$\mathbb{R}P^n$ Definition & Cell Structure

$\mathbb{R}P^n := \{ L \mid L \text{ is a 1-Dimensional vector subspace in } \mathbb{R}^{n+1}, (0,0) \}$

$\gamma$ is the topology induced by the quotient topology

$\gamma : \mathbb{R}P^n \setminus \{0\} \to \mathbb{R}P^n$

$x \mapsto [x]$ \quad $x \gamma y \iff \exists \lambda \in \mathbb{R} \text{ s.t. } x = \lambda y$

$\mathbb{R}P^1$ in $\mathbb{R}^1$ there is exactly one line through the origin

$\mathbb{R}P^0 = \{0\}$

$\mathbb{R}P^2$ in $\mathbb{R}^2$ the lines through the origin look like

$\mathbb{R}P^2 = \mathbb{D}^2/\sim \Rightarrow \mathbb{D}^2/\sim = \mathbb{S}^2 = \mathbb{O}$

$\mathbb{R}P^3$ in $\mathbb{R}^3$ the lines through the origin look like

$\mathbb{R}P^3 = \mathbb{D}^2/\sim \Rightarrow \mathbb{D}^2/\sim = \mathbb{D}^2$
The General Procedure

(i) Each line is determined by some nonzero vector in $\mathbb{R}^{n+1}$ unique up to scalar multiplication $\mathbb{R}P^n$

(ii) Restrict to unit length vectors $S^n / \sim \rightarrow S^n$

Sphere mod Antipodes

(iii) Identify Antipodes on interior of a hemisphere $D^n / \sim \rightarrow \mathbb{D}^n$

Disk with Antipodes of its boundary identified

(iv) View $\partial D^n / \sim \rightarrow \mathbb{R}P^n$ via 2-fold cover map

$\Psi: \partial D^n = S^{n-1} \rightarrow \mathbb{R}P^{n-1}$ covering space projection map

Conclude:

Each cell structure on $\mathbb{R}P^n$ having one cell in each dimension less than and including $n$.

$\mathbb{R}P^n = \{e^0, (e^1, \varphi_1), (e^2, \varphi_2), \ldots, (e^n, \varphi_n)\}$

$R^{\infty} = \bigcup_{i=0}^{\infty} \mathbb{R}P^n$

$\mathbb{R}P^{\infty}$ has one cell in every dimension.

"Space of lines through the origin in $\mathbb{R}^{\infty} = \bigcup_{i=0}^{\infty} \mathbb{R}P^n$"
Other Points of View for $\mathbb{RP}^2$

1. 

\[ D^2 / \mathbb{Z} \cong \mathbb{D}^2 \]

\[ N_1 = \langle a, b \mid a^2 = 1 \rangle \]

\[ X \cong \mathbb{Z} \] Space w/ prescribed fundamental group $\mathbb{Z}$

2. 

Möbius Strip / its boundary = Mucam

\[ \triangle \cong D^2 \]

\[ \triangle \cong \text{Möbius Strip} \]

\[ 2\text{Möbius Strip} \]

Observing this is the same as $\star 1$ because Möbius strip = mapping cylinder of 2-fold cover $S^1 \to S^1 \text{ i.e. } \mathfrak{g}$ deformation retracts onto core circle and this identifies antipodes on $\mathfrak{g}$. 
Subcomplexes of $\mathbb{RP}^n$

The $K$-skeleton of any CW-complex is always a subcomplex. Therefore $\mathbb{RP}^K \subseteq \mathbb{RP}^n$ can naturally be viewed as a subcomplex via $i: \mathbb{RP}^K \rightarrow \mathbb{RP}^n$ the natural inclusion map provided, of course, that $K \leq n$.

**Fact:** This is a complete list of the subcomplexes of $\mathbb{RP}^n$.

**Definition:** A "subcomplex" is a closed subspace $A \subseteq X$ that is the union of cells of $X$.

\[ \mathbb{RP}^1 = \mathbb{S}^1/\mathbb{Z}/2 \mathbb{Z} \]

\[ \mathbb{RP}^2 = \mathbb{S}^2/(\mathbb{Z}/2 \mathbb{Z}) \]

\[ \mathbb{RP}^3 = \mathbb{S}^3/(\mathbb{Z}/2 \mathbb{Z}) \]

\[ \mathbb{RP}^4 = \mathbb{S}^4/(\mathbb{Z}/2 \mathbb{Z}) \]

\[ S^4 = \{(x,y) \mid x^2 + y^2 = 1 \} \]

\[ S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1 \} \]

Open sets $U_i$ are open disks $e^2 \setminus \partial e^2$.

