1. Prove that the following are equivalent:
   (i) There is no retraction $D^n \to S^{n-1}$.
   (ii) Every continuous map $D^n \to D^n$ has a fixed point.

2. Suppose $X = A \cup B$ is a CW-complex, where $A$ and $B$ are contractible subcomplexes with contractible intersection. Prove that $X$ is contractible.

3. Let $G$ be a finitely generated abelian group. Find a finite-dimensional path-connected CW-complex $X_G$ such that $\pi_1(X_G) = G$.

4. Consider the 3-dimensional cube $I^3$. By a face of $I^3$, we mean one of the six square faces. Identify opposite faces of $I^3$ by rotating each face $90^\circ$ counter-clockwise (relative to the outward pointing normal) in its center and translating. Let $Y$ be the resulting quotient space. Compute the fundamental group of $Y$.

5. Determine which of the following spaces are homotopy equivalent to each other:
   (a) $S^1 \vee S^1$.
   (b) The complement in $S^3$ of the Hopf link. $T^2 = S^1 \times S^1$
   (c) The complement in $S^3$ of the unlink with two components. $S^2 \vee S^1 \vee S^1$
   (d) The complement in $\mathbb{R}^3$ of two parallel lines. $S^1 \vee S^1$
   (e) The complement in $\mathbb{R}^3$ of two intersecting lines. $S^1 \vee S^1 \vee S^1$

6. Consider the two curves $A$ and $B$ in the solid torus $T \cong D^2 \times S^1$ in $\mathbb{R}^3$ as in the figure. Draw a picture that shows that $A$ bounds a disk in $\mathbb{R}^3 - B$. Show that $A$ is not null-homotopic in $T - B$. 

\[
\begin{align*}
\text{Diagram:} & \\
\text{represents commutator:} & BCB^{-1}C^{-1}
\end{align*}
\]
Let $D^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ be the closed unit disk. Let $A = \{ z \in \mathbb{C} \mid |z + 1/2| < 1/8 \}$, let $B = \{ z \in \mathbb{C} \mid |z| < 1/8 \}$, and let $C = \{ z \in \mathbb{C} \mid |z - 1/2| < 1/8 \}$. Let $Z$ be the space obtained from $D^2 - (A \cup B \cup C)$ by identifying the four circles in the boundary so that their counterclockwise orientations are preserved. Compute the fundamental group of $Z$ and all of its homology groups.

\[ \begin{array}{c}
\pi_1(Z) \cong \langle a, b, c, d \mid aba^{-1}b^{-1}c^{-1}d^{-1} \rangle \\
H_1(Z) \cong \mathbb{Z} \oplus \mathbb{Z}^3 \\
H_0(Z) \cong \mathbb{Z} \text{ path connected}
\end{array} \]
get a map \( r : \mathbb{D}^n \to S^{n-1} \) defined by send \( x \) to the point on \( S^{n-1} = \partial \mathbb{D}^n \) which is the intersection of \( \partial \mathbb{D}^n \) with the ray passing through \( x \) and \( r(x) \) in the direction of \( x \) 

\[
 t(x - r(x)) \cap \partial \mathbb{D}^n > 0
\]

we claim \( r \) is a retraction. For this we check

- continuity
- surjectivity
- \( r^2 = \text{id} \)

Continuity

\( r(x) \) depends continuously on \( x \) and \( r(x) \) where \( r(x) \) depends on \( x \) continuously.

\[
 r(x) \text{ depends continuously on } x.
\]

Surjectivity

Let \( x \in \partial \mathbb{D}^n \), then \( r(x) = x \) since the top of \( x - r(x) \) intersects the boundary at \( x \).

Retraction

\( \text{im}(r) = \partial \mathbb{D}^n \) by definition and the proof of surjectivity shows \( r^2 = \text{id} \) since \( r|_{\partial \mathbb{D}^n} = \text{id} |_{\partial \mathbb{D}^n} \).

