfunctions should lead to control of  $\exp(itH)$ , by the spectral calculus. In this section, I explain some preliminary and very recent results in this direction, in which work is still underway.

**Definition.** The *wave operators*  $\Omega^\pm$  associated to a perturbed Hamiltonian<sup>25</sup>  $H = H_0 + V$  are

$$(87) \quad \Omega^\pm f = \lim_{t \rightarrow \mp\infty} e^{itH} e^{-itH_0} f,$$

where the limit is taken in the strong operator topology, provided it exists.

**Theorem 26.** [14] *Let  $H = H_0 + V$  on  $L^2(\mathbb{R}^+)$  with Dirichlet boundary condition at the origin. Suppose that  $V \in L^p + L^1$  for some  $1 < p < 2$ . Suppose further that*

$$(88) \quad \lim_{x \rightarrow +\infty} \int_0^x V(y) dy \quad \text{exists.}$$

*Then for each  $f \in L^2(\mathbb{R}^+)$ , the two limits (87) exist in  $L^2$  norm as  $t \rightarrow \mp\infty$ . Moreover,  $\Omega^\pm$  are bijective isometries between  $\mathcal{H} = L^2(\mathbb{R}^+)$  and  $\mathcal{H}_{ac}$ .*

Here  $\mathcal{H}_{ac}$  denotes the maximal subspace of  $\mathcal{H}$  on which  $H$  has purely absolutely continuous spectrum.

$V$  is not assumed to be nonnegative, so the supplementary hypothesis on existence of  $\int_0^\infty V$  is not a restriction on the size of  $V$ . Heuristically, a hypothesis with this flavor is needed for particles to have any chance of being asymptotically free.<sup>26</sup>

<sup>25</sup>Either on  $\mathcal{H} = L^2(\mathbb{R})$ , or on  $\mathcal{H} = L^2(\mathbb{R}^+)$  with a suitable boundary condition at the origin.

<sup>26</sup>For potentials satisfying appropriate symbol-type hypotheses (that is, the first one or few derivatives decay faster than the potentials themselves), Hörmander [24] has constructed modified wave operators which take into account long-range effects. We believe that we have obtained a similar generalization of Theorem 26, but this work is still in a preliminary phase.

Another way to state the conclusion is this: for each  $f \in \mathcal{H}_{\text{ac}}$  there exist  $g_{\pm} \in \mathcal{H}$  such that

$$(89) \quad \|e^{itH}f - e^{itH_0}g_{\pm}\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty.$$

The mappings  $f \mapsto g_{\pm}$  thus defined are isometric bijections from  $\mathcal{H}_{\text{ac}}$  to  $\mathcal{H} = L^2(\mathbb{R}^+)$ ;  $f = \Omega^{\pm}g_{\pm}$ . The *scattering operator*  $(\Omega^+)^{-1} \circ \Omega^-$  mapping  $g_-$  to  $g_+$  is a unitary isomorphism of  $L^2$ . Its physical interpretation is that any incoming particle that is asymptotically free at  $t = -\infty$  will be asymptotically free at  $t = +\infty$ , and  $(\Omega^+)^{-1} \circ \Omega^-$  describes the transition from pre-interaction to post-interaction asymptotics.

To go further, note that as a consequence of the theory developed earlier in these notes, we know that under the hypothesis (88), for almost every  $\lambda \in \mathbb{R}$  there exists a unique pair  $(u_{\lambda}, \omega(\lambda))$ , where  $u_{\lambda}$  is a generalized eigenfunction with spectral parameter  $\lambda^2$  satisfying the boundary condition  $u_{\lambda}(0) = 0$ , and  $\omega(\lambda) \in \mathbb{R}$ , with asymptotic behavior

$$(90) \quad u_{\lambda}(x) = \sin(\phi(x, \lambda)) + o(1) \quad \text{as } x \rightarrow +\infty$$

where

$$(91) \quad \phi(x, \lambda) = \lambda x + \omega(\lambda) + (2\lambda)^{-1} \int_x^{\infty} V.$$

**Theorem 27.** [14] *Under the hypotheses of Theorem 26,  $(\Omega^+)^{-1} \circ \Omega^-$  is the unitary “Fourier multiplier” operator on  $L^2(\mathbb{R}^+)$  mapping  $\sin(\lambda x)$  to  $e^{2i\omega(\lambda)} \sin(\lambda x)$  for every  $\lambda \in \mathbb{R}$ .*

There are five steps in the analysis.