$U_1 = \{ [1: y: z] \}$ dotted blue line not included

$U_2 = \{ [x: 1: z] \}$ dotted red line not included

$U_3 = \{ [x: y: 1] \}$ boundary not included.

Intersections have 2 zero coordinates.
Fundamental group & Covering spaces of $\mathbb{RP}^n$

\[ \pi_1(\mathbb{RP}^n) = \pi_1(\{pt\}) = \pi_1(\emptyset) = \{1\} \]
\[ \pi_1(\mathbb{RP}^2) = \pi_1(S^1) = \pi_1(\emptyset) \cong \mathbb{Z} \]

\[ \pi_1(\mathbb{RP}^2) : \]

Applying Van Kampen

\[ x \]
\[ \xymatrix{ a \ar@{-}[r] & b \ar@{-}[r] & a } \]
\[ \xymatrix{ \oplus \ar@{-}[r] & D^2 } \]
\[ \Rightarrow \]
\[ \mathbb{Z}/2\mathbb{Z} = \langle a, a^2 \rangle \]

\[ \phi : p^2 \to a \]
\[ \theta \mapsto a^2 \]

\[ \phi : p^2 \to ab \]
\[ \theta \mapsto (ab)^2 \]

\[ \Rightarrow \quad \mathbb{Z}/2\mathbb{Z} = \langle ab, (ab)^2 \rangle \]

\[ \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z} \quad \forall n \geq 2 \]

\[ \pi_1(\mathbb{RP}^1) \cong \mathbb{Z} \]

\[ \pi_1(\mathbb{RP}^0) \cong 1 \]

\[ \pi_1 \] only sees the 2-skeleton, all of which are the same for $\mathbb{RP}^k, k \geq 2$
By covering space theory we have $S^n \& \mathbb{RP}^n$ are the only connected covers of $\mathbb{RP}^n$ \( n \geq 2 \).

\[
\begin{array}{c}
\xymatrix{1 \ar[r] & 2} \\
\mathbb{Z}/2\mathbb{Z}
\end{array}
\]

\[\xymatrix{\text{Galois correspondence} & S^n \ar[d] & \text{Universal cover} & \text{2-Sheeted cover} & \text{1-Sheeted cover} \Rightarrow \text{homeo}}
\]

\[\mathbb{RP}^n \ar[d] \downarrow \Rightarrow \text{homeo} \]

Proof: Let $\psi: S^n \to S^n$ denote the antipodal map $x \mapsto -x$.

\[\mathbb{Z}/2\mathbb{Z} \cong \left\langle \psi \right\rangle \text{ homeo } S^n \]

is a covering space action since points in opposite hemispheres swap hemispheres.

\[S^n/\left\langle \psi \right\rangle \cong \mathbb{RP}^n \Rightarrow \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}\]

Since $\pi_1(S^n) \cong 1$ $\forall n \geq 2$

i.e. $S^n$ is simply connected and therefore the universal cover of $\mathbb{RP}^n$.

Finding a Generator for $\pi_1(\mathbb{RP}^n)$

\[
\begin{array}{c}
\alpha \times 2 \ar[r] & \mathbb{RP}^n \\
\alpha \ar[r] & \overline{\alpha}, \text{ s.t. } \alpha \overline{\alpha} = 0
\end{array}
\]

A path connecting pair of antipodes projects to a loop in $\pi_1$. Repeating this loop twice lifts to a loop whose inclusion in $S^n$ is contractible.

\[1 \neq \overline{\alpha} \in \pi_1(\mathbb{RP}^n) \]

Descends to null-homotopy in $\mathbb{RP}^n$. 
Deck Groups

\( (G(\mathbb{RP}^n) \cong 1 = \langle \text{id} \rangle ) \) since this is a 1-sheeted cover

\[
\begin{align*}
S^n & \quad \xrightarrow{x^2} \quad \mathbb{RP}^n \\
\text{Covering Space} & \quad \text{Subgroup of} \quad \pi_1(\mathbb{RP}^n)
\end{align*}
\]

* \( \mathbb{Z}/2\mathbb{Z} : 1 = 2 \)

\( \mathbb{Z}/2\mathbb{Z} \)
This is a normal covering
Deck Group

* Follows immediately from the fact that \( S^n \) is the universal cover
* \( \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \)

Now we put a cell structure on \( S^n \) so that it has exactly this group of cellular homeomorphisms.

