

A CHEAP CAFFARELLI–KOHN–NIRENBERG INEQUALITY FOR THE NAVIER–STOKES EQUATION WITH HYPER-DISSIPATION

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Abstract

We prove that for the Navier–Stokes equation with dissipation $(-\Delta)^\alpha$, where $1 < \alpha < 5/4$, and smooth initial data, the Hausdorff dimension of the singular set at time of first blow up is at most $5 - 4\alpha$. This unifies two directions from which one might approach the problem of global solvability, though it provides no direct progress on either.

1 Introduction

The Navier–Stokes equation with dissipation $(-\Delta)^\alpha$ is given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = -(-\Delta)^\alpha u, \quad (1)$$

where u is a time-dependent divergence free vector field in \mathbb{R}^3 . One sets the initial condition

$$u(x, 0) = u_0(x) \quad (2)$$

where $u_0(x) \in C_c^\infty(\mathbb{R}^3)$.

It is easy to see that classical solutions to this equation on a time interval $[0, T]$ satisfy conservation of energy, namely that

$$\|u(\cdot, T)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - \int_0^T \langle (-\Delta)^\alpha u, u \rangle.$$

The second term on the right is called the dissipation term.

Caffarelli, Kohn, and Nirenberg [CKN] showed that when $\alpha = 1$, the singular set of a generalized weak solution (a weak solution satisfying a certain generalization of conservation of energy) to the system (1),(2) has parabolic Hausdorff dimension at most 1. This could be considered a first step towards showing global strong solvability. Indeed any improvement in this upper bound on the dimension would be genuine progress towards solution of the global solvability problem.

Another well-known fact is the following. (We learned the proof which appears in the following section from Diego Cordoba. It appears to exist in the folklore and we have not succeeded in attributing it. Sinai and Mattingly [MS] recently gave a different proof.)

PROPOSITION 1.1. *If $\alpha > 5/4$, one has global strong solvability for the system (1),(2).*

Indeed Proposition 1.1 could be viewed as a first step towards the solution of the global strong solvability for $\alpha = 1$ and any improvement in the exponent $5/4$ could be viewed as genuine progress.

The purpose of this paper is to interpolate the two results so that these two paths to progress are unified. We prove

Theorem 1.2. *If T is the time of first breakdown for the system (1),(2), with $5/4 > \alpha > 1$ then the Hausdorff dimension of the singular set at time T is at most $5 - 4\alpha$.*

Heuristically, the theorem ought to be a mild generalization of Proposition 1.1. We think about the microlocal analysis of a solution u in terms of coefficients u_Q associated to cubes Q . Very roughly speaking, the coefficient u_Q should be viewed as a generalized wavelet coefficient. The proof of Proposition 1.1 goes wrong for $\alpha < 5/4$ only because of a small number of cubes with large coefficients. At any time, the set of points contained in arbitrarily small such cubes has dimension at most $5 - 4\alpha$. The difficulty lies in preventing the energy from flowing between far away cubes near the time of blow up. This is not quite so easy as it sounds because energy can flow from small cubes to large ones containing them and this is a quite non-local effect.

We were also unable to directly generalize the proof of Caffarelli, Kohn, and Nirenberg. They rely on what they call the “generalized energy inequality” which is built on an amusing property of the divergence free heat equation. Let ϕ be any compactly bump function in space and time and u a divergence free vector field, then

$$\begin{aligned} & \int \langle (\frac{\partial}{\partial t} - \Delta) u, \phi u \rangle dt \\ &= \int \left(\frac{1}{2} \frac{\partial}{\partial t} (\langle u, \phi u \rangle) + \langle \nabla u, \phi \nabla u \rangle - \left\langle \left(\frac{1}{2} \frac{\partial \phi}{\partial t} + \Delta \phi \right) u, u \right\rangle \right) dt. \end{aligned}$$

The first term represents a change in local energy. The second term represents local dissipation. The third term is an error which can be made insignificant by choosing ϕ to satisfy the backwards heat equation. It is this method of controlling the error which we are unable to generalize. The later proof of Lin [L] also uses the generalized energy inequality.

To deal with localization we combine the theory of paramultiplication, which was developed by Bony (see e.g. [B]), and Coifman and Meyer (see e.g. [CoM]), with the theory of type $(1, 1 - \epsilon)$ pseudodifferential operators

which is also described in [Ta]. The latter theory allows us to localize on cubes describing our “wavelet coefficients”. Mimicking Caffarelli–Kohn–Nirenberg’s Proposition 2 we try to prove regularity at points not contained in any squares in which too much dissipation occurs. For any such point, we derive a certain level of regularity which we refer to as critical regularity. We find a square centered at such a point on which we can apply a certain barrier estimate which guarantees arbitrary regularity on its interior. This barrier estimate can be thought of as a localized version of global existence with small data. (One can compare this estimate with results in [KT] and [Ca] in which local well-posedness is established with hypotheses slightly more relaxed than ours.) But in order for this to work, it is important that there not be too many small dissipating squares on the boundary of our square. This combinatorial issue is an ingredient which seems not to have appeared before and which restricts us to the case $\alpha > 1$.

In section 2, we describe the proof of Proposition 1.1. In section 3, we describe a way of measuring the Hausdorff dimension of sets. In section 4, we introduce a dyadic model for Navier–Stokes which illustrates the ideas of our proof. In section 5, we recall some Littlewood–Paley theory. In section 6, we describe what happens in the way of localization of dissipation and energy flow. Most of the general microlocal analysis occurs in this section. In section 7, we prove a certain minimal, what we call, critical regularity which is maintained away from a $5 - 4\alpha$ dimensional set. In section 8, we describe a frequently occurring class of squares which have no highly dissipative square on their boundary. In section 9, we apply the results of sections 7 and 8 to bootstrap the critical regularity into arbitrarily good regularity.

Remark on constants and notation. For us \langle , \rangle is always an L^2 pairing in space.

Throughout this paper the expression $A \lesssim B$ means $A \leq CB$ where C is a constant. But what is a constant? Constants may depend on T , the norms of the initial values, and on an ϵ which we keep fixed throughout this paper. They may not depend on a particular square or a particular scale which we are estimating at. (However sometimes we choose constants so that there is a particular initial scale on which constants can depend. But in that case, the point is that we make estimates depending on those constants for all scales.) Sometimes our hypotheses or our definitions have constants in them. When this is the case, our conclusions are understood to be a multi-parameter family of conclusions depending also on the constants

in our definitions. We will invoke propositions and lemmas using convenient constants in the hypotheses and we do not make any attempt to keep track of these. If we did, the paper might become unreadable.

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2 Preliminaries

We begin by recalling the proof of Proposition 1.1.

Proof of Proposition 1.1. We will use the standard notation that H^β denotes the L^2 Sobolev space over \mathbb{R}^3 with β derivatives. We will let $W^{\beta,p}$ denote the L^p Sobolev space over \mathbb{R}^3 with β derivatives.

We first recall the energy inequality obtained by pairing the equation (1) with u

$$\frac{1}{2} \frac{\partial(\|u\|_{L^2}^2)}{\partial t} = -\|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2}^2 \leq -\|u\|_{H^\alpha}^2 + \|u\|_{L^2}^2. \quad (3)$$

From this, we obtain by integrating over time (and observing that the L^2 norm is always positive), that if the solution u remains smooth up to time T , we have the estimate

$$\int_0^T \|u\|_{H^\alpha}^2 dt \lesssim (1+T). \quad (4)$$

We now pair (1) with $(-\Delta)^\beta u$ in order to estimate $\partial(\|u\|_{H^\beta}^2)/\partial t$. We obtain

$$\frac{1}{2} \frac{\partial(\|(-\Delta)^{\beta/2} u\|_{L^2}^2)}{\partial t} + \langle u \cdot \nabla u, (-\Delta)^\beta u \rangle = -\|(-\Delta)^{\frac{\alpha+\beta}{2}} u\|_{L^2}^2. \quad (5)$$

Clearly, we must estimate the nonlinear problem term

$$\langle u \cdot \nabla u, (-\Delta)^\beta u \rangle = \langle (-\Delta)^{\beta/2} (u \cdot \nabla u), (-\Delta)^{\beta/2} u \rangle.$$

Since u is divergence free, we can bound the absolute value of the above by

$$\|u\|_{W^{1,p}} \|u\|_{W^{\beta,q}} \|u\|_{H^\beta},$$

with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Now since we are assuming $\alpha > \frac{5}{4}$, we obtain by the Sobolev imbedding theorem,

$$\|u\|_{W^{1,p}} \|u\|_{W^{\beta,q}} \|u\|_{H^\beta} \lesssim \|u\|_{H^\alpha} \|u\|_{H^{\alpha+\beta}} \|u\|_{H^\beta},$$

since we must spend $> 3/2$ derivatives to get all three terms in L^2 and we have spent $2\alpha - 1$.