- Identification of the projection operator from  $\mathcal{H}$  to  $\mathcal{H}_{\text{ac}}$ . The operator from  $L^2(\mathbb{R}, d\lambda)$  to  $L^2([0, \infty))$  defined formally by

$$(92) \quad S(F)(x) = c_0 \int_0^{\infty} F(\lambda) u_{\lambda}(x) d\lambda$$

is an isometry onto  $\mathcal{H}_{\text{ac}}$ .  $S \circ S^*$  is the orthogonal projection from  $\mathcal{H}$  onto  $\mathcal{H}_{\text{ac}}$ . Thus  $\exp(itH)$  maps  $\int F(\lambda) u_{\lambda} d\lambda$  to  $\int F(\lambda) e^{it\lambda^2} u_{\lambda} d\lambda$ .

- A very easy lemma showing for  $f \in \mathcal{H}_{\text{ac}}$ ,  $e^{itH}f \rightarrow 0$  in  $L^2$  norm on any compact subset of  $\mathbb{R}^+$ , as  $|t| \rightarrow \infty$ .
- Decompose  $e^{itH}f$  as the sum of two terms. In the main term,

$$(93) \quad \int_{\mathbb{R}} e^{it\lambda^2} F(\lambda) u_{\lambda}(x) d\lambda \text{ is replaced by } \int_{\mathbb{R}} e^{it\lambda^2} F(\lambda) \sin(\phi(x, \lambda)) d\lambda.$$

The difference is shown to tend to zero in  $L^2([R, \infty))$  as  $R \rightarrow \infty$ , uniformly in  $t \in \mathbb{R}$ , for a dense subspace of  $\mathcal{H}_{\text{ac}}$ .

Namely, we take any compact subset  $\Lambda \subset \mathbb{R}$  on which all our estimates for the generalized eigenfunctions hold *uniformly* for  $\lambda \in \Lambda$ , and consider all  $F \in L^{\infty}$  supported in  $\Lambda$ . Essentially<sup>27</sup>, this works because all our multilinear expressions of degree  $\geq 1$  involve the restriction of  $V$  to  $[R, \infty)$ ; the  $L^p$  norm of the restriction

<sup>27</sup>We have two proofs for this step. One requires a refinement of our multilinear operator machinery. Namely, it works when any one of the functions on which the multilinear operator acts (more generally, sufficiently few of them) belong to  $L^2$ , provided all the others belong to  $L^p$  for some  $p < 2$ . The alternative proof uses the theorem of Lacey and Thiele on the boundedness of the bilinear Hilbert transform  $\int_{\mathbb{R}} f(x-t)g(x+t)t^{-1}dt$  from  $L^4 \otimes L^4$  to  $L^2$ . If  $|x|^{\varepsilon}V \in L^1 + L^2$  for some  $\varepsilon > 0$ , then the bilinear Hilbert transform is not needed.

tends to zero as  $R \rightarrow \infty$ . This step requires a bit more than the full strength of the multilinear operator machinery outlined in preceding sections.<sup>28</sup>

- Another easy step replaces the phase  $\lambda x + \omega(\lambda) + (2\lambda)^{-1} \int_x^\infty V$  by  $\lambda x + \omega(\lambda)$  in the main term.
- The final step, evaluation of  $\Omega^\pm$  in terms of the function  $\omega$ , is routine:

$$(94) \quad \Omega^\pm \left( \int_0^\infty F(\lambda) \sin(\lambda x) d\lambda \right) = \int_0^\infty F(\lambda) e^{\pm i\omega(\lambda)} u_\lambda(x) d\lambda.$$

The next issue to consider is that of asymptotic completeness.

**Definition.**<sup>29</sup>  $H$  is said to be *asymptotically complete* if the ranges of the wave operators  $\Omega^\pm$  are equal to  $\mathcal{H}_{\text{continuous}} = \mathcal{H} \ominus \mathcal{H}_{\text{pp}}$ .

Of course, a necessary condition for asymptotic completeness is that the singular continuous spectrum should be empty.

The physical interpretation is that for an asymptotically complete system, all states are superpositions of bound states and scattering states, the latter being those states which are asymptotically free as  $t \rightarrow \pm\infty$ . For the class of operators  $H$  under discussion here, since the range of  $\Omega^\pm$  is equal to  $\mathcal{H}_{\text{ac}}$  for either choice of sign, asymptotic completeness is equivalent to  $\mathcal{H}_{\text{sc}} = \emptyset$ . It remains an open question whether this is true.