Inductively, we create \( S^n \) from \( S^{n-1} \) by attaching two disks via homeomorphisms on the boundary (the choice of homeomorphism is irrelevant).

\[
S^0 = \cdot \quad S^1 = \quad S^2 = \quad \ldots
\]

Notice: \( \mathbb{Z}/2\mathbb{Z} \) has a subgroup of index 2 \( \Rightarrow \) Non-Orientable & \( \mathbb{E} \) connected, orientable double cover \( S^n \).

Antipodal map reverses orientation so two different orientations get sent to the same place compatibly.

\( \mathbb{RP}^n \) can't be orientable.
(Dis)connected covers of IRP^n (n ≥ 2)

The k-sheeted covers of IRP^n are just $i + j$ different connected covers of IRP^n where $i = \# 1$-sheeted covers and $j = \# 2$-sheeted covers. Then we will have $K = 2^j i$.

Since any such pair of integers defines a cover we see

\[
\begin{array}{c}
\{ \text{k-sheeted covers of IRP}^n \} / \text{Covering Space Isomorphism} \\
\end{array}
\] \[\leftrightarrow\]
\[
\{ (i, j) | K = 2^j i \}
\]

but also covering space theory tells us that

\[
\begin{array}{c}
\{ \text{k-sheeted covers of IRP}^n \} / \text{Covering Space Isomorphism} \\
\end{array}
\] \[\leftrightarrow\]
\[
\{ \text{Homomorphisms } \mathbb{Z}/2\mathbb{Z} \to S_K \}
\]

Strangely, this tells us that

\[
\begin{array}{c}
\{ \text{Homomorphisms } \mathbb{Z}/2\mathbb{Z} \to S_K \} \xrightarrow{\text{change of basis}} \{ \text{elements of order 1 or 2 in } S_K \}
\end{array}
\]

\[
\xrightarrow{\text{change of basis}} \}
\]

\[
\{ (i, j) | K = 2^j i \} \xrightarrow{\text{choice of an even \# Smaller}} \{ \text{choice of an even \# Smaller} \}
\]
$\mathbb{RP}^n$ is a closed, non-orientable manifold ($n \geq 2$)

By our first definition of $\mathbb{RP}^n$ as $\left(\mathbb{R}^{n+1}\setminus \{0\}\right)/\sim$, $\lambda \in \mathbb{R}$

We can represent $\mathbb{RP}^n$ as the set $\left\{[x_0:x_1:...:x_n] : x_i \neq 0 \right\}$ of equivalence classes of points together with the quotient topology. We need to verify:

- Locally Euclidean
- Hausdorff
- Compact
- No boundary
- Non-orientable ($n \geq 2$)

All Facts are consequences of basic covering space theory.

We simply note that $S^n$ is a 2-fold cover of $\mathbb{RP}^n$.

Locally Euclidean: $x \in \mathbb{RP}^n \Rightarrow \exists U$ containing $x$, fundamental neighborhood $\Rightarrow p^{-1}(U) \cong U_1 \sqcup U_2$ where $U_1, U_2 \cong U$ via $p|_{U_i}$.

Since $U_1$ and $U_2$ are homeomorphic to open balls in $\mathbb{R}^n$, via stereographic projection, so is $U$.

Alternatively, we produce the charts $U_i := \left\{[x_0:...:1:...:x_n] : x_i \neq 0 \right\}$

Since the map $p_i(U_i) := (x_0, ..., x_i, 1, x_{i+1}, ..., x_n)$ are homeomorphisms to $\mathbb{R}^n$.

* in particular we showed $\dim(\mathbb{RP}^n) = n$.