Conversely, if \( \exists r : \mathbb{D}^n \to S^{n-1} \) retraction we can compose it with the inclusion \( i : S^{n-1} \to \mathbb{D}^n \) and antipodal map in such a way as to construct a map \( f : \mathbb{D}^n \to \mathbb{D}^n \) with no fixed points.

\[
 f(x) \in \partial \mathbb{D}^n \quad \forall x \quad \Rightarrow \quad \text{fixed point would happen on boundary}
\]

but \( f(x) = -r(r(x)) = -x \quad \forall x \in \partial \mathbb{D}^n \).
\[ X = \bigcap_{\cap} \text{AnB}, \quad \text{A,B,AnB contractible, X is a CW complex, A,B are subcomplexes of } X(=\text{AnB}). \text{ Prove } X \text{ is contractible.} \]

Recall that CW pairs satisfy the homotopy extension property (HEP) and that a pair \((Y,Z)\) satisfies HEP \(\iff Y \times I \text{ retracts onto } (Y \times \{0\}) \cup (Z \times I).\)

Thus, we have provided an explicit homotopy equivalence from \(X\) to a point. We conclude \(X\) is contractible.

---

**Key**

- \(M_{id_X} \text{ and } M_{id(\text{AnB})}\) denote mapping cylinders of the identity maps of \(X\) to itself and \(\text{AnB}\) to itself. Since mapping cylinders deformation retract onto the target the first and seventh homotopies are justified.

- CW+HEP Recall that for CW pairs the retract guaranteed by HEP is in fact a deformation retract so the second homotopy is justified since \((X,A)\) is a CW pair as given in the hypothesis that \(A\) is a subcomplex of \(X\).

- \(A \simeq \), \(\text{AnB} \simeq \) & \(B \simeq \) Since each of these spaces is contractible \(\Rightarrow\) they have the homotopy type of a point this justifies the third, fourth and eighth homotopies.

- Recall that quotients of spaces by contractible subcomplexes are homotopy equivalences, thus since cones are contractible the \(S^0 + \{0\}\) are.
Compute $T_1(Y)$ where $Y$ is $I^3$ with opposing faces identified by a $90^\circ$ rotation counter-clockwise (according to the outward facing norm) followed by a translation.

First we put a cell structure on $I^3$ by taking products of $I$ with its usual cell structure having two 0-cells $\{0,1\}$ and one 1-cell $\{e\}$.

$I^2$ has cells $\{(0,0),(0,1),(1,0),(1,1),(0,e),(1,e),(e,0),(e,1),(e,e)\}$

there are four 0-cells $\{(0,0),(0,1),(1,0),(1,1)\}$

four 1-cells $\{(0,e),(1,e),(e,0),(e,1)\}$ and one 2-cell $\{e,e\}$

$I^3$ has cells $\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(0,0,e),(0,1,e),(1,0,e),(1,1,e),(0,e,0),(0,e,1),(1,e,0),(1,e,1),(e,0,0),(e,0,1),(e,1,0),(e,1,1),(e,e,0),(e,e,1),(e,0,e),(e,e,e)\}$

there are eight 0-cells $\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(0,1,e),(1,0,e),(1,1,e),(e,0,0)\}$

twelve 1-cells $\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(0,1,e),(1,0,e),(1,1,e),(1,e,0),(1,e,1),(e,0,0),(e,0,1),(e,1,0),(e,1,1)\}$

twelve 1-cells

six 2-cells and one 3-cell.

To avoid confusion we will trace through the identifications first just on vertices, again for edges, then for faces and lastly on 3-cells.

Identifying the Front and Back faces we get $V_0$ identified with $V_2$ $V_1$ identified with $V_3$ 

Identifying the Left and Right faces we get

Identifying the Top and Bottom faces we get

Hence $Y$ has two 0-cells when we will call $V_0$ and $V_1$. 