Applying Cauchy-Schwartz, we get immediately

$$\|u\|_{H^\alpha} \|u\|_{H^{\alpha+\beta}} \|u\|_{H^\beta} \leq \delta \|u\|_{H^{\alpha+\beta}}^2 + \frac{1}{\delta} \|u\|_{H^\alpha}^2 \|u\|_{H^\beta}^2.$$

Combining this with (5), we get

$$\frac{\partial \|u\|_{H^\beta}^2}{\partial t} \lesssim \|u\|_{H^\alpha}^2 \|u\|_{H^\beta}^2 + \|u\|_{L^2}^2.$$

In turn, combining this with (4) and with Gronwall's inequality gives global solvability. □

3 Hausdorff Dimension

The purpose of this section is to recall the definition of Hausdorff dimension and to explain a simple construction which automatically produces sets below a certain desired dimension.

Given any set $E \subset \mathbb{R}^n$, we define its d -dimensional Hausdorff measure $\mathcal{H}^d(E)$ by

$$\mathcal{H}^d(E) = \lim_{\delta \rightarrow 0} \inf_{C \in \mathcal{C}_\delta(E)} \sum_{B \in C} r(B)^d.$$

Here \mathcal{C}_δ is the set of all coverings of E by balls of radius less than or equal to δ , and if B is a ball, $r(B)$ is its radius.

We say the Hausdorff dimension of E is the infimum of the set of all d for which $\mathcal{H}^d(E)$ is 0. Note that for the purposes of the definition of dimension it does not matter what basis of objects we cover by (balls, squares, ellipsoids of a fixed multiplicity) as long as they are a regular family. (See [S].)

All of our arguments are built on the notion of discretizing at a certain scale 2^{-j} . We would like to be able to think of sets of a certain Hausdorff dimension as sets of 2^{jd} elements at scale 2^{-j} . We make this intuition precise for our purposes by the following.

LEMMA 3.1. *Let A_1, \dots, A_j, \dots be a sequence of collections of balls in \mathbb{R}^n so that each element of A_j has radius 2^{-j} . Suppose that $\#(A_j) \leq 2^{jd}$. Define*

$$E = \limsup_{j \rightarrow \infty} A_j,$$

to be the set of points in infinitely many of the $\cup_{B \in A_j} B$'s. Then the Hausdorff dimension of E is at most d .

Proof. Since the Hausdorff dimension of E is the infimum of the set of all γ for which $\mathcal{H}^\gamma(E) = 0$, it is enough to prove that $\mathcal{H}^\gamma(E) = 0$ for all $\gamma > d$.

Pick j such that $2^{-j} < \delta$. We can cover E by the $\cup_{k>j} \cup_{B \in A_k} B$.

Now

$$\mathcal{H}^\gamma(E) \leq \sum_{k>j} 2^{kd} (2^{-k})^\gamma,$$

which limits to zero as j goes to ∞ whenever $\gamma > d$. \square

4 Dyadic Heuristic

Before going into the details of the proof of Theorem 1.2, we present a heuristic to illustrate the main ideas.

We shall replace our vector valued function u by a scalar valued one. We shall represent it by a wavelet expansion. This shall be left rather abstract. We do not care which system of wavelets we use precisely so long as the size of the coefficients of a function with respect to the basis accurately reflect the regularity properties of the function. We denote our orthonormal family of wavelets $\{w_Q\}$, with w_Q the wavelet associated to the spatial dyadic cube Q of sidelength $2^{-j(Q)}$. Then u can be represented as

$$u = \sum_Q u_Q w_Q,$$

where u_Q is the wavelet coefficient associated to the cube Q .

Until the end of this section we reserve notation w_Q to denote the wavelet localized on a cube Q , while u_Q , and v_Q are reserved for wavelet coefficients associated to a cube Q .

We need a few definitions in order to make our model precise. A cube Q in \mathbb{R}^3 is called a dyadic cube if its sidelength is integer power of 2, 2^l , and the corners of the cube are on the lattice $2^l \mathbb{Z}^3$. A dyadic child of a dyadic cube Q with sidelength 2^l is any of 2^3 dyadic cubes Q' contained in cube Q and such that their sidelengths are 2^{l-1} .

We will call (Q, Q', Q'') a cascade if Q' is a dyadic child of Q , and Q'' is a dyadic child of Q' .

Now we are ready to define an operator which will mimick the behavior of nonlinear term $u \cdot \nabla u$. Note that we have

$$\|w_Q\|_{L^\infty} \sim 2^{3j(Q)/2}. \quad (6)$$

On the other hand

$$\|\nabla w_Q\|_{L^2} \sim 2^j, \quad (7)$$

since our wavelets w_Q 's are orthonormal and localized to the frequencies around 2^j .

Having in mind (6) and (7) we define a cascade down operator as follows:

$$c_d(u, v) = \sum_{cascades} 2^{3j(Q)/2} 2^{j(Q)} u_Q v_{Q'} w_{Q''} .$$

Similarly we define a cascade up operator as

$$c_u(u, v) = \sum_{cascades} 2^{3j(Q)/2} 2^{j(Q)} u_{Q''} v_{Q'} w_Q .$$

(Note that the choice of definition of the cascade down operator is somewhat arbitrary. We are happy to take any bilinear operator with few coefficients which shifts energy to high frequency only a few scales at a time. Once cascade down is chosen, cascade up must be one of its adjoints.)

Also let us define

$$c(u, v) = c_d(u, v) - c_u(u, v) .$$

Obviously,

$$\langle c_d(u, u), u \rangle = \langle c_u(u, u), u \rangle ,$$

which implies

$$\langle c(u, u), u \rangle = 0 . \tag{8}$$

Having defined operator $c(u, v)$ we can speak about a dyadic model for the Euler and the Navier–Stokes equations. More precisely by a dyadic version of the Euler equation we mean

$$\frac{du}{dt} + c(u, u) = 0 .$$

For the remainder of this section, we define Laplacian as $\Delta(w_Q) = 2^{2j} w_Q$. Now we introduce a dyadic version of the Navier–Stokes equation as

$$\frac{du}{dt} + c(u, u) + \Delta u = 0 .$$

Also we could speak about a dyadic version of the Navier–Stokes equation with hyper-dissipation by which we mean

$$\frac{du}{dt} + c(u, u) + (\Delta)^\alpha u = 0 . \tag{9}$$

A simple consequence of (8) is conservation of energy for all three equations, which is an important feature of these equations preserved in the dyadic model described here.

Now we consider a dyadic model for the Navier–Stokes equation with hyper-dissipation (9). We would like to estimate Hausdorff dimension of the set of singular points at the first time of blow up, T . The nonlinear term $c(u, u)$ on scale j looks like $2^{5j/2} u_Q^2$ (if we imagine for a moment that all neighboring cubes have coefficients of roughly the same size), while dissipation term gives decay like $2^{2\alpha j} u_Q$. This means that, as long as

$u_Q < 2^{-\frac{j}{2}(5-4\alpha)}$, the growth of u_Q is under control. But let us check what happens if $u_Q > 2^{-\frac{j}{2}(5-4\alpha)}$. (This idea is made more precise by Lemma 9.1. The reader is encouraged to work out its simpler dyadic analogue). We call this bound on the coefficients critical regularity and this is what we shall prove away from a $5 - 4\alpha$ dimensional set.