Compact + Hausdorff: we appeal to the following general fact $p : \tilde{X} \to X$ cover w/ $p^{-1}(x) \neq \emptyset$ and finite $\forall x \in X$ implies $\tilde{X}$ compact Hausdorff $\iff X$ compact Hausdorff

(Exercise 3 section 1.3 page 77 of Hatcher)

$S^n$ compact (closed + bounded subspace of $\mathbb{R}^{n+1}$)

and Hausdorff (subspace topology of Hausdorff space $\mathbb{R}^{n+1}$)

hence $\mathbb{RP}^n$ compact + Hausdorff.

Non-orientable: $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ when $n \geq 2 \Rightarrow \pi_1(\mathbb{RP}^2)$ has a subgroup of index 2 corresponding to an oriented double cover, namely, the only double cover $S^n$. The deck group of $S^n$ is $\mathbb{Z}/2\mathbb{Z}$
generated by the antipodal map. Thus, the antipodal map commutes with the covering space projection and thus opposite orientations are compatibly identified.

Alternatively, since our oriented double cover is connected we can't have consistent choice of orientation.

No boundary: This was settled by the proof that \( \mathbb{R}P^n \) is locally Euclidean. Every point has a fundamental neighbourhood homeomorphic to \( \mathbb{R}^n \). A point on \( \partial \mathbb{R}P^n \) would be homeomorphic to \( \mathbb{R}^{n-1} \).

Compact + Hausdorff:

Alternatively, we know \( \mathbb{R}P^n \) is a CW-complex with one cell in each dimension \( \leq n \). CW-complex \( \Rightarrow \) Hausdorff

Finite CW-complex \( \Rightarrow \) Compact
\[ \mathbb{R}P^\infty = K(\mathbb{Z}/\mathbb{Z}, 1) \]

Fundamental Group Determines Homotopy Type of a \( K(G, 1) \)
- \( \pi_1(\mathbb{R}P^\infty) \cong \mathbb{Z}/2\mathbb{Z} \)
- \( S^\infty \to \mathbb{R}P^\infty \) universal cover is contractible
  \[
  (x_1, x_2, ...) \xrightarrow{\text{linear}} (0, x_1, x_2, ...) \xrightarrow{\text{normalize linear map}} (1, 0, ...) = \text{pt}.
  \]

\[ \mathbb{R}P^1 = S^1 = K(\mathbb{Z}, 1) \]
- \( \pi_1(\mathbb{R}P^1) = \pi_1(S^1) \cong \mathbb{Z} \)
- \( \mathbb{R} \) is universal cover
  \( \mathbb{R} \) is contractible

\[ T^n = S^1 \times S^1 \times \ldots \times S^1 = K(\mathbb{Z}^n, 1) \]
- \( \pi_1(T^n) = \prod_{i=1}^{n} \pi_1(S^1) = \mathbb{Z}^n \)
- universal cover of \( T^n \) is \( \prod_{i=1}^{n} \mathbb{R} \)

Since products of covering maps are covering maps and fundamental groups of products are products of fundamental groups.

In General \( K(G \times H, 1) \cong K(G, 1) \times K(H, 1) \)
\[
S^\infty / (Z/mZ) = K(Z/mZ, 1)
\]
\[
S^\infty \cong C^\infty, \quad \frac{Z/mZ \times \eta}{\text{homeo}} S^\infty
\]
\[
(\varepsilon_0, \varepsilon_1, \cdots) \mapsto e^{2\pi i \varepsilon_0} (\varepsilon_0, \varepsilon_1, \cdots)
\]
Covering space action
\[
\Rightarrow \pi_1(S^\infty / (Z/mZ)) \cong Z/mZ
\]
\[
S^\infty \rightarrow S^\infty / (Z/mZ) \text{ universal cover}
\]
\[
S^\infty \text{ contractible.}
\]

Closed surface other than \( S^2 \) & \( \mathbb{R}P^2 = K(G, 1) \)

have infinite \( \pi_1 \).
Recall, only simply connected surfaces are \( \mathbb{R}^2 \) and \( S^2 \).
Since \( \tilde{X} \rightarrow X \) must be infinite sheeted, \( \tilde{X} \) not compact \( \Rightarrow \mathbb{R}^2 = \tilde{X} \) for all closed surfaces w/ \( \pi_1 \), infinite.