No new identifications
Identifying the front and back faces we get there are 8 different edges

Identifying the right and left faces we get there are 4 different edges

Identifying the top and bottom faces we get No new identifications

Hence $Y$ has four 1-cells which we will call $e_1, e_2, e_3, e_4$

The faces of $I^3$ get pairwise identified into three distinct 2-cells when we call FB, RL and TB (for front/back, right/left, top/bottom respectively.)

The 3-cell of $I^3$ has no other 3-cell to be identified with so the cells of $Y$ are $\{v_0, v_1, e_1, e_2, e_3, e_4, FB, RL, TB, (e, e, e)\}$

Since $\Pi_1$ of a space only sees the 1-skeleton we will find the attaching maps and finish with a computation of $\Pi_1(Y)$

$$SK_0(Y) = \begin{array}{c}
v_0 \\
v_1
dendarray$$

$$SK_1(Y) = \begin{array}{c}
v_0 \\
e_1 \\
e_2 \\
e_3 \\
e_4 \\
v_1 
dendarray$$

$$SK_2(Y) = SK_1(Y) \sqcup \begin{array}{c}
D^2 \\
\bullet \\
D^2 \\
\bullet \\
D^2 \\
\bullet 
dendarray$$

quotienting by the contractible subspace $e_4$ we obtain a space homotopy equivalent to $Y$ whose attaching maps $\bar{q}_i$ factor through the quotient $Z := Y/e_4$

$$SK_0(Z) = \begin{array}{c}
\circ 
dendarray$$

$$SK_1(Z) = \begin{array}{c}
e_2 \\
\bullet \\
e_1 
dendarray$$

$$SK_2(Z) = SK_1(Z) \sqcup \begin{array}{c}
D^2 \\
\bullet \\
D^2 \\
\bullet \\
D^2 \\
\bullet 
dendarray$$

where the attaching maps are given by the following words

$\bar{q}_1 : e_1 e_2 e_3 e_4$

$\bar{q}_2 : e_1 e_3 e_4 e_2$

$\bar{q}_3 : e_1 e_3 e_4 e_2 e_3^{-1}$

$\tilde{q}_1 = e_1 e_2 e_3$

$\tilde{q}_2 = e_1 e_3 e_2$

$\tilde{q}_3 = e_1 e_2 e_3^{-1}$
Now, by the corollary to Van-Kampen's theorem

\[ \pi_1(Y) \cong \pi_1(Z) \cong \langle e_1, e_2, e_3 \mid e_1 e_2 e_3 = e e_3 e_2 = e e_2 e_3^{-1} = 1 \rangle \]

\( \pi_1 \) is homology invariant.

Using these relations we will be able to recognize this group as the quaternion group of order 8.

\[ e e_3^{-1} e_2 = 1 \Rightarrow e e_3^{-1} = e_2 \]

Substituting this into the third relation we get

\[ e_1 (e_1 e_3^{-1}) e_3 = 1 \Rightarrow e_1 = e_2 \]

Similarly,

\[ e_1 e_2 e_3 = 1 \Rightarrow e_1 e_2 = e_3^{-1} \]

thus \( e_1 (e_1 e_2) e_2 = 1 \)

\[ \Rightarrow e_1^2 = e_2 \]

To summarize what we know so far we have

\[ e_1^2 = e_3^2 = e_2^{-2} \]

Since \( e_1 e_2 e_3 = 1 \) it follows that \( e_1 = e_3^{-1} e_2^{-1} \)

\[ \Rightarrow e_2 = e_3 e_3^{-1} \]

Using this on the relation \( e_1 e_3^{-1} e_2 = 1 \)

we can rewrite our group as

\[ \pi_1(Z) \cong \langle e_1, e_3 \mid e_1 e_3 e_3^{-1} e_2 = 1, e e_3 = e_3^{-1} e_1, e_1^2 = e_3^2 \rangle \]

\[ e_1^2 = e_3^2 \Rightarrow e_3^{-1} e_1 e_1 = e_3 \Rightarrow e_1 e_3 e_1 = e_3 \]

\[ \Rightarrow e_1 e_3 = e_3 e_1^{-1} \]

\[ 1 = e_1^2 e_3^{-2} = e_1 e_1 e_3^{-1} e_3 = e_1 e_3^{-1} e_3^{-1} = e_3^{-1} e_1 e_3^{-1} e_3^{-1} \]