We have obtained analogous results for Schrödinger operators on the whole real line instead of the half-line, but the statements are slightly more complicated and are omitted here. Another case that can be treated by our methods is that of certain Dirac-type operators, which arise in the inverse scattering method for the nonlinear Schrödinger equation. The unperturbed Hamiltonian is now

$$(95) \quad H_0 y = \begin{pmatrix} -i\partial_x & 0 \\ 0 & i\partial_x \end{pmatrix} y$$

where  $y$  takes values in the space  $\mathbb{C}^2$  of column vectors, and  $\partial_x = d/dx$ . The perturbed Hamiltonian is

$$(96) \quad H = H_0 + \begin{pmatrix} 0 & V \\ \bar{V} & 0 \end{pmatrix}$$

where  $V$  is complex-valued. We assume that  $V \in L^1 + L^p(\mathbb{R})$  for some  $p < 2$ .

The theory for this equation is closely parallel to that for  $-\partial_x^2 + V$ , the main difference being a simplification: no WKB phase shift term  $(2\lambda)^{-1} \int_0^x V$  appears in the exponentials. Consequently we are able to prove [14] the existence of wave operators for arbitrary  $V \in L^1 + L^p$ , without any supplementary hypothesis. Precise statements will not be given here.

For certain classes of random  $V \in L^1 + L^p$ , it is known, by the techniques of [31],[30], that the spectrum is almost surely absolutely continuous. As a corollary we obtain asymptotic completeness for almost every operator in these classes.

### 13. THREE VARIATIONS ON A THEME OF STRICHARTZ

In this section we briefly discuss three different ways in which estimates of Strichartz type are relevant to our subject matter. The first was alluded to earlier: the linear maximal

<sup>28</sup>It works in general, without the supplementary hypothesis on the existence of the improper integral  $\int_0^\infty V$ .

<sup>29</sup>B. Simon has informed me that different authors define asymptotic completeness differently, and that many require stronger properties such as discrete point spectrum which certainly do not hold, in general, for the class of potentials under discussion in these notes.

function theory allows one to deduce one Strichartz estimate from another, an application first observed by Tao [59]. For the free Laplacian  $H_0$ , the following three inequalities are all valid for all  $f \in L^2(\mathbb{R})$ ,  $g \in L^{6/5}(\mathbb{R}^{1+1})$ :

$$(97) \quad \|e^{itH_0} f(x)\|_{L_{x,t}^6} \leq C \|f\|_{L_x^2}$$

$$(98) \quad \left\| \int_{-\infty}^{\infty} e^{i(t-t')H_0} g(t') dt' \right\|_{L_{x,t}^6} \leq C \|g\|_{L_{x,t}^{6/5}}$$

$$(99) \quad \left\| \int_0^t e^{i(t-t')H_0} g(t') dt' \right\|_{L_{x,t}^6} \leq C \|g\|_{L_{x,t}^{6/5}}.$$

In the latter two lines,  $g(t')$  denotes a function of  $x' \in \mathbb{R}$ , and  $e^{i(t-t')H_0} g(t')$  is what one gets by applying the indicated operator to that function, and evaluating at  $x$ . The first inequality implies the second, by dualizing and then composing an operator with its adjoint. The third is of interest, because the quantity whose norm appears on the left-hand side appears in Duhamel's formula.

It is in deducing (99) from (98) that Corollary 7 is useful. Regard functions of  $(x, t)$  as being functions of  $t \in \mathbb{R}$ , taking values in auxiliary Banach spaces  $L^p(\mathbb{R}, dx)$ . The left-hand side of (99), evaluated at  $t$ , is obtained by applying the operator  $Tg(t) = \int_{\mathbb{R}} \exp(i(t-t')H_0)g(t') dt'$  to  $g$  times the characteristic function of  $[0, t]$ . (98) asserts that  $T$  is bounded from the space  $L_t^p(X)$  of  $X$ -valued functions in  $L^p$  to  $L_t^q(Y)$ , where  $X, Y$  equal  $L^{6/5}(\mathbb{R}), L^6(\mathbb{R})$ , respectively, and  $p = 6/5 < q = 6$ . Thus Corollary 7, extended to Banach space-valued functions, says that (98) directly implies (99). This extension to Banach spaces follows from the same proof as in the scalar case.