Graphs = \( K(F_n, 1) \)

non-closed surfaces deformation retract onto graphs
universal cover is a graph \( \Rightarrow \) tree \( \Rightarrow \) contractible.
Simplicial Homology of $\mathbb{RP}^2$

\[ 0 \rightarrow \mathbb{Z}^2 \xrightarrow{(1,1,1)} \mathbb{Z}^3 \xrightarrow{(1,1,0)} \mathbb{Z}^2 \rightarrow 0 \]

\[ \alpha \mapsto b-a+c \quad \nu \mapsto 0 \]

\[ \forall \nu \rightarrow 0 \]

\[ a \mapsto w-v \]

\[ b \mapsto w-v \]

\[ c \mapsto v = v-v \]

\[ \left( \begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{cc} -1 & 1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \]

we have a chain complex

\[ A = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad B = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \]

Note: $A(-1,1)B = A(-1,0)B = (0,0)$

Smith form

\[ 0 \rightarrow \mathbb{Z}[u,v] \xrightarrow{(1,1)} \mathbb{Z}[a,b,c] \xrightarrow{(1,1)} \mathbb{Z}[u,v] \rightarrow 0 \]

\[ b \]

\[ c \]

\[ \alpha \mapsto w-v \]

\[ b \mapsto w-v \]

\[ c \mapsto v = v-v \]

$\det(A) = -1 \in \mathbb{Z}^*$

$\det(B) = -1 \in \mathbb{Z}^*$

$\Rightarrow A^{-1} \& B^{-1}$ exist

$B^{-1}(-1,1) = \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$

\[ C = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \]

$\det(C) = -1 \Rightarrow C$ is a.

$H_*(\mathbb{RP}^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

with $\mathbb{Z}/2\mathbb{Z}$-coefficients

\[ 0 \rightarrow (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{(1,0)} (\mathbb{Z}/2\mathbb{Z})^3 \xrightarrow{(0,1)} (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow 0 \]

$H_0(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

$H_1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

$H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$
0 \rightarrow \mathbb{Z}[u,v] \rightarrow \mathbb{Z}[a,b,c] \rightarrow \mathbb{Z}[v,w] \rightarrow 0

u \mapsto b-c+a \quad a \mapsto 0 \quad v \mapsto 0
V \mapsto c-b+a \quad b \mapsto w-v \quad c \mapsto w-v

this is the chain complex from before with the
roles of a and c reversed. the computation will
be almost identical to that on the reverse side.

Euler Characteristic of \( \mathbb{R}P^2 \)

\[ \chi(\mathbb{R}P^2) = (-1)^0 \text{Rank}(H_0(\mathbb{R}P^2)) + (-1)^1 \text{Rank}(H_1(\mathbb{R}P^2)) \]
\[ = 1+0 = 1 \]

computing with \( \mathbb{Z}/2\mathbb{Z} \) coefficients

\[ \chi(\mathbb{R}P^2) = (-1)^0 \dim(H_0(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})) + (-1)^1 \dim(H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})) \]
\[ = 1-1+1 = 1 \]

⚠️ Note: \( \dim(H_i(X; \mathbb{R})) \neq \text{Rank}(H_i(X; \mathbb{Z})) \) \( \mathbb{R} \) a field

However, you can still compute Euler characteristic
by passing free ranks over \( \mathbb{Z} \) to dimension of \( \mathbb{R} \)-vector
spaces.

Application: \( \mathbb{R}P^2 \rightarrow Y \) covering map, \( Y \) CW-complex
\[ \Rightarrow p \text{ is a homeomorphism.} \]

Proof: \( \chi(\mathbb{R}P^2) = m \chi(Y) \) where \( m = \# \text{Sheets of cover} \)
\[ 1 = m \chi(Y) \Rightarrow m=1 \text{ & } \chi(Y) = 1 \]
Since \( m > 0 \) an integer
\( \mathbb{R}P^2 \) compact \( \Rightarrow \) cover is finite
\[ m=1 \Rightarrow p \text{ is a homeomorphism.} \]
Second proof that \( \mathbb{R}P^2 \xrightarrow{p} Y \) covering \( Y \) CW-complex
\[ \Rightarrow \text{ } p \text{ homeomorphism} \]