So \( e_1^{-1} e_1^{-1} = e_2 \)

\[ \Rightarrow (e_1^2)^{-1} = e_3^2 = e_2 \]

\[ \Rightarrow e_1^4 = e_3^4 = 1 \]

\[ \pi_1(Z) \cong \langle e_1, e_2 \mid e_1^4 = e_3^4 = 1, e_1 e_3 = e_3 e_1, e_1 e_2 = e_3, e_2 = e_3^2 \rangle \]
Let $X$ be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Calculate $\pi_i(X)$ and $H_*(X)$.

Notice that the map $f: \mathbb{R}^3 \setminus \{0\} \to S^2$ given by $x \mapsto 1/\|x\|$ sends any vector lying on a coordinate axis to another vector on the same coordinate axis. It follows that $f$ restricts to our space $X$. Let $\tilde{f}: X \to S^2$ be this restriction. Clearly $\tilde{f}$ is continuous since $f$ was and moreover $\text{im}(\tilde{f}) = S^2 \setminus \{x_0, x_1, x_2, x_3, x_4, x_5\}$.

Next, we can apply the stereographic projection map from the north pole (since the point on the sphere is missing) to see that our space can also be viewed as $\mathbb{R}^2 \setminus \{x_1, x_2, x_3, x_4, x_5\}$.

By picking a point on each boundary circle and connecting all of these points via a path to the chosen point on the circle centered at the origin, we have a tree, call it $T$.

Since $T$ is a contractible subspace of $\mathbb{R}^2 \setminus \{x_1, x_2, x_3, x_4, x_5\}$, we will not change the homotopy type of our space when passing to the quotient.

Conclude $X \cong \bigvee_{i=1}^{5} S^1_i$. 

$$X \cong \bigvee_{i=1}^{5} S^1_i \Rightarrow \pi_i(X) \cong \bigvee_{i=1}^{5} \mathbb{Z}, \ H_0(X) \cong \mathbb{Z} \leftarrow \text{path connected}, \ H_1(X) \cong \mathbb{Z}^5 \leftarrow \text{abelianize}$$
Second Proof

Since $S^3$ is the one point compactification of $\mathbb{R}^3$ we can view $\mathbb{R}^3 \setminus \{x, y, z\text{-axes}\}$ as the complement of a graph in $S^3$. Specifically, if we let $S^3 = \mathbb{R}^3 \cup \{\infty\}$ then $\mathbb{R}^3 \setminus \{x, y, z\text{-axes}\} \cong S^3 \setminus (\{x, y, z\text{-axes}\} \cup \{\infty\})$

Since the point $\infty$ is not in our space we may go back to thinking of the situation in $\mathbb{R}^3$. Because $\infty$ is the only point of intersection of these lines in the 3-sphere this space is the same as $\mathbb{R}^3 \setminus \{5\text{ non-intersecting lines}\}$. W.L.O.G. we may assume the lines in this complement are parallel to each other.

\[ \mathbb{R}^3 \setminus \{5\text{ non-intersecting lines}\} \cong \mathbb{R}^2 \setminus \{x_1, x_2, x_3, x_4, x_5\} \]

which puts us in the situation from before.
prove or disprove the following statement.
If $X$ is a CW-complex with finitely many cells, then $\pi_2(X)$ is a finitely generated abelian group.

**Claim:** The statement is false.

Consider the space $S^2 \vee S^1$. The universal cover is $\mathbb{R}$ with $S^2$ attached (at a single point) at every integer in $\mathbb{R}$. Since we can always pass to universal covers when computing fundamental groups, and contracting $\mathbb{R}$ to the origin makes this universal cover look like a countable wedge of 2-spheres, $\pi_2(\mathbb{R})$ has a countably many infinite cyclic summands, one coming from each copy of $S^2$ at the wedge point since $\pi_2(S^2) \cong \mathbb{Z}$.