We can rewrite equation (9) in terms of wavelets coefficients as follows:

$$\frac{du_Q}{dt} = \sum_{(Q', Q'')} c_{(Q, Q', Q'')} 2^{5j(Q)/2} u_{Q'} u_{Q''} - 2^{2\alpha j(Q)} u_Q, \tag{10}$$

where $c_{(Q, Q', Q'')} = -1$ if (Q, Q', Q'') is a cascade, where $c_{(Q, Q', Q'')} = 32$ if (Q', Q'', Q) is a cascade and vanishes otherwise.

Having assumed $u_Q \gtrsim 2^{-\frac{j}{2}(5-4\alpha)}$ for some time t and assuming that at the initial time $t = 0$ it is much smaller, by the smoothness assumption on the initial condition, we integrate (10) in time on the interval $[0, T]$ and obtain for one of the choices of (Q', Q'') giving a non-vanishing coefficient,

$$2^{\frac{5j}{2}} \int_0^T |u_{Q'} u_{Q''}| dt \gtrsim 2^{-\frac{j}{2}(5-4\alpha)},$$

which by Cauchy–Schwartz implies

$$2^{\frac{5j}{2}} \left(\int_0^T u_Q^2 dt \right)^{1/2} \left(\int_0^T u_{Q''}^2 dt \right)^{1/2} \gtrsim 2^{-\frac{j}{2}(5-4\alpha)}.$$

The last expression can be rewritten as

$$2^{2j\alpha} \left(\int_0^T u_Q^2 dt \right)^{1/2} \left(\int_0^T u_{Q''}^2 dt \right)^{1/2} \gtrsim 2^{-j(5-4\alpha)}.$$

This can happen if either

$$2^{2j\alpha} \int_0^T u_Q^2 dt \gtrsim 2^{-j(5-4\alpha)}, \tag{11}$$

or

$$2^{2j\alpha} \int_0^T u_{Q''}^2 dt \gtrsim 2^{-j(5-4\alpha)}. \tag{12}$$

However having in mind conservation of energy we have

$$2^{2j\alpha} \int_0^T \sum_{Q \text{ at scale } j} u_Q^2 dt \lesssim 1.$$

Thus we conclude that (11) or (12) could happen in at most $\lesssim 2^{j(5-4\alpha)}$ cubes Q . Now we invoke Lemma 3.1 and conclude that the Hausdorff dimension of the set of points of the equation (9) at which critical regularity fails is at most $5 - 4\alpha$.

In the remainder of the paper, we shall prove the same result for the actual Navier–Stokes with hyper-dissipation and we shall prove that this

critical regularity implies stronger regularity at points away from the bad set.

We would like to remark that the dyadic model seems quite interesting in that it possesses all the features of Navier–Stokes which we shall use in this paper. But moreover the nonlinear term possesses a built-in dispersive feature – namely that cascade down passes energy equally to all the squares Q'' which are children of Q' and grandchildren of Q . The model is easier to understand than Navier–Stokes, however, and recently Fedja Nazarov informed us of an ingenious proof of global solvability for it which uses its tendency to disperse energy amongst different scales.

5 Littlewood–Paley Theory and Pseudodifferential Operators

We shall use a standard Littlewood–Paley partition of frequency space. That is, we shall have Fourier multipliers P_j (on $L^2(\mathbb{R}^3)$) so that their symbols $p_j(\xi)$ are smooth and supported in $\frac{2}{3}2^j < |\xi| < 3(2^j)$, so that $p_j(\xi) = p_0(2^{-j}\xi)$ and so that

$$\sum_j p_j(\xi) = 1.$$

We define $\tilde{P}_j = \sum_{k=-2}^2 P_{j+k}$. Notice that $\tilde{P}_j P_j = P_j$, since \tilde{P}_j is the sum of all Littlewood–Paley projections the support of whose symbols intersects the support of $p_j(\xi)$. We can analogously define the symbols $\tilde{p}_j = \sum_{k=-2}^2 p_{j+k}$.

The Littlewood–Paley operators satisfy the inequality

PROPOSITION 5.1. $\|P_j f\|_{L^\infty} \lesssim 2^{3j/2} \|P_j f\|_{L^2}.$

Proof. Let $\phi_j = \tilde{p}_j$. Then (since we are working in \mathbb{R}^3) we have

$$\phi_j(x) = 2^{3j} \phi_0(2^j x).$$

Now $\tilde{P}_j f = \phi_j * f$ so by Young's inequality, we have

$$\|P_j f\|_{L^\infty} = \|\tilde{P}_j P_j f\|_{L^\infty} \leq \|\phi_j\|_{L^2} \|P_j f\|_{L^2},$$

which is finite since ϕ_j is a Schwartz function. But by the relation between ϕ_j and ϕ_0 , we have

$$\|\phi_j\|_{L^2} = 2^{3j/2} \|\phi_0\|_{L^2},$$

which proves the proposition. \square

The idea of Littlewood–Paley theory is that $P_j f$ is like a combination of wavelets supported on cubes of length 2^{-j} and that the previous proposition is sharp only when a large proportion of the L^2 energy of $P_j f$ is concentrated in a single one of these cubes. Hence one is led to try to localize $P_j f$ in space on such cubes and this is marginally consistent with the Heisenberg uncertainty principle. However we have some room for error in our dimension estimate since it is a closed condition. Therefore we fix an $\epsilon > 0$ and never try to localize better than within $2^{-j(1-\epsilon)}$.

Our localization is nearly perfect if we neglect negligible quantities. For the remainder of the paper, whenever we are at scale j , we neglect quantities of size $\lesssim 2^{-100j}$ since they will not affect our estimates. Similarly we neglect operators whose norms are smaller than 2^{-100j} provided they will only be applied to functions whose norms are $\lesssim 1$. The choice of 100 is arbitrary. It is large enough so that it does not affect the L^∞ norm of u . But in fact our techniques for showing quantities are negligible rely on Schwartz function properties and could give arbitrary exponent with loss in the constant. The current approach works well principally because of conservation of energy, since the L^2 norm of u is certainly $\lesssim 1$.

For any cube Q with sidelength greater than $2^{-j(1-2\epsilon)}$, we define a bump function $\phi_{Q,j}$ which is positive, is bounded above by 1, equals 1 on Q and is 0 outside of $(1 + 2^{-j\epsilon})Q$. Further, we require for each multiindex α that there is a constant C_α independent of Q so that

$$|D^\alpha \phi_{Q,j}| \leq C_\alpha 2^{|\alpha|j(1-\epsilon)}. \quad (13)$$

We say that any bump function which satisfies the estimates (13) is of type j .

The map $\phi_{Q,j} P_j$ acts much like a projection, and we shall treat $\|\phi_{Q,j} P_j f\|_{L^2}$ as if it were a wavelet coefficient. When we deal with a cube Q of sidelength $2^{-j(1-2\epsilon)}$, we shall define $j(Q) = j$ and we shall denote $\|\phi_{Q,j(Q)} P_j f\|_{L^2}$ by f_Q . Further if $j(Q) = j$, we say that Q is at level j .

PROPOSITION 5.2. *Given f with $\|f\|_{L^2} \lesssim 1$, and ϕ a bump function of type j we have that the quantity*

$$\|\phi P_j f - \tilde{P}_j \phi P_j f\|_{L^2}$$

is negligible.

Proof. Let us define $\phi = \phi_1 + \phi_2$, where $\hat{\phi}_2(\xi) = \chi_{\{|\xi| > \frac{1}{100} 2^j\}} \hat{\phi}(\xi)$. We have

$$\phi P_j f - \tilde{P}_j \phi P_j f = \phi_1 P_j f - \tilde{P}_j \phi_1 P_j f + \phi_2 P_j f - \tilde{P}_j \phi_2 P_j f.$$

By our estimates on the derivatives of ϕ , we can get

$$\|\phi_2(\xi)\|_{L^\infty} = \text{negligible},$$

while, because of the Fourier transform supports,

$$\tilde{P}_j \phi_1 P_j f = \phi_1 P_j f.$$

Thus the proposition is proved. \square

Combining Propositions 5.1 and 5.2, we get the following extremely useful lemma.