Since \( p \) is a finite cover (compactness of \( \mathbb{R}P^2 \), finite CW-structure) the composition \( S^2 \xrightarrow{\text{cover}} \mathbb{R}P^2 \xrightarrow{p} Y \) is a covering map

hence \( \pi_1(Y) \otimes S^2 \), an even dimensional sphere \[ \Rightarrow \pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z} \]

Since \( \mathbb{Z}/2\mathbb{Z} \) is at least a subgroup of \( \pi_1(Y) \) because \( \mathbb{R}P^2 \) covers it. But then \( p \) is 1-sheeted \( \Rightarrow p \) homeomorphism.

proof that we can pass Euler Characteristic computations to dimensions of vector spaces by taking coefficients in a field.

there are at least two ways to do this. one is to directly show that the sum is unchanged by repeating the proof of Euler characteristic but with different coefficients.

The other uses only the universal coefficient theorem

Since \( H^i(X; \mathbb{R}) \cong \text{Ext}(H_{i-1}(X), \mathbb{R}) \oplus \text{Hom}(H_i(X), \mathbb{R}) \)

set \( \text{char}(\mathbb{R}) = p \), then \( \text{Hom}(H_i(X), \mathbb{R}) \) has a \( \mathbb{R} \)

summand for each \( \mathbb{Z} \) summand in \( H_i(X) \) as well as for each cyclic summand whose order is divisible by \( p \)

Since this would be \( \mathbb{Z}/p^n\mathbb{Z} \) for some \( k \) and sending

1 to 1 \( \mathbb{R} \) acts via the quotient by \( \langle p^{k-1} \rangle \) is a non-trivial homomorphism and in fact the only possible one (correspondence theorem). Now, these extra dimensions will cancel with the non-zero parts of \( \text{Ext}(H_i(X), \mathbb{R}) \) in \( H^i(X; \mathbb{R}) \) since \( \mathbb{R}/m\mathbb{R} \cong \begin{cases} 0, \text{ p nonzero} \\ \mathbb{R}, \text{ p 1 nonzero} \end{cases} \)
the latter has an application to manifolds.

\[ M \text{ an odd dimensional compact manifold } \Rightarrow \chi(M) = 0 \]

**Proof:** We use \( \mathbb{Z}/2\mathbb{Z} \)-coefficients since \( \mathbb{Z}/2\mathbb{Z} \) is a field (we can pass to \( \mathbb{Z}/2\mathbb{Z} \) without changing Euler characteristic.) and all manifolds are \( \mathbb{Z}/2\mathbb{Z} \)-orientable.

By Poincaré duality, \[ H^i(M; \mathbb{Z}/2\mathbb{Z}) \cong H_{n-i}(M; \mathbb{Z}/2\mathbb{Z}) \] so their dimensions are equal.

\[ \chi(M) = \sum_{i=0}^{2k+1} (-1)^i \dim(H^i(M; \mathbb{Z}/2\mathbb{Z})) \]

\[ = \sum_{i=0}^{K} (-1)^i (\dim(H^i(M; \mathbb{Z}/2\mathbb{Z})) - \dim(H_{n-i}(M; \mathbb{Z}/2\mathbb{Z}))) = 0 \]

Since each term vanishes.

We can repeat the idea of the proof on the reverse side to see that the non-free contributions to \( \dim(H_{n-i}(M; \mathbb{Z}/2\mathbb{Z})) \) come in pairs by UCT and therefore cancel out contributing nothing to the final sum.

**Observe:** Since \( \text{rank}(H^i(M; \mathbb{Z})) = \text{rank}(H_i(M; \mathbb{Z})) \) by UCT the case \( M \) compact ORIENTABLE and odd dimensional is immediate from P.D.
Cohomology Groups of $\mathbb{RP}^2$

To find $H^*(\mathbb{RP}^2)$ & $H^*(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$ from the $\Delta$-complex structure is a purely algebraic computation. We dualize the complex

$$\begin{array}{ccccccc}
0 & \to & \mathbb{Z}^2 & \xrightarrow{(1,0)} & \mathbb{Z}^3 & \xrightarrow{(0,1)} & \mathbb{Z}^2 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & \\
\end{array}$$

by applying $\text{Hom}(\cdot, \mathbb{Z})$.

Matrices now multiply on the left and vectors in chain groups are column vectors (note the convention we were using before in homology, was row vectors in chain groups and matrices multiply on the right).