$S^2 \vee S^1$ is a CW-complex with a minimum of $3$ total cells, but $\pi_2(S^2 \vee S^1) \cong \pi_2(\mathbb{R}) \cong \pi_2(\vee^\infty S^2)$.

We could argue geometrically that a map $S^2 \to S^2$ generating $\pi_2(S^2)$ into one sphere in the wedge is distinct (homotopically) to that of any other in the wedge or on $S^2$. 

$S^2$ simply connected $\Rightarrow$ Hurewicz implies $H_2(\vee^\infty S^2) \cong \pi_2(\vee^\infty S^2)$.

Covering maps out of the universal cover of a space induce isomorphisms on $\pi_n$ for all $n > 2$. 

\[ \infty \text{-sheeted covering map} \]

\[ \cong S^2 \vee S^1 \]
Claim: Maps $f: S^n \to S^n$ have a fixed point unless $\deg(f) = (-1)^{n+1}$

Proof 1: We prove $f$ has no fixed point $\implies \deg(f) = (-1)^{n+1}$

Suppose $f(x) \neq x \quad \forall x \in S^n$. Then the line joining $f(x)$ to $-x$ does not pass through the origin, i.e., $(1-t)f(x) - tx \neq 0 \quad \forall t \in [0,1] \quad x \in S^n$

hence $g(x, t) := \frac{(1-t)f(x) - tx}{\| (1-t)f(x) - tx \|}$ is a well-defined homotopy from $f$ to the antipodal map. We divide by the length here so that $g(x, s): S^n \to S^n$ for each fixed $s$.

Proof 2: Lefschetz fixed point theorem

Consider $\zeta(f)$ where $f$ is any map $S^n \to S^n$.

Since $H_0(S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$, the first factor of $\mathbb{Z}$ coming from $H_0(S^n)$ and the second coming from $H_n(S^n)$, the Lefschetz number $\zeta(f)$ is the sum $(-1)^n \text{tr}(f_*: H_0(S^n) \to H_0(S^n)) + (-1)^n \text{tr}(f_*: H_n(S^n) \to H_n(S^n))$

This is because $H_0(S^n)$ is represented by a point. The image under $f$ of a point can be chosen to be a basis of $H_0(S^n)$ at the target.

Thus $\zeta(f) = 1 + (-1)^n K = 0$ $\iff$ $(-1)^n K = -1$

$K = -1$ for $n$ even

$K = 1$ for $n$ odd

$n$ even $\implies$ degree of antipodal map $= -1$

$n$ odd $\implies$ degree of antipodal map $= 1$

Since $\zeta(f) = 0$ $\implies f$ has a fixed point.
We construct a free resolution of \( \mathbb{Z}_2 \) as a \( \mathbb{Z}_4 \)-module.

To begin we construct an exact sequence beginning with the map \( \mathbb{Z}_2 \to 0 \) by repeatedly finding generators for the kernel. The result is

\[
\cdots \to \mathbb{Z}_4 \to \mathbb{Z}_4 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0
\]

This gives us a chain complex by dropping the degree zero term.

\[
\cdots \to \mathbb{Z}_4 \to \mathbb{Z}_4 \to \mathbb{Z}_4 \to 0
\]

Applying the contravariant hom \( \text{Hom}_{\mathbb{Z}_4}(\cdot, \mathbb{Z}_2) \)

\[
\begin{matrix}
\cdots \leftarrow \text{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \leftarrow \text{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \leftarrow 0 \\
\text{Nil}^2 \\
\cdots \leftarrow \mathbb{Z}_2 \leftarrow \mathbb{Z}_2 \leftarrow 0
\end{matrix}
\]

we are left with a cochain complex having \( \mathbb{Z}_2 \) concentrated in every degree and all zero maps \( \Rightarrow \text{Ext}_{\mathbb{Z}_4}^n(\mathbb{Z}_2, \mathbb{Z}_2) \)

\[
\text{Nil}^2(\cdots \leftarrow \mathbb{Z}_2 \leftarrow \mathbb{Z}_2 \leftarrow 0) \\
\text{Nil}^2 \mathbb{Z}_2 \neq 0 \text{ for n.}
\]

Let \( \Phi \in C'(X; G) \) be regarded as a function from paths in \( X \) to \( G \).

