Rectifiable Paths

\[ C: [a,b] \rightarrow (X,d) \quad \sup \sum_{i=0}^{n-1} d(c(t_{i}), c(t_{i+1})) < \infty \]

Set \( \lambda(c) = \ell \).

Reparametrize by arc length \( C: [a,b] \xrightarrow{\lambda} [0,\lambda] \xrightarrow{\tilde{c}} X \)

\[ \lambda \begin{cases} 
\text{continuous} \\
\text{weakly monotonic} \\
\text{subjective}
\end{cases} \]

The reparametrization by arc length is \( \tilde{C}: [0,\lambda] \rightarrow X \) & it satisfies

\[ \lambda(\tilde{C}|[0,\lambda]) = \lambda \quad \tilde{c}(\lambda) = c(x) \]

where \( \lambda(x) = \lambda \)

or \( \lambda(C|[0,\lambda]) = \lambda \)

Note: Monotonicity of \( \lambda \Rightarrow \text{im}(c) = \text{im}(\tilde{c}) \)

Reparametrize proportional to arc length

\[ \forall C: [a,b] \rightarrow X \quad \exists \tilde{C}: [0,1] \rightarrow X \]

s.t. \( \text{im}(c) = \text{im}(\tilde{c}) \)

\( \tilde{c} \) parametrized proportional to arc length.

(i.e. \( \lambda\tilde{c}: [0,1] \rightarrow [0,\lambda] \) is linear)

\[ \lambda(C|[a,\lambda(t)]) = \lambda(c([a,\lambda(t)])) = \lambda \]

First reparametrize w.r.t. arc length. This doesn't change the image so throw away the original function. Map linearly from \([0,1] \xrightarrow{\lambda} [0,\lambda] \) via

\[ t \mapsto \lambda t \]

which is monotone increasing since \( \frac{d}{dt} \lambda t = \lambda > 0 \)

Hence, we have not changed lengths and \( \lambda\tilde{c}: [0,1] \rightarrow [0,\lambda] \)

\[ t \mapsto \lambda(c([0,\lambda t])) = \lambda t \]

Linear.

\[ C: [a,b] \rightarrow (X,d) \quad \lambda(c) = \ell \]

\[ \Rightarrow \exists \tilde{C}_1: [0,\lambda] \rightarrow (X,d) \quad \& \quad \tilde{C}_2: [0,1] \rightarrow (X,d) \]

s.t. \( \text{i) im}(c) = \text{im}(\tilde{C}_1) = \text{im}(\tilde{C}_2) \)

\( \lambda\tilde{C}_1 = \text{id} [0,\lambda] \)

\( \lambda\tilde{C}_2 \) is linear

If moreover, \( d(c(a), c(b)) = \ell \) then \( C \) is the image of a geodesic segment in \( X \) connecting \( c(a) \) to \( c(b) \).