LEMMA 5.3. *Let ϕ be a bump function of type j . Then*

$$\|\phi P_j f\|_{L^\infty} \lesssim 2^{\frac{3j}{2}} \|\phi P_j f\|_{L^2} + \text{negligible}.$$

Proof. We neglect negligible terms. Then we have $\phi P_j f = \tilde{P}_j \phi P_j f$, by Proposition 5.2. We estimate $\|\tilde{P}_j \phi P_j f\|_{L^\infty}$ by Proposition 5.1. \square

PROPOSITION 5.4. *For any cube Q , we have that for any f with $\|f\|_{H^\beta} \lesssim 1$ with $\beta < 10$ that*

$$(1 - \phi_{(1+2^{-\epsilon j(Q)/2})Q, j(Q)}) P_j \phi_{Q, j(Q)} f,$$

is negligible.

Proof. $(1 - \phi_{(1+2^{-\epsilon j(Q)/2})Q, j(Q)}) P_j \phi_{Q, j(Q)}$, is a composition of type $(1, 1 - \epsilon)$ pseudodifferential operators whose symbols have disjoint support. Thus it is smoothing. \square

Notice that the above proposition really says that we can move bump functions across Littlewood–Paley projections as long as the bump functions proliferate and increase in support. In other words, the proposition can be rewritten as

$$\phi_{(1+2^{-\epsilon j(Q)/2})Q, j(Q)} P_j \phi_{Q, j(Q)} = P_j \phi_{Q, j(Q)} + \text{negligible}.$$

Similarly, this can be used to remove inconvenient bump functions, provided there is a smaller bump function already present in the expression.

PROPOSITION 5.5. *Let \mathcal{C} be a covering of a set E by cubes of sidelength $2^{-j(1-2\epsilon)}$ for some fixed j . Then for any $f \in L^2$, we have*

$$\|\chi_E P_j f\|_{L^2}^2 \leq \sum_{Q \in \mathcal{C}} f_Q^2.$$

Proof. We simply observe that

$$\sum_{Q \in \mathcal{C}} f_Q^2 = \int \left(\sum_{Q \in \mathcal{C}} \phi_{Q, j}^2 \right) |P_j f|^2 \geq \int \chi_E |P_j f|^2. \quad \square$$

Essentially what we have done in this section is to use approximations to projections which are uniformly pseudodifferential operators of type $(1, 1 - \epsilon)$. The negligible interaction of distant squares can be just as well derived from the asymptotic formula for composition of such operators. The composition of two $(1, 1 - \epsilon)$ operators whose symbols have disjoint support is infinitely smoothing and hence negligible.

6 Dissipation, Trichotomy, and Forbidden Squares

In this section, with u a solution to the Navier–Stokes with hyperdissipation, we write down estimates involving the growth of the u_Q 's. This is the only section in which we work directly with the equation. By the end of this section, we will have reduced our problem to a combinatorial one about ordinary differential equations on graphs.

Recall we have

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = -(-\Delta)^\alpha u, \tag{14}$$

together with

$$\nabla \cdot u = 0. \tag{15}$$

In light of (15), we can rewrite (14) as

$$\frac{\partial u}{\partial t} + T(u \cdot \nabla u) = -(-\Delta)^\alpha u, \tag{16}$$

where T is the projection into divergence free vector fields. The operator T is a singular integral operator and is also a Fourier multiplier. We pick a cube Q of side length $2^{-j(1-2\epsilon)}$ and compute the L^2 pairing of the equation with $P_j \phi_{Q,j}^2 P_j u$. We obtain the energy estimate

$$\frac{1}{2} \frac{d}{dt} u_Q^2 = \langle -T(u \cdot \nabla u), P_j \phi_{Q,j}^2 P_j u \rangle - \langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle. \tag{17}$$

We shall estimate the two terms on the right-hand side of (17) separately.

As before we use the notation

$$\tilde{P}_j = \sum_{k=-2}^2 P_{j+k}.$$

This has the advantage that

$$P_j = \tilde{P}_j P_j.$$

We make some definitions. We define for each cube Q , the set $\mathcal{N}^1(Q)$, the nuclear family of Q to be a union of sets A_Q, B_Q, C_Q, D_Q, E_Q , where A_Q, B_Q, C_Q, D_Q, E_Q are covers of $\tilde{Q} = (1 + 2^{-\epsilon j/4})Q$ by fewer than 1024 cubes each at levels respectively $j - 2, j - 1, j, j + 1$, and $j + 2$. We define recursively $\mathcal{N}^l(Q)$ to be the union of all $\mathcal{N}^1(Q')$ for all $Q' \in \mathcal{N}^{l-1}(Q)$. Thus in particular, we have

$$\#(\mathcal{N}^l(Q)) \leq 2^{13l},$$

since $5(2^{10}) < 2^{13}$.

PROPOSITION 6.1. *Let Q be a cube and $j = j(Q)$. Then*

$$\begin{aligned} &\langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle \\ &\geq c 2^{2\alpha j} u_Q^2 - C 2^{(2\alpha-2\epsilon)j} \sum_{Q' \in \mathcal{N}^1(Q)} u_{Q'}^2 - \text{negligible}. \end{aligned} \tag{18}$$

Proof. Note that

$$\begin{aligned} \langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle &= \langle (-\Delta)^\alpha \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle \\ &\quad + \langle [\phi_{Q,j} P_j, (-\Delta)^\alpha] u, \phi_{Q,j} P_j u \rangle = X + Y. \end{aligned}$$

Note that by Proposition 5.2, we have

$$X \gtrsim 2^{2\alpha j} u_Q^2 - \text{negligible}.$$

To estimate Y , we observe that $[\phi_{Q,j} P_j, (-\Delta)^\alpha]$ is of order $2\alpha - \epsilon$ since $\phi_{Q,j} P_j$ is of type $(1, 1 - \epsilon)$. Further by Proposition 5.2, we have

$$[\phi_{Q,j} P_j, (-\Delta)^\alpha] \tilde{P}_j - [\phi_{Q,j} P_j, (-\Delta)^\alpha] = \text{negligible}.$$

Even further, applying Proposition 5.4, we get

$$Y = \langle [\phi_{Q,j} P_j, (-\Delta)^\alpha] \phi_{\tilde{Q},j} \tilde{P}_j u, \phi_{Q,j} P_j u \rangle + \text{negligible}.$$

Now applying the mapping properties of operators of order $2\alpha - \epsilon$ and Cauchy-Schwartz, we get for any number K ,

$$|Y| \lesssim \frac{1}{K} 2^{2\alpha j} u_Q^2 + K 2^{(2\alpha - 2\epsilon)j} \|\phi_{\tilde{Q},j} \tilde{P}_j u\|_{L^2}^2.$$

Now applying Proposition 5.5, we get the desired result. \square

Let us define for every l ,

$$u_{\mathcal{N}^l(Q)}^2 = \sum_{Q' \in \mathcal{N}^l(Q)} u_{Q'}^2.$$

Then we have

COROLLARY 6.2. *Let Q be a cube and $j = j(Q)$. Then for any l ,*

$$\begin{aligned} \sum_{Q' \in \mathcal{N}^l(Q)} \langle (-\Delta)^\alpha u, P_j \phi_{Q',j}^2 P_j u \rangle \\ \geq c 2^{2\alpha j} u_{\mathcal{N}^l(Q)}^2 - C 2^{(2\alpha - 2\epsilon)j} u_{\mathcal{N}^{l+1}(Q)}^2 - \text{negligible}. \end{aligned} \quad (19)$$

Proof. Simply sum Proposition 6.1 over $\mathcal{N}^l(Q)$. \square

We want to use Proposition 6.1 to fight against the growth of u_Q . Thus the term $2^{(2\alpha - 2\epsilon)j} u_Q^2$ would appear to be a serious nuisance. However we are able to show

LEMMA 6.3. *For any interval of time $J \subset [0, T]$ and any cube Q we may find an $l < 400/\epsilon$ so that*

$$\int_J u_{\mathcal{N}^l(Q)}^2 dt + \text{negligible} \gtrsim 2^{-\epsilon j} \int_J u_{\mathcal{N}^{l+1}(Q)}^2 dt. \quad (20)$$

Proof. By conservation of energy, for every $l \leq 400/\epsilon$ and every t we have,

$$u_{\mathcal{N}^{l+1}(Q)}^2 \lesssim 1.$$

Since our constants can depend on T this means

$$\int_J u_{\mathcal{N}^{l+1}(Q)}^2 dt \lesssim 1.$$

Now suppose the lemma is false. Applying the opposite of (20), for $l = 1 \dots 400/\epsilon$, we get

$$\int_J u_{\mathcal{N}^2(Q)}^2 dt + negligible \lesssim 2^{-399j} = negligible,$$

which implies that (20) holds for $l = 1$. □

We will be able to apply Lemma 6.3 together with Corollary 6.2 to show that for any cube and any interval in time, there is an l th iterated nuclear family for small l which is undergoing dissipation.