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{chain complex}$$

$$H^0(\mathbb{RP}^2) = \text{Ker} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cong \mathbb{Z}$$

$$H^1(\mathbb{RP}^2) = \text{Ker} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} / \text{im} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cong 0$$

$$H^2(\mathbb{RP}^2) \cong \text{coker} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cong \mathbb{Z}/2\mathbb{Z}$$

With $\mathbb{Z}/2\mathbb{Z}$ coefficients we get the complex

$$\begin{array}{ccccccc}
0 & \leftarrow & (\mathbb{Z}/2\mathbb{Z})^2 & \leftarrow & (\mathbb{Z}/2\mathbb{Z})^3 & \leftarrow & (\mathbb{Z}/2\mathbb{Z})^2 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \\
\end{array}$$

$$H^0(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \text{Ker} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cong \mathbb{Z}/2\mathbb{Z}$$

$$H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \text{Ker} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} / \text{im} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cong \mathbb{Z}/2\mathbb{Z}$$

$$H^2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \text{coker} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cong \mathbb{Z}/2\mathbb{Z}$$
Universal Coefficient Theorem

Another purely algebraic way of getting at the cohomology groups of $\mathbb{RP}^2$ once $H_\ast(\mathbb{RP}^2; G)$ is known is the universal coefficient theorem.

\textbf{Z-coeff}

(i) $H_\ast(\mathbb{RP}^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$H^0(\mathbb{RP}^2) \cong \mathbb{Z}$ since $\mathbb{RP}^2$ connected

$H^1(\mathbb{RP}^2) \cong \text{Ext}(H_0(\mathbb{RP}^2), \mathbb{Z}) \oplus \text{Hom}(H_1(\mathbb{RP}^2), \mathbb{Z})$

\[
\begin{array}{c}
\cong \mathbb{Z} \\
\cong \mathbb{Z}/2\mathbb{Z}
\end{array}
\]

This makes sense because passing to cohomology moves torsion up one degree and preserves free parts.

\textbf{Z/2Z-coeff}

$H_\ast(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$H^0(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ since $\mathbb{RP}^2$ connected

$H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \text{Ext}(H_0(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \oplus \text{Hom}(H_1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$

\[
\begin{array}{c}
\cong \mathbb{Z}/2\mathbb{Z} \\
\cong \mathbb{Z}/2\mathbb{Z} \\
\cong \mathbb{Z}/2\mathbb{Z}
\end{array}
\]
Cellular Homology of $\mathbb{R}P^2$

Since we know $\mathbb{R}P^2$ has a CW structure with one $0, 2$ and 2 cell.

$$\partial_2 = 0 \quad \text{since there is one 0-cell}$$

$$\partial_2(e^2) = \deg\left( f: e^2 \to \mathbb{R}P^1 / \mathbb{R}P^0 \right) = \sum \text{(local degrees)}$$

the attaching map factors through the quotient identifying antipodes on $e^2$. This is because $a$ and $b^{-1}$ get sent to the same place.

$$\deg(f) = \sum \text{(local degrees)}$$

$$\partial e^2 \to \mathbb{R}P^1$$

consider a small enough nbhd of any point on $\mathbb{R}P^1$, Its preimage is two disjoint nbhds homeomorphic to it. (This is a two-sheeted covering map).

Forming the quotient $\partial e^2 \cap f_1(e^1)$ we get

Since $f_2 = f, \Psi$ where $\Psi$ is the antipodal map $\Psi: S^1 \to S^1$ and $\deg(f_1) = 1$

we have $\deg(f) = \deg(f_1) + \deg(f_2) = 1 + 1 = 2$

$$\partial_2 = \times 2$$
Cup Product Structure on $\mathbb{RP}^2$

We use $\mathbb{Z}_2 = \{-1, 1\}$-coefficients but suppress the notation. We also take for granted the chain complex below as it needed to be recomputed for finding generators for the homology groups. (Tensoring gets you the answer but not the basis)

$$
\begin{align*}
0 & \rightarrow \mathbb{Z}_2[u, v] \xrightarrow{(1, 1)} \mathbb{Z}_2[a, b, c] \xrightarrow{(1, 0)} \mathbb{Z}_2[uv, w] \rightarrow 0 \\
& \uparrow (10) \quad \downarrow (01) \quad \quad \quad \downarrow (00) \quad \quad \quad \downarrow (01) \\
0 & \rightarrow \mathbb{Z}_2[u, u+v] \xrightarrow{(1, 1)} \mathbb{Z}_2[a+b+c, b, c] \xrightarrow{(00)} \mathbb{Z}_2[uv, v] \rightarrow 0
\end{align*}
$$