A second way to bring Strichartz and Fourier restriction inequalities into the subject is to consider the following physically artificial situation. Consider a one-parameter family of potentials

$$(100) \quad V_s(x) = W(x) \cos(sx^2)$$

where  $W$  is real-valued and fixed. Let  $H_s = -\partial_x^2 + V_s$ .

**Theorem 28.** *Suppose that  $W \in L^p + L^1(\mathbb{R})$  for some  $p < 4$ . Then for almost every  $s \in \mathbb{R}$ , an essential support for the absolutely continuous spectrum of  $H_s$  is  $\mathbb{R}^+$ . For almost every pair  $(s, \lambda)$ , all generalized eigenfunctions of  $H_s$  with spectral parameter  $\lambda^2$  are bounded and have WKB asymptotic behavior.*

The basic point here is that the operator  $f \mapsto \int_{\mathbb{R}} \exp(-i\lambda x + isx^2)f(x) dx$  maps  $L^p$  to  $L^q$  for all  $p < 4$ , with  $q = q(p) > 4$ . This can be generalized to incorporate the WKB phase correction. Otherwise the analysis is essentially the same as in the proof of Theorem 2. We have not established the presence of a negative power of  $n!$  in the analogue of Theorem 13, but as explained in §9, these conclusions can be obtained without it.

One cannot expect to have the Strichartz estimate (97) with the free Laplacian replaced by  $H = H_0 + V$  for general  $V \in L^1 + L^p$ ,  $1 < p < 2$ , for two reasons. Firstly<sup>30</sup>, as already pointed out, bound states can occur, indeed the point spectrum can be dense in  $\mathbb{R}^+$ , and they destroy any such dispersion inequality. Secondly, although one could ask for such an estimate only for all  $f \in \mathcal{H}_{ac}$ , that is unlikely to hold. The problem is that our estimates are far from uniform in the spectral parameter  $\lambda$ , and are very weak; we know only that

<sup>30</sup>“The preference for *first* over *firstly* in formal enumerations is one of the harmless pedantries in which those who like oddities because they are odd are free to indulge, provided that they abstain from censuring those who do not share the liking.” H. W. Fowler [21].

$\log \sup_x |u(x, \lambda)|$  is locally integrable in  $\lambda$ . The following seems nearly the best that is likely to be true.

**Problem 1.** Suppose that  $V \in L^1 + L^p$ . Show that there exists a nonnegative function  $w$ , strictly positive almost everywhere, such that for any function  $f$  satisfying  $\int_{\mathbb{R}} |f(\lambda)|^2 w(\lambda) d\lambda < \infty$ , the function  $g(y) = \int f(\lambda) u(y, \lambda) d\lambda$  satisfies  $\exp(itH)g(x) \in L^6_{x,t}$ .

Here  $u(y, \lambda)$  denotes a generalized eigenfunction with WKB asymptotics at  $y = +\infty$ .

I believe that such a result follows by combining ingredients from our analysis of wave operators with the usual derivation of the  $L^{4-\delta}$  restriction theorem in  $\mathbb{R}^2$ ; work on this is in progress. However, at the time of writing of these notes, the proof has not been completed.

#### 14. OPEN PROBLEMS

The following are some of the principal open problems, for the one-dimensional case, related to the results discussed in these notes.

**Problem 2.** *Square integrable potentials.* Extend all results from  $L^p$ ,  $p < 2$ , to  $L^2$  (and hence to  $L^2 + L^1$ , by rather easy supplementary arguments). As is clear from the discussion, this amounts to a nonlinear extension of Carleson's theorem on almost everywhere convergence of Fourier transforms and series.

Carleson showed<sup>31</sup> that the map

$$(101) \quad f \mapsto \sup_y \left| \int_{-\infty}^y e^{ix\xi} \hat{f}(\xi) d\xi \right|$$

maps  $L^2(\mathbb{R})$  to weak<sup>32</sup>  $L^2$ . Since the Fourier transform is an invertible isometry on  $L^2$ , by setting  $f = \hat{V}$  we deduce that  $V \mapsto \sup_y \left| \int_{-\infty}^y e^{ix\xi} V(\xi) d\xi \right|$  is bounded. The first-order term in our expansion<sup>33</sup> is this, with the added complication that the phase  $x\xi$  is replaced by  $x\xi - (2x)^{-1} \int_0^\xi V(t) dt$ .