Now we turn our attention to the term

$$G_Q = \langle -T(u \cdot \nabla u), P_j \phi_{Q,j}^2 P_j u \rangle.$$

We rewrite it as

$$G_Q = \langle -\phi_{Q,j} \tilde{P}_j T P_j (u \cdot \nabla u), \phi_{Q,j} P_j u \rangle.$$

Now we use the “trichotomy”. We can write

$$P_j(u \cdot \nabla u) = H_{j,th} + H_{j,hl} + H_{j,hh} + H_{loc},$$

where the low-high part is given by

$$H_{j,th} = \sum_{k < j - \frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_j \nabla u),$$

the high-low part is given by

$$H_{j,hl} = \sum_{k < j - \frac{1000}{\epsilon}} P_j((\tilde{P}_j u) \cdot P_k \nabla u),$$

the high-high part is given

$$H_{j,hh} = \sum_{k > j + \frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{k > j + \frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u)$$

(technically that was the end of the trichotomy) and the local part is given by

$$H_{loc} = \sum_{j - \frac{1000}{\epsilon} < k < j + \frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{j - \frac{1000}{\epsilon} < k < j + \frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u).$$

Now we break up G_Q in the obvious way:

$$G_Q = G_{Q,th} + G_{Q,hl} + G_{Q,hh} + G_{Q,loc},$$

where

$$\begin{aligned} G_{Q,th} &= \langle -\phi_{Q,j} \tilde{P}_j T H_{j,th}, \phi_{Q,j} P_j u \rangle, \\ G_{Q,hl} &= \langle -\phi_{Q,j} \tilde{P}_j T H_{j,hl}, \phi_{Q,j} P_j u \rangle, \end{aligned}$$

$$G_{Q, hh} = \langle -\phi_{Q,j} \tilde{P}_j T H_{j, hh} \phi_{Q,j} P_j u \rangle ,$$

$$G_{Q, loc} = \langle -\phi_{Q,j} \tilde{P}_j T H_{j, loc} \phi_{Q,j} P_j u \rangle .$$

Now fixing Q and j we would like to estimate each of $G_{Q, lh}$, $G_{Q, hl}$, $G_{Q, hh}$, and $G_{Q, loc}$.

We take a moment to be careful about how we localize. For any $k < j$, define $Q_k = 2^{(j-k)(1-2\epsilon)}Q$. For any $k \geq j$ (including j) define $Q_k = (1 + 2^{-\epsilon k/2})Q$. We retain this notation only for this section and the following one.

Then we have

LEMMA 6.4. For any $\delta > 0$

$$|G_{Q, lh}| + |G_{Q, hl}| \lesssim \sum_{k=\delta j}^{j-\frac{1000}{\epsilon}} 2^{\frac{3k}{2}+j} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + 2^{j(1+\frac{3\delta}{2})} u_{\mathcal{N}^1(Q)} u_Q + negligible .$$

Proof. We consider for $k < j - \frac{1000}{\epsilon}$, the relevant expression

$$G_{Q, lh, k} = \langle -\phi_{Q,j} P_j T (P_k u \tilde{P}_j \nabla u), \phi_{Q,j} P_j u \rangle .$$

We divide this into two cases which are $k < \delta j$ and $k \geq \delta j$. In the second case, we use the idea of Proposition 5.4 (we just have to use a higher degree of smoothing than is used in that proposition) to observe

$G_{Q, lh, k} = \langle -\phi_{Q,j} P_j T ((\phi_{Q_k, k} P_k) u \cdot \nabla (\phi_{Q_j, j} \tilde{P}_j u)), \phi_{Q,j} P_j u \rangle + negligible .$
 (This is because ∇ acts on $\phi_{Q_j, j}$ only where it has a negligible effect on the whole quantity by Proposition 5.4.) Now we simply observe by Proposition 5.3 that

$$\|\phi_{Q_k, k} P_k u\|_{L^\infty} \lesssim 2^{3k/2} u_{Q_k} + negligible ,$$

and by Proposition 5.4, the proof of Proposition 5.2 (to control the action of ∇) and Proposition 5.5 that

$$\|\nabla \phi_{Q_j, j} \tilde{P}_j u\|_{L^2} \lesssim 2^j u_{\mathcal{N}^1(Q)} + negligible .$$

On the other hand for the first case, there is no point in localizing to Q_k because k is too small and so the error is too big. Thus in in this case, we simply estimate

$$\|P_k u\|_{L^\infty} \lesssim 2^{3\delta j/2} u_{Q_k} \lesssim 2^{3\delta j/2} .$$

This last application of conservation of energy accounts for the peculiar homogeneity of our result. We are pleased to control the energy in very large scales by 1.

Summing these estimates gives the desired bound for $G_{Q, lh}$. The bound for $G_{Q, hl}$ proceeds likewise (and gives a better estimate since the derivative falls on the level k term.) □

LEMMA 6.5. $|G_{Q,loc}| \lesssim 2^{5j/2} u_Q u_{\mathcal{N}^{1000/\epsilon}(Q)}^2 + negligible.$

Proof. From the definition of $G_{Q,loc}$ we have

$$G_{Q,loc} = \sum_{l=-2}^2 \left[\sum_{k=j-\frac{1000}{\epsilon}}^{j+\frac{1000}{\epsilon}} \langle \phi_{Q,j} P_j T(P_{k+l} u \cdot \nabla P_k u), \phi_{Q,j} P_j u \rangle + \sum_{k=j-\frac{1000}{\epsilon}}^{j+\frac{1000}{\epsilon}} \langle \phi_{Q,j} P_j T(P_k u \cdot \nabla P_{k+l} u), \phi_{Q,j} P_j u \rangle \right].$$

Since the above sum has only $\lesssim 1$ many terms by applying Proposition 5.4 we observe that for some particular values of k, l

$$|G_{Q,loc}| \lesssim |\langle \phi_{Q,j} P_j T(\phi_{Q_j,j} P_k u \cdot \phi_{Q_j,j} \nabla P_{k+l} u), \phi_{Q,j} P_j u \rangle| + negligible.$$

Now we apply Proposition 5.3 together with Cauchy–Schwartz to obtain

$$\|\phi_{Q_j,j} P_k u\|_{L^\infty} \lesssim 2^{3j/2} \|\phi_{Q_j,j} P_k u\|_{L^2} + negligible.$$

Using Proposition 5.5, we observe that

$$\|\phi_{Q_j,j} P_k u\|_{L^2} \lesssim u_{\mathcal{N}^{1000/\epsilon}(Q)}.$$

Finally direct calculation shows

$\|\phi_{Q_j,j} \nabla P_{k+l} u\|_{L^2} \lesssim \|\nabla \phi_{(1+2^{-2\epsilon j/3})Q_j,j} P_{k+l} u\| \lesssim 2^j u_{\mathcal{N}^{1000/\epsilon}(Q)} + negligible,$ where the first inequality comes from the fact that $\phi_{(1+2^{-2\epsilon j/3})Q_j,j} = 1$ on the support of $\phi_{Q_j,j}$ and the second inequality comes again from Proposition 5.5. Combining all these estimates proves the lemma. \square

LEMMA 6.6. $|G_{Q,hh}| \lesssim \sum_{k>j+\frac{1000}{\epsilon}} u_Q 2^{\frac{3j}{2}+k} \|\phi_{Q_j,j} P_k u\|_{L^2}^2.$

Proof. We begin similarly to before by estimating

$$\langle \phi_{Q,j} P_j T(P_k u \cdot \nabla \tilde{P}_k u), \phi_{Q,j} P_j u \rangle.$$

With the other terms one can proceed likewise. Now, similarly to what we have already done, we observe that we can write this as,

$$\langle \phi_{Q,j} P_j T(\phi_{Q_j,j} P_k u \cdot \nabla \phi_{Q_j,j} \tilde{P}_k u), \phi_{Q,j} P_j u \rangle + negligible.$$

Now we estimate

$$\|\phi_{Q,j} P_j u\|_{L^\infty} \lesssim 2^{3j/2} u_Q + negligible,$$

$$\|\nabla \phi_{Q_j,j} \tilde{P}_k u\|_{L^2} \lesssim 2^k \sum_{l=-2}^2 \|\phi_{Q_j,j} P_{k+l} u\|_{L^2} + negligible.$$

Combining these estimates gives the desired inequality. \square

In fact, we get a slightly better estimate, for instance by the div-curl lemma, but it does not seem to be necessary.