Assuming \( \Phi \) is a cocycle

\[
\text{Claim: } \Phi(f \cdot g) = \Phi(f) + \Phi(g)
\]

homotopic to the third edge of the triangle.

\[
\text{Claim: } \Phi(\text{constant path}) = 0
\]

If \( \Delta \to \ast \) denotes the constant map from a 2-simplex to a point then \( \partial(\Delta \to \ast) = \text{constant path} \) for some choice of 3. Thus \( \Phi(\text{constant path}) = \Phi(\partial(\Delta \to \ast)) \) cocycle

\[
\Rightarrow \Phi(f \cdot g) = \Phi(f) + \Phi(g)
\]
\( \in \mathcal{C}(X; G) \) cocycle

\( \Rightarrow \mathcal{C}(f) = \mathcal{C}(g) \) when \( f = g \)

we apply the last two facts to a triangle whose edges

are \( f, \) constant path, \( g^{-1} \) (going clockwise around the boundary)

Since \( f = g \) as paths they share endpoints
and they can be viewed as bounding a disk
since the homotopy interpolating between these
paths factors through a disk with \( f \cdot g^{-1} \) as
its boundary circle.

We know \( 0 = \mathcal{C}(f \circ g^{-1}) = \mathcal{C}(g \circ f) = \mathcal{C}(f \cdot g \cdot f) \)

\( = \mathcal{C}(f) - \mathcal{C}(g) + \mathcal{C}(f) \)

\( \Rightarrow \mathcal{C}(g) = \mathcal{C}(f) \)

Claim: cocycles are coboundaries \( \Leftrightarrow \mathcal{C}(f) \) depends only on endpoints
of \( f \) for any path \( f. \)

Proof: Suppose \( \mathcal{C} = \delta \psi \) for some \( \psi \in \mathcal{C}^0(X; G) \) a function on
points of \( X \) to \( G. \) Then \( \mathcal{C}(f) = \delta \psi(f) = \psi(\text{end } f) - \psi(\text{start } f) \)

thus \( \mathcal{C} \) depends only on the endpoints of \( f \) for any path \( f. \)

Conversely, if \( \mathcal{C}(f) \) depends only on the endpoints of \( f \) for any path \( f \)
pick a base point \( x_0 \in X \) (here we are reducing to the case
\( X \) is path connected) for every other point \( x \in X \) there is
a path \( f_x \) from \( x_0 \) to \( x \) once we choose an arbitrary
value for \( \psi(x_0) \) taking \( \psi(\bar{x}) = \psi(f) + \psi(x_0) \) ensures
that \( \mathcal{C}(f) = \psi(x) - \psi(x_0) = \psi(f) = \delta \psi(f) \).
Fact: \( X \text{ path connected} \Rightarrow H^1(X; G) \overset{\sim}{\longrightarrow} \text{Hom}(\pi_1(X), G) \)

define the map \( \Psi: H^1(X; G) \longrightarrow \text{Hom}(\pi_1(X), G) \)

\[
\Psi f \mapsto \varphi|_{\pi_1(X)}
\]

This is surjective since any map \( \varphi: \pi_1(X) \longrightarrow G \) can be extended to be defined on all paths in \( X \) by setting the unassigned values to the identity in \( G \), say, then extended linearly. \( \varphi|_{\pi_1(X)} \in \text{Hom}(\pi_1(X), G) \) since

\[
\varphi|_{\pi_1(X)}(f \cdot g) = \varphi|_{\pi_1(X)}(f) + \varphi|_{\pi_1(X)}(g)
\]

as just computed.