A subproblem<sup>34</sup> is to obtain estimates in  $L_y^\infty L_x^{q,\infty}$ , where  $L^{q,\infty}$  denotes the space weak  $L^q$  and where for the multilinear term of degree  $m$ ,  $q = 2/m$ ; a subsubproblem is to do so with the phase correction  $(2x)^{-1} \int V$  omitted. For  $m = 1$ , this is a consequence of Plancherel's theorem; for  $m = 2$  it boils down to Plancherel's theorem plus the weak type  $(1, 1)$  boundedness of the Hilbert transform. The first nontrivial case is  $m = 3$ ; this has recently been successfully analyzed by Muscalu, Tao, and Thiele [37].

The next problem is taken from a list of problems proposed by Simon [51].

**Problem 3.** *Existence of singular continuous spectrum.* Can there exist singular continuous spectrum, for potentials which are  $O(|x|^{-r})$  for some  $r > 1/2$ , or more generally, for potentials in  $L^2$ ? Can the spectral measure have singular components of dimension  $0 < \alpha < 1$ ?

<sup>31</sup>Subsequently extended to  $1 < p < 2$  by Hunt, with further refinements near  $L^1$  by Sjölin, an influential second proof by C. Fefferman, and recently a superb short analysis by Lacey and Thiele.

<sup>32</sup>It actually is bounded from  $L^2$  to  $L^2$ , as was shown by Rubio de Francia via a simple application of weighted norm inequalities and extrapolation.

<sup>33</sup>One should beware the perils of reductionism; we have seen that certain fundamental properties of the generalized eigenfunctions and scattering coefficients are obscured when individual terms of this multilinear expansion are examined in isolation.

<sup>34</sup>This is of interest, because suitable estimates of this weaker type suffice to imply existence of ac spectrum, as in the work of Deift and Killip.

If  $(1 + |x|)^\gamma V \in L^p$  and  $1 \leq p \leq 2$ , then we have shown that WKB asymptotics hold for all energies except an exceptional set of Hausdorff dimension  $\leq 1 - \gamma p'$  (provided this quantity is  $\geq 0$ ). On the other hand, Remling and Kriecherbauer [33] have shown that WKB asymptotics can indeed fail for a set of energies of precisely this dimension. However, in order to obtain spectrum of this dimension, according to an analogue of the criterion (20), one needs to construct sufficiently many generalized eigenfunctions with appropriate decay. Essentially, one needs

$$(102) \quad \limsup_{R \rightarrow \infty} R^{1-\alpha} \int_{|x| \leq R} |u_E(x)|^2 dx < \infty.$$

**Problem 4.** *Asymptotic completeness.* Are Schrödinger operators with  $L^p$  potentials necessarily asymptotically complete?

According to the discussion in §12, this is actually essentially the same as the preceding problem.

**Problem 5.** *Stability of dynamical systems* under time-dependent perturbations. To what extent do our results extend to more general dynamical systems?

Our asymptotic analysis may be viewed as an almost sure stability result for perturbations of a (completely integrable) dynamical system. The unperturbed system has state space  $\mathbb{C}^2 \times \Lambda$ , where  $\Lambda \subset \mathbb{R}$  is any fixed compact interval. The time evolution is given by

$$(103) \quad \frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} v \\ -\lambda^2 u \\ 0 \end{pmatrix},$$

which is simply a reformulation of the Schrödinger equation  $-u'' = \lambda^2 u$ . All orbits are periodic, with periods  $2\pi/\lambda$ . Now consider the perturbed evolution

$$(104) \quad \frac{d}{dt} \begin{pmatrix} u \\ v \\ \lambda \end{pmatrix} = \begin{pmatrix} v \\ (V(t) - \lambda^2)u \\ 0 \end{pmatrix}.$$

For  $V \in L^1$ , each trajectory of the perturbed system is asymptotic to some trajectory of the unperturbed system. For  $V \in L^1 + L^p$ ,  $1 < p < 2$ , our results imply that for almost every initial condition at  $t = 0$ , the resulting trajectory is asymptotic to some unperturbed trajectory (though in a weaker sense, with a change of clock, which takes into account the WKB phase shift).

Preliminary work in this direction is underway.

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