The previous three lemmas are somewhat wasteful, particularly in the definition of the local part. We are using more squares than we need to cover. For the result of the final section we need a somewhat more efficient decomposition, which is proved in exactly the same way. We formulate it here.

For Q a cube at level j , we define the ancestors of Q ,

$$\mathcal{A}(Q) = \{Q_k\}_{\epsilon j < k < j-4}.$$

For $k \geq j-4$, we define the collection $\mathcal{S}_k(Q)$ to be a covering (with overlap $\lesssim 1$) of the cube Q_{j-4} by cubes at level k . We define the strict extended family

$$\mathcal{E}(Q) = \bigcup_{k=j-4}^{j+\frac{1000}{\epsilon}} \mathcal{S}_k(Q),$$

and we define $\mathcal{F}(Q)$, the followers of Q by

$$\mathcal{F}(Q) = \bigcup_{k>j+\frac{1000}{\epsilon}} \mathcal{S}_k(Q).$$

Then applying the same estimates as in the proofs of Lemmas 6.4, 6.5 and 6.6, to a slightly different decomposition we obtain

COROLLARY 6.7. *For any $\delta > 0$ we can estimate*

$$G_Q \lesssim Z_{Q,lhhl} + Z_{Q,loc} + Z_{Q,hh} + 2^{j(1+\frac{3}{2}\delta)} u_Q u_{\mathcal{N}^1(Q)},$$

where

$$Z_{Q,lhhl} = \sum_{k<j-4} \sum_{Q' \in \mathcal{N}^1(Q)} 2^{j+\frac{3k}{2}} u_{Q_k} u_{Q'} u_Q,$$

and

$$Z_{Q,loc} = 2^{\frac{5j}{2}} \sum_{Q', Q'' \in \mathcal{E}(Q)} u_Q u_{Q'} u_{Q''},$$

and finally

$$Z_{Q,hh} = \sum_{k>j+\frac{1000}{\epsilon}} 2^{\frac{3j}{2}+k} u_Q \sum_{Q' \in \mathcal{S}_k(Q)} u_{Q'}^2.$$

Proof. This time we define

$$H'_{j,th} = \sum_{k<j-4} P_j((P_k u) \cdot \tilde{P}_j \nabla u),$$

$$H'_{j,hl} = \sum_{k<j-4} P_j((\tilde{P}_j u) \cdot P_k \nabla u),$$

$$H'_{j, hh} = \sum_{k > j + \frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{k > j + \frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u),$$

and

$$H'_{loc} = \sum_{j-4 < k < j + \frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{j-4 < k < j + \frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u).$$

Now we break up G_Q in the obvious way:

$$G_Q = G'_{Q, lh} + G'_{Q, hl} + G'_{Q, hh} + G'_{Q, loc},$$

where

$$\begin{aligned} G'_{Q, lh} &= \langle -\phi_{Q, j} \tilde{P}_j T H'_{j, lh}, \phi_{Q, j} P_j u \rangle, \\ G'_{Q, hl} &= \langle -\phi_{Q, j} \tilde{P}_j T H'_{j, hl}, \phi_{Q, j} P_j u \rangle, \\ G'_{Q, hh} &= \langle -\phi_{Q, j} \tilde{P}_j T H'_{j, hh} \phi_{Q, j} P_j u \rangle, \\ G'_{Q, loc} &= \langle -\phi_{Q, j} \tilde{P}_j T H'_{j, loc}, \phi_{Q, j} P_j u \rangle. \end{aligned}$$

Now we simply estimate each of the G' 's as before. □

Now we are ready to describe our singular set. We will say that a cube Q of sidelength $2^{-j(1-2\epsilon)}$ is *bad* if

$$\int_0^T \int \sum_{k \geq j} 2^{2\alpha k} |\phi_{Q, k} P_k u|^2 \gtrsim 2^{-(5-4\alpha)j - 100\epsilon j}. \tag{21}$$

Let E_j be the union of $2^{3000/\epsilon} Q$ for all cubes Q of sidelength $2^{-j(1-2\epsilon)}$ which are bad.

We will need the following well-known covering lemma of Vitali (see [S]):

LEMMA 6.8. *Let \mathcal{C} be any collection of cubes, then there is a subcollection \mathcal{C}' so that any two cubes in \mathcal{C}' are pairwise disjoint and so that*

$$\bigcup_{Q \in \mathcal{C}} Q \subset \bigcup_{Q \in \mathcal{C}'} 5Q.$$

PROPOSITION 6.9. *There is covering \mathcal{Q}_j of E_j by cubes of sidelength $2^{-j(1-2\epsilon)}$ so that*

$$\#(\mathcal{Q}_j) \lesssim 2^{(5-4\alpha)j + 100\epsilon j}.$$

Proof. Let \mathcal{C} be the collection of cubes $2Q$ where Q is a bad cube at level j . From Lemma 6.8 we know that there are disjoint cubes $\{2Q_\rho\}_{\rho \in \mathbb{Z}}$ such that the collection consisting of $\{10Q_\rho\}$ covers the set E_j . Any cube of sidelength $10 \cdot 2^{-j(1-2\epsilon)}$ can be covered by 1000 cubes of sidelength $2^{-j(1-2\epsilon)}$ (and 1000 is a constant.) We will define \mathcal{Q}_j to be the covering formed by the union of the thousandths of the elements of $\{10Q_\rho\}$.

In order to count cubes in \mathcal{Q}_j it is enough to count the disjoint cubes used in the construction of \mathcal{Q}_j .

However we know since the $2Q_\rho$'s are disjoint that

$$\sum_{\rho} \sum_k \int_0^T 2^{2\alpha k} |\phi_{Q_{\rho,k}} P_k u|^2 \lesssim \int_0^T \int |\Delta^\alpha u|^2 \lesssim 1 \tag{22}$$

by conservation of energy, while we know

$$\sum_{\rho} \sum_k \int_0^T 2^{2\alpha k} |\phi_{Q_{\rho,k}} P_j u|^2 \gtrsim \#(\mathcal{Q}_j) 2^{-(5-4\alpha)j-100\epsilon_j} \tag{23}$$

by the badness of the cubes.

Combining the inequalities (22) and (23) implies the claim. □

We pause for a moment to apply Lemma 3.1.

COROLLARY 6.10. *The dimension of $E = \limsup_{j \rightarrow \infty} E_j$ is bounded by $5 - 4\alpha + O(\epsilon)$.*

If we could show that if $x \notin E$ then x is a regular point of u at time T – in other words that

$$\limsup_{t \rightarrow T} |u(x, t)| < \infty.$$

(Indeed if we could show it is a regular point for a derivative of u of any fixed order) this would immediately imply Theorem 1.2.

Unfortunately, we will show something slightly more complicated – the same statement for a somewhat larger collection \mathcal{Q}'_j .

In the following section, we will investigate what are the immediate consequences of $x \notin E$. Saying that $x \notin E$ is the same as saying that there exists a j so that for any $k > j$, we have $x \notin E_k$. Denote F_j as the set of points with this property. To prove a regularity statement about E , it suffices to show that that statement holds for any j_0 , provided $x \in F_{j_0}$. However fixing j , we may change our constants in the definition of bad square so that x is not contained in any bad squares. Thus we may as well assume x is contained in no bad squares. This will be our hypothesis in the following section. (And as promised, the constants will now depend on j_0 .)