Notice all vertical arrows square to the identity

$$
\begin{align*}
H_0(\mathbb{RP}^2) &= \langle \nu \rangle, \\
H_1(\mathbb{RP}^2) &= \langle \alpha \rangle, \\
H_2(\mathbb{RP}^2) &= \langle u+v \rangle
\end{align*}
$$

Dualizing we get generators $\langle \tilde{\nu} \rangle = H^0(\mathbb{RP}^2)$, $\langle \tilde{\alpha} \rangle = H^1(\mathbb{RP}^2)$, $\langle \tilde{\beta} \rangle = H^2(\mathbb{RP}^2)$ via dual bases (i.e. $\tilde{\nu}(w+v) = 0$, $\tilde{\nu}(v) = 1$

$\alpha(a+b+c) = 0$, $\alpha(b) = 0$, $\alpha(c) = 1$

$\beta(u) = 0$, $\beta(u+v) = 1$

Our goal is to show $\alpha \cup \nu = \beta$. $\alpha(a) = \alpha((a+b+c)+b+c) = \alpha(a+b+c)+\alpha(b)+\alpha(c) = 1$

$$
\begin{align*}
(\alpha \cup \nu)(u) &= \alpha(c) \cdot \nu(b) = 1 \cdot 0 \\
(\alpha \cup \nu)(u+v) &= (\alpha \cup \nu)(u) + (\alpha \cup \nu)(v) = 0 + \alpha(c) \cdot \nu(a) = 1
\end{align*}
$$

Therefore $\alpha \cup \nu = \beta$ and we have shown the

**Cohomology Ring of $\mathbb{RP}^2 \cong \mathbb{Z}[\alpha]/(\alpha^3)$**

Geometrically, $\alpha$ is the dotted line.

This line can be homotoped to intersect $b$ rather than $a$ but always must intersect $c$ a minimum of once. Reversing the roles of $a$ and $b$ corresponds to the choice of basis.

$\mathbb{Z}[v, u+v]$
Since the chain complex for $\mathbb{RP}^2$ with $\Delta$-complex structure below is so similar to the one just studied, we simply interpret the results in this setting rather than redoing all of the computations.

There are two ways to view the generator of $H^1(\mathbb{RP}^2)$, $\alpha$

Model $1^*$

Model $2^*$

Model $3^*$

These pictures convince us that $\alpha \cup \alpha \neq 0$ since two copies of the dotted line must intersect at least once.

We now do the computation on our last model for $\mathbb{RP}^2$

$e$ generates $H_1(\mathbb{RP}^2)$, so the generator of $H^1(\mathbb{RP}^2)$ is represented by a cocycle $\Phi : C_1(\mathbb{RP}^2) \rightarrow \mathbb{Z}/2\mathbb{Z}$ s.t. $\Phi(e) = 1$. The cocycle condition then says $0 = \delta(\Phi)$ in particular

$$0 = \delta(\Phi)(T_1) = \Phi(\partial T_1) = \Phi(e) - \Phi(e_2) + \Phi(e_1)$$

& $0 = \delta(\Phi)(T_2) = \Phi(\partial T_2) = \Phi(e) - \Phi(e_1) + \Phi(e_2)$

$\Rightarrow \Phi(e_1) = \Phi(e_2)$ we choose the value $\Phi(e_1) = 1$, $\Phi(e_2) = 0$.

Now, $(\Phi \cup \Phi)(T_1) = \Phi(e_1) \cdot \Phi(e) = \Phi(e_2) \cdot \Phi(e) = 0$

and $(\Phi \cup \Phi)(T_1 + T_2) = (\Phi \cup \Phi)(T_1) + (\Phi \cup \Phi)(T_2) = 1 + 0 = 1$

hence $\Phi \cup \Phi$ represents a generator for $H^2(\mathbb{RP}^2)$ since $T_1 + T_2$ represents a generator for $H_2(\mathbb{RP}^2)$.