\( \varphi|_{\pi_1(X)} \) is well-defined on elements of \( \pi_1(X) \) since we also just showed \( \varphi|_{\pi_1(X)}(f) = \varphi|_{\pi_1(X)}(g) \) if \( f \equiv g \).

It remains to show that this map is injective. We want to show that any cocycle \( \Xi \), i.e., \( \varphi|_{\pi_1(X)} \equiv 0 \) must be a coboundary and thus represent a cohomology class homologous to 0.

We characterized cocycles that are coboundaries earlier so what we want to show reduces to

\[
\exists \text{ cocycle, } \varphi|_{\pi_1(X)} \equiv 0 \quad \Rightarrow \quad \varphi(f) \text{ depends only on the endpoints of } f \text{ \forall paths } f.
\]

Let \( f \) be a path in \( X \) and \( g \) be any other path in \( X \) sharing endpoints with \( f \). Either \( f \cdot g \) or \( f \cdot \overline{g} \) forms a loop in \( \pi_1(X) \) based, say, at the initial point of \( f \).

Hence either \( 0 = \varphi(f \cdot g) = \varphi(f) + \varphi(g) \) or \( 0 = \varphi(f \cdot \overline{g}) = \varphi(f) - \varphi(g) \), depending on the orientation of \( g \). It follows that the value of \( \varphi(f) \) depends only on the endpoints of the path.

Apparently there is also an argument using universal coefficient theorem to deduce this isomorphism. Basically, \( \text{Hom}(\pi_1(X), G) \) descends to \( \text{Hom}(H_1(X), G) \) since \( G \) is abelian.
Second proof: $H^i(X; G) \cong \text{Hom}(\pi_i(X), G)$ provided $X$ path connected.

by the universal coefficient theorem $H^i(X; G) \cong \text{Ext}(H_0(X), G) \oplus \text{Hom}(H_1(X), G)$

Since $H_0(X)$ free
Since $X$ path connected $\Rightarrow \pi_i(X) = H_i(X)$
and $G$ abelian
So maps $\pi_i(X) \rightarrow G$
factors through $H_i(X)$.

We now use both LES of pair and MV sequence
in homology & cohomology to compute $H_i(S^n; G)$ & $H^i(S^n; G)$.

$H_0(S^n; G) \cong G$ since there are 2 path components
$H_i(S^n; G) = 0 \forall i \neq 0$ because there are no cells of dimension $> 0$.

Looking now at the long exact sequence of the pair $(D^n, S^{n-1})$
we see $\cdots \rightarrow H_i(S^{n-1}; G) \rightarrow H_i(D^n; G) \rightarrow H_i(D^n, S^{n-1}; G) \rightarrow H_{i-1}(S^{n-1}; G) \rightarrow \cdots$
is exact. Since $H_i(D^n; G) = 0 \forall i > 0$ it follows that

$H_i(D^n, S^{n-1}; G) \rightarrow H_i(S^n; G)$ (by exactness) as long as $n > 1$.

For $n = 1$ we get $\mathbb{Z}_2$s in the long exact sequence for all
$i > 1$ and the remaining part is $\cdots \rightarrow 0 \rightarrow H_1(S^0; G) \rightarrow H_1(D^1; G) \rightarrow H_1(D^1, S^0; G)$

$\rightarrow H_0(S^0; G) \rightarrow H_0(D^1; G) \rightarrow H_0(D^1, S^0; G) \rightarrow 0$

Plugging in the known values for these homology groups we have

$\cdots \rightarrow 0 \rightarrow H_1(D^1, S^0; G) \rightarrow G \rightarrow \cdots$

$\cong H_0(D^1, S^0; G) \rightarrow 0$ $\cong H_0(D^1, S^0; G) \rightarrow 0$ $\cong H_0(D^1, S^0; G) \rightarrow 0$

$G$ path connectedness.