7 Critical Regularity

Heuristically, the worst part of G_Q is $G_{Q,loc}$ which looks at scale j like $2^{5j/2} u_Q^3$. On the other hand, dissipation gives decay like $2^{2\alpha j} u_Q^2$. This should mean that as long as $u_Q < 2^{2(\alpha - \frac{5}{4})j}$, the growth of u_Q ought to be under control. It is this estimate which we will show in this section for any Q not contained in E_k for any $k > j_0$.

Theorem 7.1. *Let Q be as above, then there is a constant C , depending only on T , the initial conditions for u , the constant in the definition of badness, and j_0 so that*

$$u_Q(t) < C2^{2(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j(Q)}.$$

Proof. We proceed by contradiction. Suppose the theorem is false. We let T_0 be the first time and Q be the largest cube so that

$$u_Q(T_0) > C2^{2(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j}.$$

Now since our initial data is smooth, at the initial time, (since j is chosen sufficiently large), we have

$$u_Q(0) \lesssim 2^{-1000j}.$$

Therefore, it must be the case that

$$\int_0^{T_0} (G_Q(t) - c2^{2\alpha j}u_Q^2 + C2^{(2\alpha-2\epsilon)j} \sum_{Q' \in \mathcal{N}^1(Q)} u_{Q'}^2) dt \gtrsim 2^{4(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j}.$$

By using Proposition 6.3, we can replace Q by an extended nuclear family $\mathcal{N}^l(Q)$, with $l < 400/\epsilon$ for which

$$\int_0^T (-c2^{2\alpha j}u_{\mathcal{N}^l(Q)}^2 + C2^{(2\alpha-2\epsilon)j}u_{\mathcal{N}^{l+1}(Q)}^2) dt \lesssim - \int_0^T 2^{2\alpha j}u_{\mathcal{N}^l(Q)}^2.$$

For the current theorem, this is all we need. Since $u_{\mathcal{N}^l(Q)}$ also begins $\lesssim 2^{-1000j}$, we must have

$$\int_0^{T_0} G_{\mathcal{N}^l(Q)}(t) dt \gtrsim 2^{4(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j} + \int_0^T 2^{2\alpha j}u_{\mathcal{N}^l(Q)}^2,$$

where we define

$$G_{\mathcal{N}^l(Q)}(t) = \sum_{Q_1 \in \mathcal{N}^l(Q)} G_{Q_1}(t).$$

Since there are only $\lesssim 1$ cubes in $\mathcal{N}^l(Q)$, for this to be the case, there must be a \tilde{Q} so that

$$\int_0^{T_0} G_{\tilde{Q}}(t) dt \gtrsim 2^{4(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j} + \int_0^T 2^{2\alpha j}u_{\mathcal{N}^l(\tilde{Q})}^2,$$

We contradict this by using Lemmas 6.4, 6.5, and 6.6 to estimate $G_{\tilde{Q}}$. By our definition of E_j , the cube \tilde{Q} is contained in no larger bad squares.

Each estimate contains a factor of $u_{\tilde{Q}}$ which we take out using the estimate $|u_{\tilde{Q}}| \lesssim 2^{2(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j}$, which we get from the definition of T_0 .

First let Q_1 be a nuclear family member of \tilde{Q} and Q_2 be a distant ancestor at level k . Suppose $k < \delta j$, we see that $2^{(1+\frac{3\delta}{2})j}u_{Q_1} \lesssim 2^{2\alpha j}u_{\mathcal{N}^l(\tilde{Q})}$. Thus we need not worry about this term.

Suppose $k > \delta j$. We must estimate

$$\int_0^T 2^{\frac{3k}{2}+j} u_{Q_1} u_{Q_2} dt.$$

Note that both Q_1 and Q_2 are good squares. Thus we have the estimates

$$\int_0^T 2^{3k} u_{Q_2}^2 \leq 2^{(2\alpha-2)k-10\epsilon k},$$

and

$$\int_0^T 2^{2j} u_{Q_1}^2 \leq 2^{(2\alpha-3)j-10\epsilon j}.$$

Applying Cauchy-Schwartz, we get (using $\alpha \geq 1$)

$$\int_0^T 2^{\frac{3k}{2}+j} u_{Q_1} u_{Q_2} dt \leq 2^{(\frac{4\alpha-5}{2})j-10\epsilon j}.$$

Summing over (there are only j terms) provides the desired estimate on $G_{\tilde{Q},hl} + G_{\tilde{Q},lh}$. We can estimate $G_{\tilde{Q},loc}$ in the same way (by allowing k as large as $j + \frac{1000}{\epsilon}$.)

We are left to estimate $G_{\tilde{Q},hh}$. We fix a scale $k > j$ and pick k_2 within 2 of k . Now we are left to estimate

$$\int_0^{T_0} 2^{\frac{3j}{2}+k} \|\phi_{\tilde{Q},k} P_k u\|_{L^2} \|\phi_{\tilde{Q},k_2} P_{k_2} u\|_{L^2}.$$

However since \tilde{Q} is a good square, we have

$$\int 2^{2\alpha k} |\phi_{\tilde{Q},k} P_k u|^2 \leq 2^{(4\alpha-5)j-10\epsilon j}.$$

Thus

$$\int 2^{k+\frac{3j}{2}} |\phi_{\tilde{Q},k} P_k u|^2 \leq 2^{(\frac{4\alpha-5}{2})j-10\epsilon j-(2\alpha-1)(k-j)},$$

which is an estimate that decays geometrically in k when $\alpha > 1/2$. By using the similar estimate for k_2 , applying Cauchy Schwartz and summing over k , we get the desired result. □

8 Combinatorics

In this section, we begin with the set E_j which is the union of a collection \mathcal{Q}_j of cubes with sidelength $2^{-(1-2\epsilon)j}$ having cardinality $\lesssim 2^{(5-4\alpha+100\epsilon)j}$. (We assume $4\alpha - 4 > 200\epsilon$.)

Theorem 8.1. *There exists a sequence of collections \mathcal{Q}'_j of cubes of sidelength $2^{-(1-2\epsilon)j}$ with $\#(\mathcal{Q}'_j) \lesssim 2^{(5-4\alpha+100\epsilon)j}$, so that for any Q of length $2^{-(1-2\epsilon)j}$ which does not intersect any element \mathcal{Q}'_j there exists a number*

$1/2 < r < 1$ with the following property: For no $k > j$ is there $\tilde{Q} \in \mathcal{Q}_k$ so that

$$100\tilde{Q} \cap \partial(rQ) \neq \emptyset. \tag{24}$$

Proof. We refer to the elements of \mathcal{Q}_j as the bad cubes. We say that a cube of sidelength $2^{-(1-2\epsilon)j}$ is very bad if either it intersects a bad cube of the same length or it intersects more than $c2^{(5-4\alpha+150\epsilon)(k-j)}$ elements of $100\mathcal{Q}_k$ for some $k > j$ with c a small constant to be specified later. Let E'_j be the union of all very bad cubes of length $2^{-(1-2\epsilon)j}$, then by the estimates on the cardinality of the \mathcal{Q}_k 's and by the Vitali lemma, we can see that E'_j can be covered by $\lesssim 2^{(5-4\alpha+100\epsilon)j}$ cubes of length $2^{-(1-2\epsilon)j}$. We refer to these cubes as \mathcal{Q}'_k . Now we need only prove (24).

Let Q be a cube of length $2^{-(1-2\epsilon)j}$ which does not intersect E'_j . Let $\mathcal{D}^k(Q)$ be the set of elements of $100\mathcal{Q}_k$ which intersect Q . Then we have the estimate

$$\#(\mathcal{D}^k(Q)) \leq c2^{(5-4\alpha+150\epsilon)(k-j)}.$$

Let $f_k(r)$ be the function defined from $1/2$ to 1 which counts how many elements of $\mathcal{D}^k(Q)$ intersect $\partial(rQ)$. For each $Q' \in \mathcal{D}^k(Q)$ define $r_{Q'}$ to be that number so that the center of Q' lies on $\partial(r_{Q'}Q)$. Then

$$f_k(r) \leq \sum_{Q' \in \mathcal{D}^k(Q)} \chi_{(r_{Q'}-100(2^{(1-2\epsilon)(j-k)}), r_{Q'}+100(2^{(1-2\epsilon)(j-k)}))}.$$

Thus

$$\|f_k(r)\|_{L^1} \leq 200c2^{(4\alpha-4-152\epsilon)(j-k)}.$$

Since $4\alpha-4 > 200\epsilon$, this estimate decays geometrically with k . By choosing c sufficiently small, we may arrange that

$$\left\| \sum_{k>j} f_k \right\|_{L^1} < \frac{1}{4}.$$

Thus by Tchebychev's inequality, there must be a value of r between $1/2$ and 1 so that $f_k(r) = 0$ for all $k > j$. This is the value of r that we choose. \square

9 Barrier Estimate

In this section, we prove regularity on the interior of a cube Q , provided that one has critical regularity for cubes containing it and cubes Q' for which $\partial Q \cap 100Q' \neq \emptyset$.

If Q is a cube and $Q_1 \subset Q$, we define $d(Q_1)$, the graph distance of Q_1 to the boundary of Q by $d(Q_1) = k - 1$, where k is the smallest positive

integer so that

$$2^k Q_1 \cap \partial Q \neq \emptyset.$$

LEMMA 9.1. *Let Q be a cube. Suppose we know that for all $t < T$ we have that for any cube Q' so that $Q \subset Q'$ with sidelength of Q' being $2^{-l(1-2\epsilon)}$, we have that*

$$|u_{Q'}(t)| \lesssim 2^{-l(\frac{5-4\alpha+2\epsilon}{2})},$$

and suppose further that for any Q' so that $100Q' \cap \partial Q \neq \emptyset$ with sidelength of Q' being $2^{-l(1-2\epsilon)}$ and $l > j - 2$, we have that

$$|u_{Q'}(t)| \lesssim 2^{-l(\frac{5-4\alpha+2\epsilon}{2})},$$

then for any $Q_1 \subset Q$ of length $2^{-k(1-2\epsilon)}$, we have the estimate

$$|u_{Q_1}(t)| \lesssim 2^{-k\rho(Q_1)}, \tag{25}$$

where

$$\rho(Q_1) = \min \left(10, \frac{5 - 4\alpha + 2\epsilon}{2} + \frac{\epsilon(d(Q_1) - 5)}{50} \right).$$

Proof. At time $t = 0$ for all Q_1 we have

$$u_{Q_1}(t) \lesssim 2^{-1000k},$$

when Q_1 has sidelength $2^{-(1-2\epsilon)k}$. Thus at time 0, the lemma is satisfied. Let t_0 be the first time at which the lemma fails and Q_1 be one of the cubes for which it fails. It must be the case by hypothesis that $32Q_1$ does not intersect ∂Q . Then we have

$$u_{Q_1}(t_0) \sim 2^{-k\rho(Q_1)}.$$

Let $t_1 < t_0$, be the last time before t_0 when

$$u_{Q_1}(t_1) \lesssim 2^{k(-\rho(Q_1) + \frac{\epsilon}{10})}.$$

Then we have

$$\int_{t_1}^{t_0} \frac{d}{dt}(u_{Q_1}^2) \gtrsim 2^{-2k\rho(Q_1)}. \tag{26}$$

However on the time interval (t_1, t_0) the lemma is satisfied.

We will invoke Corollary 6.7. Now for any $Q_2 \in \mathcal{E}(Q_1)$ we have

$$d(Q_2) \geq d(Q_1) - 5. \tag{27}$$

This is because $Q_2 \subset 32Q_1$. Further for any ancestor $Q_3 \in \mathcal{A}(Q_1)$ with Q_3 having sidelength $2^{-(1-2\epsilon)l}$, with $\epsilon k < l < k - 4$,

$$d(Q_3) \geq d(Q_1) - 5(k - l). \tag{28}$$

Further for any follower $Q_4 \in \mathcal{F}(Q_1)$ (that is a cube which contributes to $G_{Q_1, hh}$ and is in particular contained in $\frac{3}{2}Q_1$) of Q_1 with Q_4 having sidelength $2^{-(1-2\epsilon)l}$ with $l > k + \frac{1000}{\epsilon}$, we have

$$d(Q_4) \geq d(Q_1) + \left(\frac{l - k}{2} \right). \tag{29}$$

Applying (29), we conclude that $\rho(Q_4) = 10$. Thus we conclude

$$Z_{Q_1, hh} \lesssim \sum_{l>k} 2^{\frac{5k}{2}} 2^{3(l-k)} 2^{-20l} 2^{(-\rho(Q_1) + \frac{\epsilon}{10})k},$$

by counting the elements of $\mathcal{S}_l(Q)$ by $2^{3(l-k)}$. Calculating, we find

$$Z_{Q_1, hh} \lesssim 2^{-(\frac{35}{2} + \rho(Q_1))k},$$

which since $\rho(Q_1) \leq 10$, cannot possibly account for (26).

Now applying (27) to Proposition 6.1, we observe that for any t between t_1 and t_0 , we have dissipation at Q_1 of $\gtrsim 2^{2\alpha k} u_{Q_1}^2(t) \gtrsim 2^{(2\alpha - 2\rho(Q_1) + \frac{\epsilon}{5})k}$. Thus to reach a contradiction, it suffices to show that on this time interval

$$Z_{Q_1, hllh} + Z_{Q_1, loc} \lesssim 2^{(2\alpha - 2\rho(Q_1) + \frac{\epsilon}{5})k}.$$

For this reason we can also ignore the ultra-low term $2^{j(1 + \frac{3}{2}\delta)} u_{Q_1} u_{\mathcal{N}^1(Q_1)}$ in Corollary 6.7.

Using (27), we observe that for any Q_2 in $\mathcal{E}(Q_1)$ we have the estimates

$$u_{Q_2} \lesssim 2^{(-\rho(Q_1) + \frac{\epsilon}{10})k},$$

as well as

$$u_{Q_2} \lesssim 2^{(4\alpha - 5 - 2\epsilon)k/2},$$

by the lower bound on ρ . Now we calculate

$$Z_{Q_1, loc} \lesssim \sum_{Q_2, Q'_2 \in \mathcal{E}(Q_1)} 2^{\frac{5k}{2}} u_{Q_2} u_{Q'_2} u_{Q_1} \lesssim 2^{(2\alpha - \frac{4\epsilon}{5} - 2\rho(Q_1))k}.$$

Thus $Z_{Q_1, loc}$ cannot contribute to the growth.

Now to estimate $Z_{Q_1, hllh}$, we observe that for $Q'_1 \in \mathcal{N}^1(Q_1)$ we have the estimate

$$u_{Q'_1} \lesssim 2^{(-\rho(Q_1) + \frac{\epsilon}{10})k},$$

while for ancestor Q_4 of sidelength $2^{-(1-2\epsilon)l}$, we apply (28) (as well as the hypotheses of the lemma for squares larger than Q)

$$u_{Q_4} \lesssim 2^{\frac{(4\alpha - 5 - 2\epsilon)k}{2} + (\frac{3}{2} - 100\epsilon)(k-l)},$$

since of course $\frac{3}{2} > \frac{5-4\alpha}{2} + 200\epsilon$. Now we just estimate

$$Z_{Q_1, hllh} \lesssim \sum_l \sum 2^{\frac{3l}{2} + k} u_{Q'_1} u_{Q_4} u_{Q_1} \lesssim 2^{(2\alpha - \frac{4\epsilon}{5} - 2\rho(Q_1))k}.$$

Thus u_{Q_1} could not have grown which is a contradiction. \square

But now we have in fact proven the main theorem. We define E'_j as in section 8. We need to show that u is regular at any x not contained in any E'_j with j larger than some integer j_x . By changing our constants, we can say x is not in any E'_j . Now by Theorems 7.1 and 8.1, for any cube Q centered at x , we can find a cube almost half as large which satisfies

the hypotheses of Lemma 9.1. But the conclusion of the lemma implies that x is a regular point. Thus any singular point must be contained in $E = \limsup E'_j$. By lemma 3.1, we have $\dim(E) < 5 - 4\alpha + 20\epsilon$. Letting ϵ tend towards 0, we get Theorem 1.2.

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